# Quantum-mechanical analysis of electron transport in a cylindrical crossed-field vacuum diode with a periodic boundary potential

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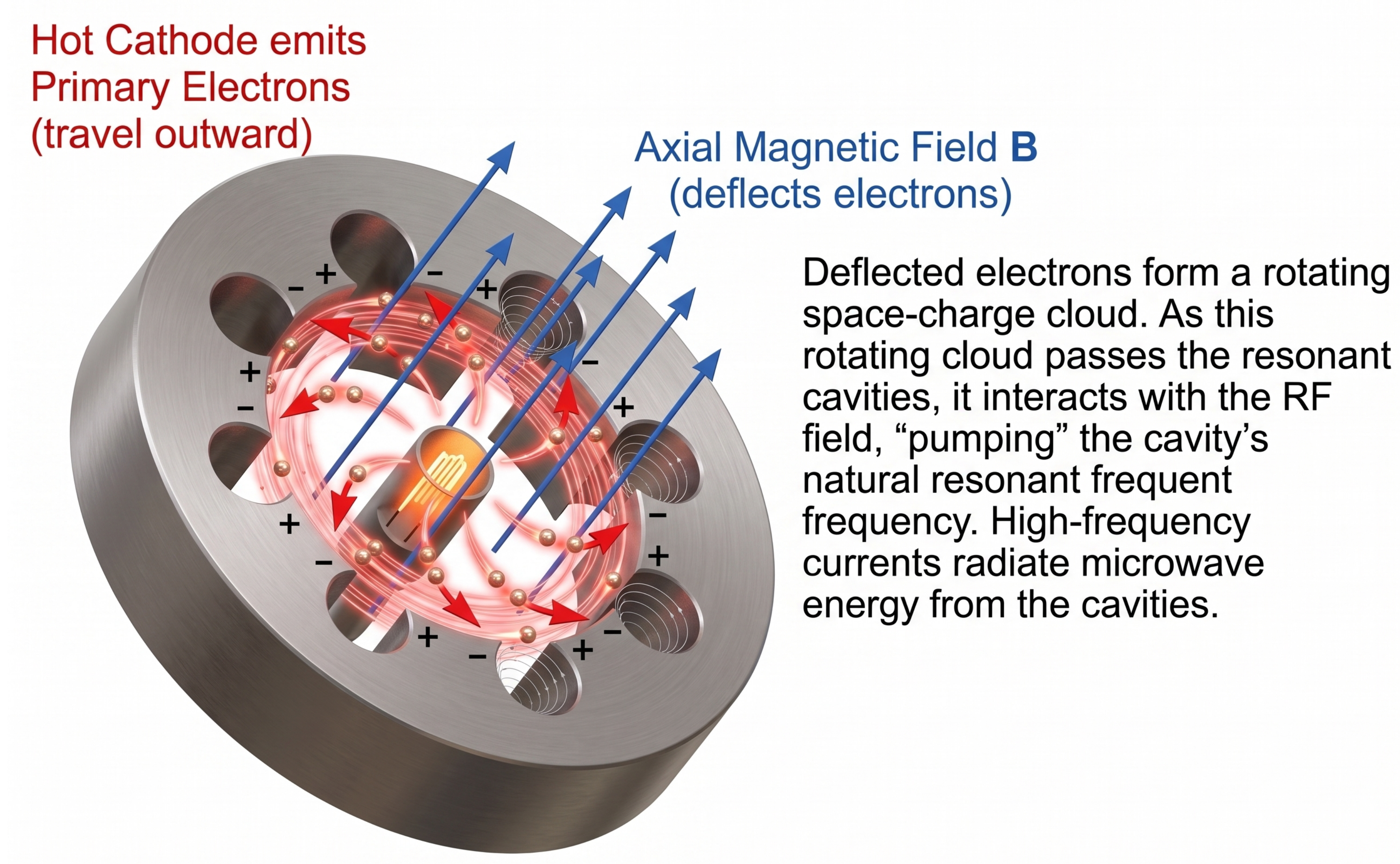
This study investigates the dynamics of the electron cloud in a cylindrical vacuum diode under stationary crossed fields $\mathbf{E} \perp \mathbf{B}$.
**Magnetic Field:** Uniform and axial $\mathbf{B} = (0, 0, B_z)$. The vector potential in the symmetric gauge is $\mathbf{A} = (0, B_z r/2, 0)$.
**Electric Field:** Radial, created by the potential difference $U_a$ between the coaxial cylinders $r_c$ and $r_a$. The anode operates as a periodic resonant structure.




*Schematic cross-section of a cylindrical vacuum diode (magnetron)*

**Classical Description:** Governed by the Lorentz force equation for an electron of mass $m$ and charge $-e$:

$$\frac{d\vec{v}}{dt} = -e\vec{E} - e[\vec{v} \times \vec{B}]. \tag{1}$$

This yields the classical Hull cutoff magnetic field, predicting absolute zero current if $B > B_{Hull} = \sqrt{\frac{8m_e U_a}{e r_a^2 (1 - r_c^2/r_a^2)^2}}$.

**Quantum Mechanical Description:** Governed by the Schrödinger equation for the wave function $\Psi(\mathbf{r}, t)$:

$$i\hbar\frac{\partial \Psi}{\partial t} = \left[\frac{1}{2m}(-i\hbar\nabla + e\mathbf{A})^2 - e\Phi(r)\right]\Psi. \tag{2}$$

Expanding the kinetic energy operator in cylindrical coordinates $(r, \phi, z)$, the Hamiltonian becomes:

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right) + \frac{eB_z}{2m}\hat{L}_z + \frac{e^2 B_z^2 r^2}{8m} - e\Phi(r), \tag{3}$$

where $\hat{L}_z = -i\hbar\frac{\partial}{\partial \phi}$ is the angular momentum operator.
Since $[\hat{H}, \hat{L}_z] = 0$, we separate variables:

$$\Psi_{n,l}(r, \phi) = \frac{1}{\sqrt{2\pi}} R_{n,l}(r) e^{il\phi}, \tag{4}$$

where $l = 0, \pm 1, \pm 2 \ldots$ is the magnetic quantum number.
**Boundary Conditions:** $R_{n,m}(r_c) = 0, \quad R_{n,m}(r_a) = 0$. Due to the infinite potential walls at the physical borders, the radial spectrum is forced to resolve into discrete steps indexed by $n = 0, 1, 2 \ldots$

Why choose a quantum mechanical description for a macroscopic device?

1. **Sub-Barrier Leakage:** Classical theory strictly prohibits current below the Hull cutoff. Quantum mechanics accurately predicts the exponentially decaying "dark current" via under-barrier tunneling.
2. **Discrete Energy Transfer (Smith-Purcell):** Interaction with periodic slow-wave structures (e.g., diffraction gratings or anode cavities) involves the absorption/emission of discrete momentum quanta $\hbar q$.
3. **Fundamental Noise Limits:** Shot noise in crossed fields fundamentally arises from discrete hopping events between quantized Landau orbits, which classical theory treats merely as a continuous fluid instability.

Substituting $\Psi_{n,l}$ into the Schrödinger equation yields a 1D radial equation with an effective potential $V_{eff}(r)$:

$$\left[-\frac{\hbar^2}{2m}\Delta_r + \underbrace{\frac{\hbar^2 l^2}{2mr^2} + \frac{m\omega_c^2 r^2}{8} + \frac{\hbar\omega_c l}{2} - e\Phi(r)}_{V_{eff}(r)}\right] R_{n,l}(r) = E_{n,l} R_{n,l}(r) \tag{5}$$

where $\omega_c = eB_z/m$ is the cyclotron frequency, $\Phi(r) = \frac{\ln(r/r_c)}{\ln(r_a/r_c)} U_a$.
**Solutions:** Near the equilibrium guiding center, the solutions are generalized Laguerre polynomials $L_n^{|l|}$. The energy spectrum (Landau levels in crossed fields) is:

$$E_{n,l} = \hbar\omega_c\left(n + \frac{1}{2} + \frac{l + |l|}{2}\right) - e\Phi(r_0), \tag{6}$$

where $n$ is the radial quantum number and $r_0$ is the radial coordinate of the guiding center.
The quantum probability current density is defined as:

$$\mathbf{j} = -\frac{i\hbar}{2m_e}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \frac{e}{m_e}\mathbf{A}|\psi|^2. \tag{7}$$

For stationary state $\Psi = R(r)e^{il\phi}$, so the radial component $J_r = 0$, and

$$J_\phi(r) = \sum_{n,m,k_z} f(n, m, k_z)\left(\frac{\hbar m}{m_e r} + \frac{\omega_c r}{2}\right)|R_{n,m}(r)|^2. \tag{8}$$

In perfectly uniform, unperturbed crossed fields, the stationary radial current is strictly zero. Electrons orbit infinitely without reaching the anode, perfectly matching the classical Hull cutoff condition.
To establish a non-vanishing current to the anode ($I_a \neq 0$), the symmetry of the stationary states must be broken. Quantum mechanics achieves this through time-dependent superposition states driven by high-frequency fields $\hat{V}(t) = -eV_{RF}(\vec{r}, t)$:

$$\Psi(\vec{r}, t) = \sum_k c_k(t)\psi_k(\vec{r})e^{-iE_k t/\hbar}. \tag{9}$$

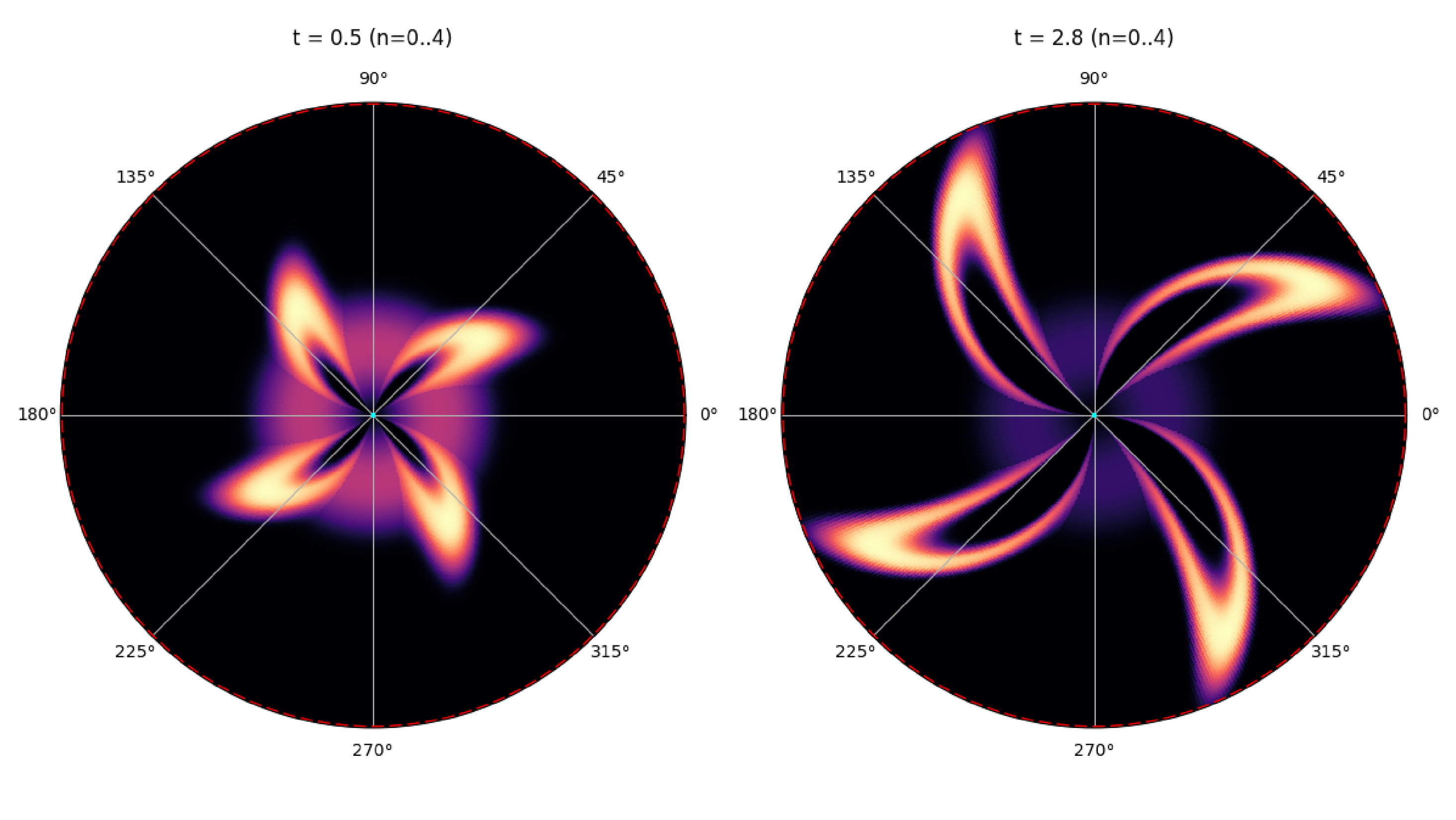
Evaluating $j_r$ using this time-dependent superposition reintroduces cross-terms:

$$j_r(r, t) = \frac{\hbar}{2m_e i}\sum_{k,f} c_k^*(t)c_f(t)\left(\psi_k^*\frac{\partial\psi_f}{\partial r} - \psi_f\frac{\partial\psi_k^*}{\partial r}\right)e^{i(E_k - E_f)t/\hbar} \neq 0. \tag{10}$$

To generate radial current, we must break the azimuthal symmetry. We introduce a perturbation: the slow-wave periodic RF potential of the anode structure:

$$\hat{V}_{RF}(r, \phi, t) = V_A\left(\frac{r}{r_a}\right)^{n_h}\cos(n_h\phi - \omega t), \tag{11}$$

where $n_h$ is the spatial harmonic number (e.g., $n_h = N/2$ for the $\pi$-mode).




*Space charge distribution for different moments of time*