Anomalous singularity of the solution of the vector Dyson equation in the critical case

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Let $S = (s_{k,l})_{k,l=1}^{n} \in [0,+\infty)^{n\times n}$ be a symmetric matrix. The paper studies the *vector Dyson equation*

$$-\frac{1}{m_k(z)} = z + \sum_{l=1}^{n} s_{k,l} m_l(z), \quad k = 1, \dots, n, \tag{1}$$

where $m(z) = (m_1(z), \dots, m_n(z)) \in \mathbb{C}^n$ and $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$. The equation (1) has a unique solution satisfying $\operatorname{Im} m_k(z) > 0$ for all $k = 1, \dots, n$. This solution is denoted by m in the following. It can be shown that there is a unique probability measure ρ on $\mathbb{R}$ whose Stieltjes transform is given by

$$z \mapsto \frac{1}{n}\sum_{k=1}^{n} m_k(z).$$

The measure ρ is called the self-consistent density of states.

Consider an $nN \times nN$ hermitian matrix H with a block structure $H = \left(H^{k,l}\right)_{k,l=1}^{n}$, where $H^{k,l}$ are $N \times N$ matrices. Let the entries of H be centred independent (up to a hermitian symmetry) random variables with finite variation such that $E\left|\left(H^{k,l}\right)_{a,b}\right|^2 = s_{k,l}$ for all $k, l = 1, \dots, n$ and $a, b = 1, \dots, N$. It is known that the weak limit of empirical eigenvalue distributions of H as N goes to infinity is exactly ρ.

In the paper we consider the class of matrices S with strictly positive entries on the anti-diagonal and just above the anti-diagonal, vanishing entries below the anti-diagonal and nonnegative entries above it. For this class we prove that

$$\lim_{z\to 0}\left(m_k(z) z^{-\left(1-\frac{2k}{n+1}\right)}\right) = c_k e^{i\frac{\pi k}{n+1}}, \quad \lim_{E\to 0} \rho(E)|E|^{-\frac{n-1}{n+1}} > 0$$

for some positive constants c_k, $k = 1, \dots, n$. Here by $\rho(E)$ we denoted the density of ρ at E.

We also consider the case when S in (1) is replaced by a block matrix $\mathcal{S} \in [0,+\infty)^{nN\times nN}$. Under the assumption of a block structure analogous to the structure of S detailed above and with additional bounds on the entries of $\mathcal{S}$ we show that

$$\mathscr{C}_1 < |m_k(z)| \cdot |z|^{-\left(1-\frac{2}{n+1}\left\lceil \frac{k}{N}\right\rceil\right)} < \mathscr{C}_2, \quad k = 1, \dots, nN$$

in some neighbourhood of zero which can depend on N, where $\mathscr{C}_1, \mathscr{C}_2$ do not depend on N.