The convolution of Laurent formal series and the fundamental solution of an implicit linear differential equation over an arbitrary ring

A. Goncharuk

The results presented below have been published in [1].

Let K be an arbitrary ring, which is not necessarily commutative, with an identity element. We consider a convolution in the ring $K((\frac{1}{x}))$ of a Laurent formal series with a finite number of terms with positive powers. This operation is an algebraic analogue for the Hurwitz product of series, which is widely used in the theory of functions (see [2], [3]).

Definition 1. Define the convolution of two formal series $A(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \ldots$ and $B(x) = \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \ldots$ as the formal series

$$(A * B)(x) = a_1 B(x) - a_2 B'(x) + \frac{a_3}{2!} B''(x) + \ldots = \sum_{i=1}^{\infty} \frac{a_i}{(i-1)!} B^{(i-1)}(x).$$

Similarly, we can define the convolution of a formal series $A(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \ldots$ and a polinom.

Let $b \in K$ and $Q(x) \in K((\frac{1}{x}))$. Consider equation

$$by' + Q(x) = y. \tag{1}$$

Since the element b isn't necessarily invertible, then the equation (1) implicitly. It can be shown that this equation has a unique solution in $K((\frac{1}{x}))$.

Theorem 1. *Let $\varepsilon_b(x) = \frac{1}{x} - \frac{1!b}{x^2} + \frac{2!b^2}{x^3} - \frac{3!b^3}{x^4} \ldots$ be the Euler series. The unique solution to equation (1) such that this one belongs to $K((\frac{1}{x}))$, has the form of convolution of the $\varepsilon_b(x)$ with the inhomogeneity $Q(x)$:*

$$y(x) = (\varepsilon_b * Q)(x).$$

By this formulas we can view the Euler series $\varepsilon_b(x)$ as the fundamental solution of differential equation (1) in $K((\frac{1}{x}))$.

Let $\frac{1}{x}K[\frac{1}{x}]$ be the ring of Laurent polynomials of the following form: $\frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \ldots + \frac{a_n}{x^n}$, where $a_n \neq 0$.

Consider the equation

$$by' + R(x) = y, \tag{2}$$

where $R(x) \in \frac{1}{x}K[\frac{1}{x}]$.

Theorem 2. *The following conditions are equal:*

1. $a_n = (n-1)ba_{n-1} - (n-1)(n-2)b^2 a_{n-2} + \ldots + (-1)^{n-1}(n-1)!b^n a_1$;
2. $\varepsilon_b * R \in \frac{1}{x}K[\frac{1}{x}]$;
3. *equation (2) has the solution from the ring of Laurent polynomials.*

Suppose now K is a field of characteristic zero. Then occur following theorem:

Theorem 3. *The equation (2) has a rational solution if and only if the condition (**??**) hold.*

We can obtain the following generalization of condition (3):

Theorem 4. *Consider following inhomogeneous equation*

$$by' + R(x) = y,$$

$$\text{where } R(x) = \sum_{j=1}^{N} \sum_{k=1}^{m_j} \frac{a_{kj}}{(x-\beta_j)^{k_j}}. \tag{3}$$

This equation has the solution in the form (3) if and only if following condition hold:

$$a_{nj} = (n-1)ba_{(n-1)j} - (n-1)(n-2)b^2 a_{(n-2)j} + \ldots + (-1)^{n-1}(n-1)!b^n a_{1j}, \text{ for all } j = 1, \ldots, m.$$

Theorem 5. *Let $K \in \mathbb{C}$. Consider equation $y' = ay + R(x)$, $a \neq 0$. Then this equation has a solution in the elementary functions if and only if this equation has a rational solution.*

References

[1] Gefter, S.L. and Goncharuk, A.B. Fundamental Solution of an Implicit Linear Inhomogeneous First Order Differential Equation Over an Arbitrary Ring, J. Math. Sci. 219 (2016), 922-935.

[2] A. Hurwitz. Sur un théorème de M. Hadamard, C. V. Acad. Sci. (Paris), 1899.

[3] L. Bieberbach. Analitische Fortsetzung, Springer – Verlag, 1955.