Annotation on paper by Anton Pryimak " The parametrix problem for Toda equation with steplike initial data"

arXiv:1801.06213

The paper concerns with the justification of the long-time asymptotics for the Toda rarefaction wave by use of the Riemann-Hilbert problem approach. The Toda rarefaction wave is the Cauchy problem solution for the doubly infinite Toda lattice

$$\begin{aligned} \dot{b}(n,t) &= 2(a(n,t)^2 - a(n-1,t)^2), \\ \dot{a}(n,t) &= a(n,t)(b(n+1,t) - b(n,t)), \end{aligned} \qquad (n,t) \in \mathbb{Z} \times \mathbb{R}_+, \tag{1}$$

with steplike initial data

$$\begin{aligned} a(n,0) \to a, \quad b(n,0) \to b, \quad \text{as } n \to -\infty, \\ a(n,0) \to \frac{1}{2} \quad b(n,0) \to 0, \quad \text{as } n \to +\infty, \end{aligned} \tag{2}$$

where $a > 0$ and $b \in \mathbb{R}$ satisfy the condition

$$1 < b - 2a. \tag{3}$$

The study of the long time asymptotics of the Toda rarefaction wave in the regime when $n \to \infty$, $t \to +\infty$, with $\xi := \frac{n}{t}$ slowly varying, is called the Toda rarefaction problem. This problem was studied by Deift et all in [**1**] in a small transition region $\xi \approx 0$. In this pioneering work the nonlinear steepest descent method for the vector oscillatory Riemann-Hilbert problem was first applied to the Toda lattice. By use of the same method, in [**2**] the first and the second terms of the asymptotical expansion for the solution with respect to large t were obtained in all principal regions of (n,t) half plane under an additional condition

$$\sum_{n=1}^{\infty} \mathrm{e}^{\nu n} \big(|a(-n,0) - a| + |b(-n,0) - b| + |a(n,0) - \tfrac{1}{2}| + |b(n,0)|\big) < \infty, \tag{4}$$

for some $\nu > 0$. It was shown that for $t \to +\infty$ the solution of (1)–(4)

- In ihe region $n > t$ is asymptotically close to the right constants $\{\frac{1}{2}, 0\}$ plus a sum of solitons corresponding to the eigenvalues $\lambda_j < -1$;
- in the region $0 < n < t$:

$$a(n,t) = \frac{n}{2t} + O\Big(\frac{1}{t}\Big), \quad b(n,t) = 1 - \frac{n}{t} + O\Big(\frac{1}{t}\Big).$$

- In the region $-2at < n < 0$:

$$a(n,t) = -\frac{n}{2t} + O\Big(\frac{1}{t}\Big), \quad b(n,t) = b - 2a - \frac{n}{t} + O\Big(\frac{1}{t}\Big);$$

- in the region $n < -2at$, the solution is close to the left background constants $\{a, b\}$ plus a sum of solitons corresponding to the eigenvalues $\lambda_j > b + 2a$.

The second terms of order $O(t^{-1})$ are quite cumbersome and are not represented here. in [**2**], they were obtained under a conjecture, that these terms got contributions from main transformations of the Riemann-Hilbert problem approach only, and did not feel any impact from the parametrix problem solutions. This hypothesis was not justified at all. Moreover, in [**2**] any proof that the formulas above were indeed the asymptotics for the solution of (1)-(4) was absent.

The goal of the paper is to justify rigorously the asymptotics obtained in [**2**], and to prove the conjecture that solution of the parametrix problem indeed does not contribute in the second terms of asymptotic expansion. To this end we solve the parametrix problem and perform in all details a conclusive asymptotic analysis. Moreover, in the present paper the Toda rarefaction problem (1)-(3) is solved completely in the middle principal regions for the general case which admits resonances at the edges of the background spectra.

An analogous asymptotic analysis for the KdV equation was made in [**3**]. However, the analysis of the Toda lattice is different from the KdV case due to specific normalization conditions for the RH problem solution at infinity, which requires additional considerations.

References

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[2] I. Egorova, J. Michor and G. Teschl *Rarefaction waves for the Toda equation via nonlinear steepest descent*, Discrete Contin. Dyn. Syst. **38**, 2007-2028 (2018).

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