Self-Induced Topological Interface and Non-Hermitian Localization of Fluctuations in an Asymmetric Exclusion Process

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Model & Background

We study kinetics of gas-density fluctuations near a nonequilibrium steady state gas structure in a driven lattice gas with a single partially penetrable impurity site (ASEP on a ring of L sites) [1, 2].

The mean-field stachastic Smoluchowski equation for site occupation $n_k(t)$ reads:

$$\partial_t n_k = J_{k-1,k} - J_{k,k+1} + \delta I_k, \quad J_{k,k+1} = \nu^+ n_k h_{k+1} - \nu^- n_{k+1} h_k, \tag{1}$$

where $\nu^\pm = \nu(1 \pm g)$, $h_k = 1 - u_k - n_k$, $u_k = U\delta_{k,0}$ is impurity distribution, and δI_k is the Langevin source of current fluctuations. Linearising $n_k = n_k^s + \delta n_k$ around the nonequilibrium steady state (NESS) n_k^s yields a **non-Hermitian stochastic equation** for gas density fluctuations δn_k:

$$\partial_t \, \delta\mathbf{n} = \hat{C}\, \delta\mathbf{n} + \delta\mathbf{I}, \quad \hat{C}_{kj} = c_k^+ \delta_{j,k-1} + c_k^- \delta_{j,k+1} - [c_{k-1}^- + c_{k+1}^+]\delta_{kj}, \tag{2}$$

with asymmetric hopping rates $c_k^\pm = 1 - u_k \pm q_k$ governed by the **effective driving field** $q_k = g(1 - u_k - 2n_k^s)$, which changes sign across the dense/rarefied gas phases. In this regard, the system demonstrates spectral properties inherent to quantum non-Hermitian ones.

At certain critical values $(\bar{n}_c, U_c, g_c)$ the system demonstrates non-equilibrium transition [1, 3]. Above a critical field g_c, NESS n_k^s develops a two-domain gas structure with a self-induced domain wall (DW), and with dense domain ahead impurity. see [1] Dense and rarefied phases separately form two topological phases for fluctuations with opposite winding numbers $w = \pm 1$, producing **non-Hermitian skin effect induced by the DW**(localization of fluctuation eigenstates $\boldsymbol{e}_\alpha$)

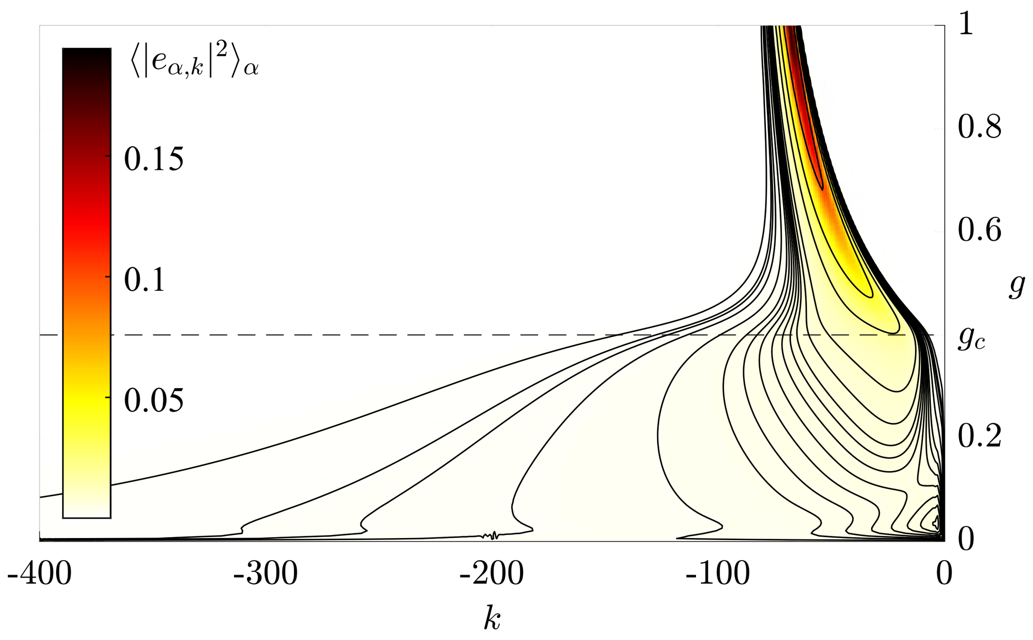


Figure 1: Non-hermitian skin localization: The averaged eigenvector vs. position k at varying driving field g.

$\mathcal{PT}$-Breaking Transition & Exceptional Points

Below the transition ($g < g_c$), the spectrum of $\hat{C}$ undergoes a $\mathcal{PT}$-**symmetry breaking** for finite chain $L < \infty$. At $g < g_0$, the spectrum and eigenstates are real. At g_0 the spectrum complexification is beginning, formation of complex-conjugate pairs via a cascade of exceptional points (EPs).

The onset of complexification is quantified by the fraction of complex eigenvalues $\eta(g) = N_{\mathrm{Im}}(g)/L$ and the eigenvector complexification measure: $\gamma = \frac{1}{L^2}\sum_{\alpha=1}^{L} \sqrt{\mathrm{Im}[\mathbf{e}_\alpha] \cdot \mathrm{Im}[\mathbf{e}_\alpha]}$, see [4].

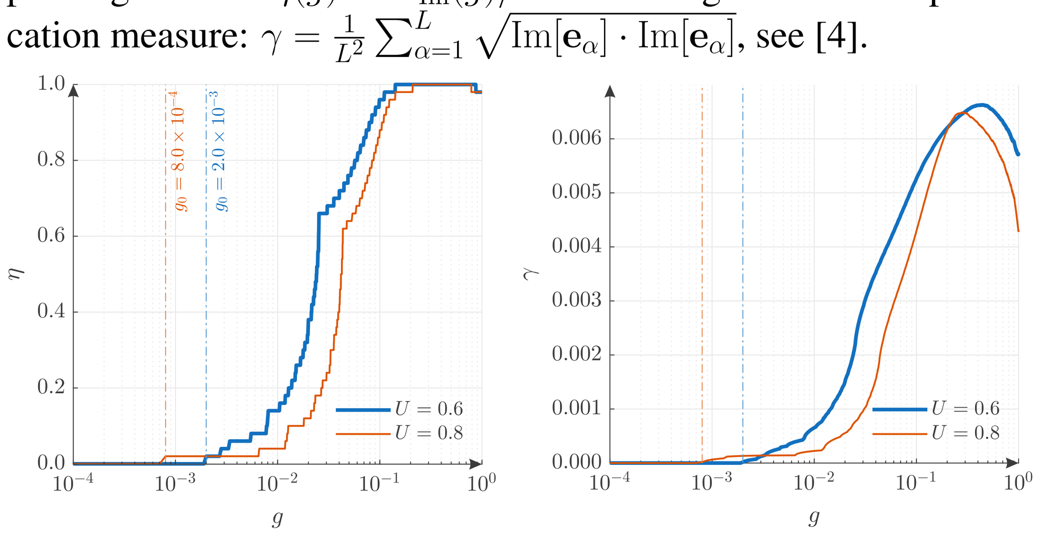


Figure 2: Spectrum complexification measures $\gamma(g)$ and $\eta(g)$.

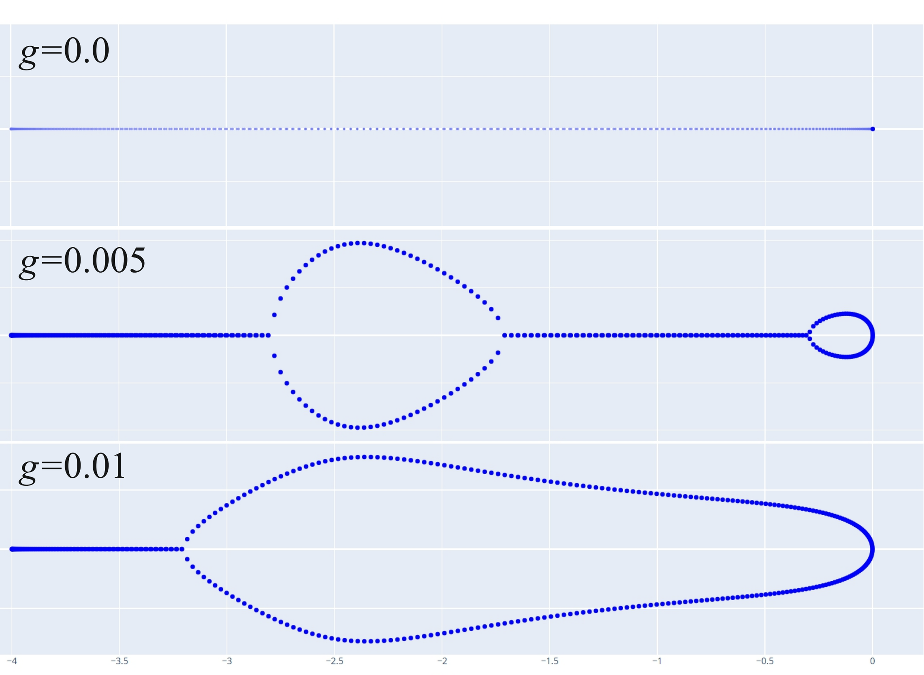


Figure 3: Eigenspectrum of C on a complex plane $(\mathrm{Re}\lambda_i, \mathrm{Im}\lambda_i)$ in the under-critical regime $g < g_c$. For $g > g_c$, see Figs. ======⇒

EP coalescence is confirmed via the **overlap** between conjugate eigenvector pairs: $\mathcal{O}_{\alpha,\bar{\alpha}} = \cos\theta = \frac{|\mathbf{e}_\alpha \cdot \mathbf{e}_{\bar{\alpha}}|}{\|\mathbf{e}_\alpha\| \, \|\mathbf{e}_{\bar{\alpha}}\|}$. At an EP: $\mathcal{O} \to 1$; for orthogonal modes: $\mathcal{O} = 0$.

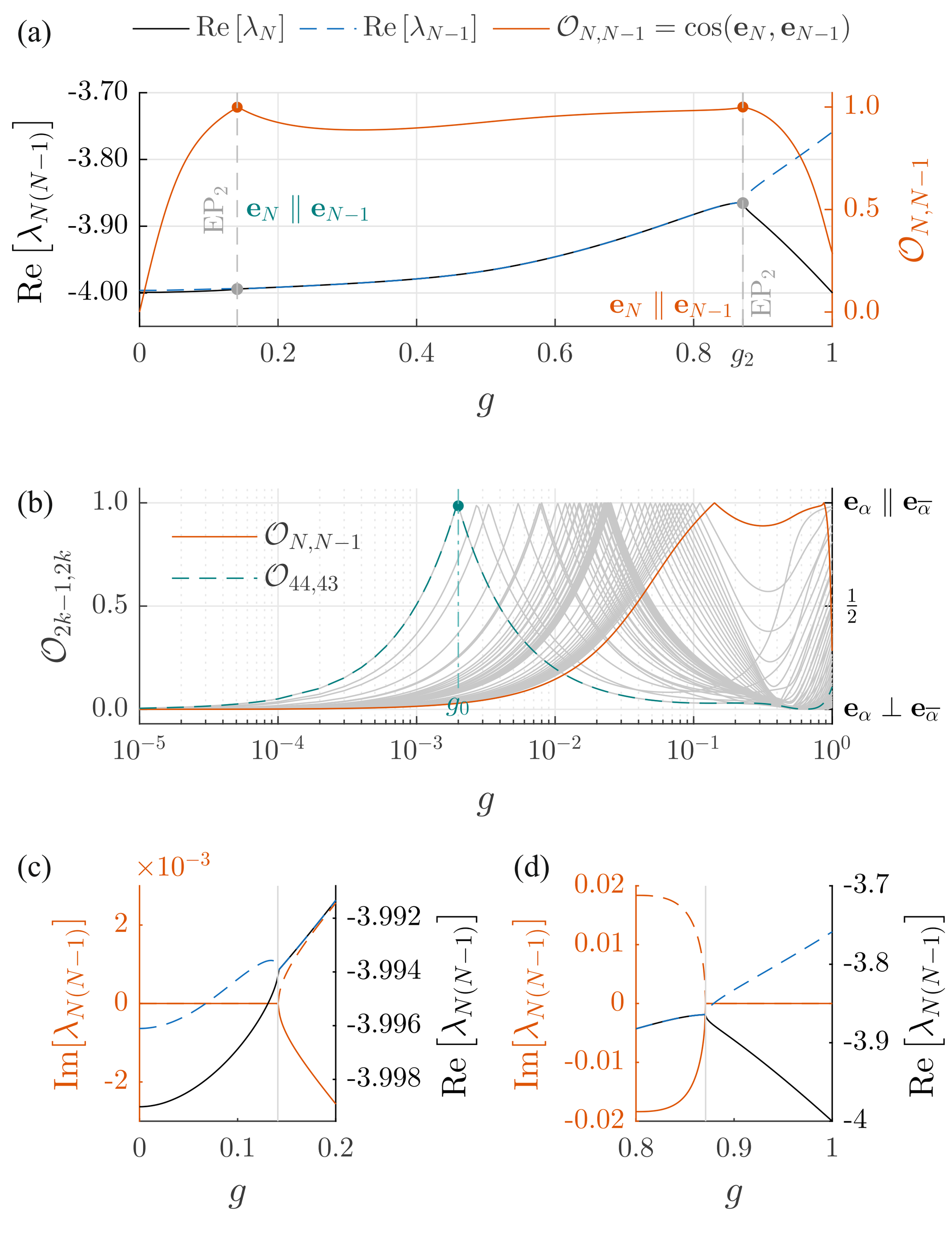


Figure 4: Overlap $\mathcal{O}_{\alpha,\bar{\alpha}}(g)$ for conjugate eigenvector pairs, showing the **cascade of EPs** as g increases. Each peak confirms eigenvector coalescence at the corresponding EP. Insets: real and imaginary parts of $(\lambda_L, \lambda_{L-1})$ exhibiting two EP transitions.

The $\mathcal{PT}$-breaking does **not** survive in the thermodynamic limit: $g_0(L) \approx 1/L^3 \to 0$ as $L \to \infty$.

Two Spectral Gaps Above Transition

Above g_c, the spectrum of $\hat{C}$ develops **two non-closing spectral gaps** (Fig. 5):

- $\Delta(g) = \min_{\alpha\neq 0}|\mathrm{Re}\,\lambda_\alpha|$ — **asymptotic relaxation rate** separating the forbidden bound state $\mathbf{e}_0$ ($\lambda_0 = 0$) from the bulk DW-localised skin modes.
- $\Delta'(g) = |\lambda_L - \lambda_{L-1}|$ — **second gap** separating the DW-localised bulk modes from a *delocalized* mode $\mathbf{e}_L$.

$$\Delta(g) \underset{L\to\infty}{\longrightarrow} \Delta_\infty > 0 \quad (g > g_c), \qquad \Delta(g) \sim L^{-2} \quad (g < g_c). \tag{3}$$

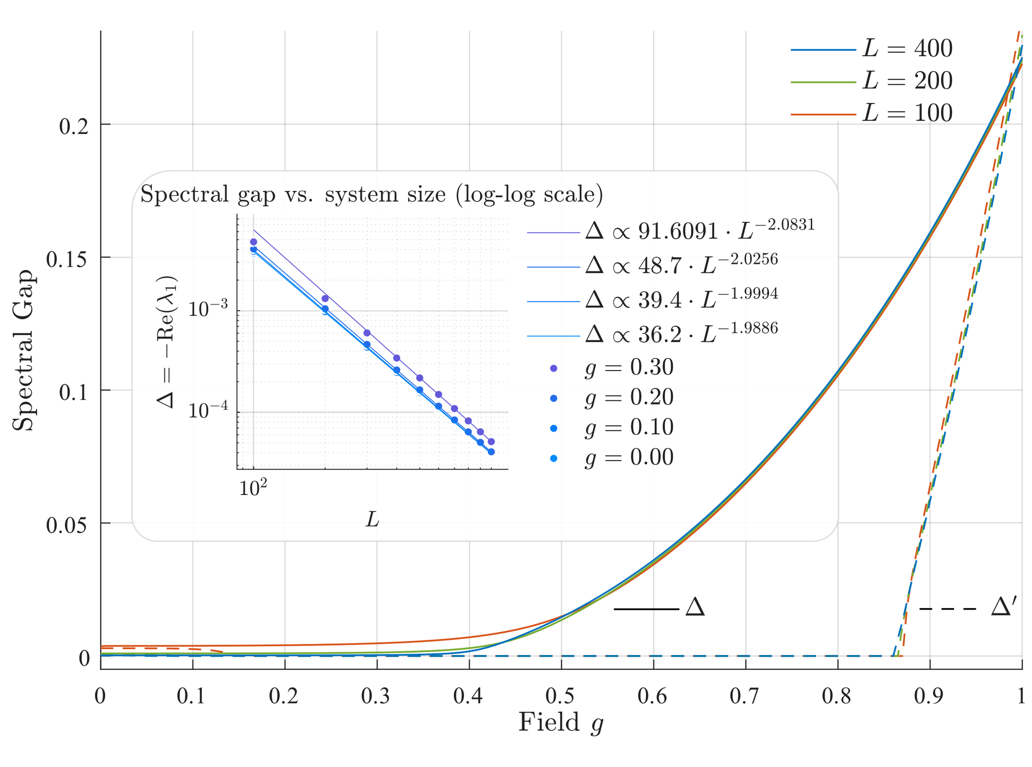
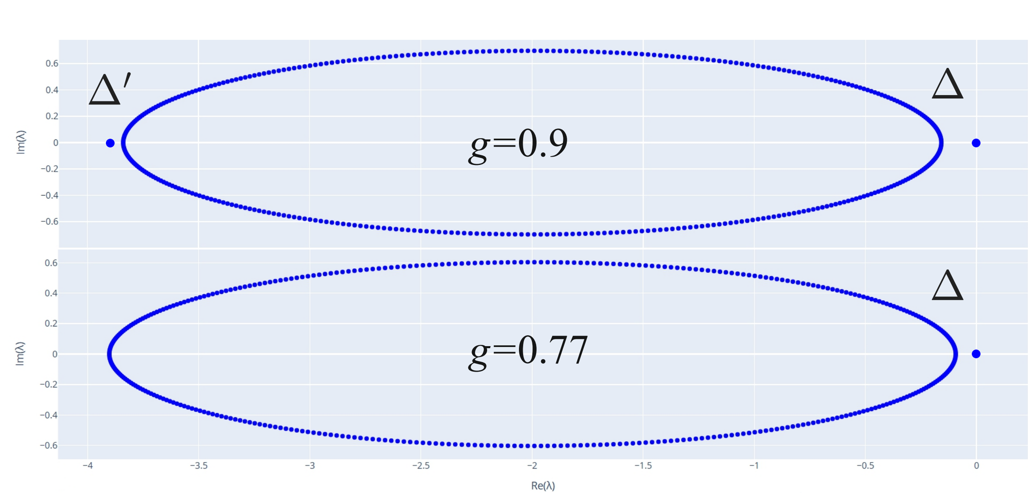


Figure 5: Spectral (relaxation) gap Δ (right gap) and second gap Δ' (left gap) vs. driving field g, for system size $L_0 = 401$, $\bar{n} = 0.3$, $U = 0.6$. Both gaps are non-closing in the thermodynamic limit above g_c.

Delocalized mode $\mathbf{e}_L$**:** with increasing $g > g_c$, the eigenstate $\mathbf{e}_L$ loses its localisation and becomes *fully delocalized* as $g \to 1$, while bulk modes remain strongly localised near the DW.



Long-Time Fluctuation Dynamics

The long-time dispersion of induced fluctuations is:

$$\langle \delta n_k^2 \rangle_{t\to\infty} \approx -\sum_{\alpha,\beta\neq 0} \frac{W_{\alpha\beta}}{\lambda_\alpha + \lambda_\beta} e_{\alpha,k}\, e_{\beta,k}, \tag{4}$$

with $W_{\alpha\beta}$ determined by the noise correlation matrix.

Since the spatial distribution of $\langle \delta n_k^2 \rangle$ is controlled by eigenstate localisation:

- **Above** g_c**:** DW-induced skin effect ⇒ fluctuations *pinned* near the domain wall; *suppressed* at impurity site. For driving-field noise: **total** suppression at $k = 0$ — the impurity behaves as a *dark state*.
- **Below** g_c**:** $\Delta \to 0$ as L^{-2} ⇒ long-lived fluctuations *accumulate* in time ($\langle x_\alpha x_\beta \rangle \propto t$ for $|\lambda_{\alpha(\beta)}| \approx \Delta \to 0$) — **linear instability** of the subcritical NESS.

After transition $(g > g_c)$, The state of impurity site loses sensitivity to external driving field and its noise, and behaves like dark-state in relation to the external driving field:

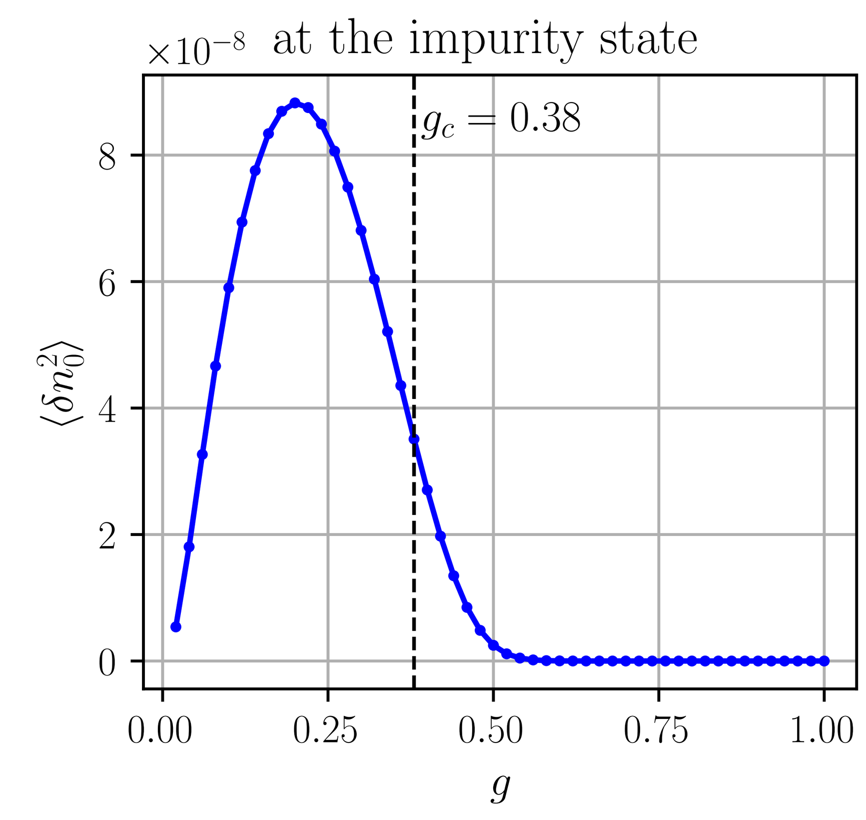


Figure 6: $\delta n_0^2(g)$ for the case of local density fluctuations induced by external driving field g.

Conclusions

- The non-Hermitian relaxation matrix $\hat{C}$ exhibits a **hierarchical cascade of exceptional points** below g_c; PT-breaking vanishes in the thermodynamic limit.
- Above g_c, **two non-closing spectral gaps** (Δ and Δ') emerge, separating the forbidden bound state, DW-localised skin modes, and a new *delocalized* mode.
- Particle-number conservation (U(1) symmetry) makes the $\lambda_0 = 0$ bound state **forbidden**, and the Liouvillian gap Δ **protects** the nonlinear two-domain NESS.
- **Below** g_c: closing of Δ signals a dissipative phase transition and leads to **linear instability** with fluctuation accumulation over time.
- **non-Hermitian skin localization effect** is shown for eigenvectors of **the spectrum of fluctuations**.

References

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