Annotation
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The study of zero distribution of entire functions is one of the essential areas of mathematical analysis and was investigated by many mathematicians. One of the most important classes of entire functions is the Laguerre-Pólya class (see, for example, [3] or [4]).

Definition. A real entire function f is said to be in the *Laguerre-Pólya class*, written $f \in \mathcal{L} - \mathcal{P}$, if it can be expressed in the form

$$f(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 - \frac{x}{x_k}\right) e^{x x_k^{-1}}, \tag{1}$$

where $c, \alpha, \beta, x_k \in \mathbb{R}$, $x_k \neq 0$, $\alpha \geq 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} x_k^{-2} < \infty$. As usual, the product on the right-hand side can be finite or empty (in the latter case the product equals 1).

Various significant properties and characterizations of the Laguerre-Pólya class are described in [3], [10] and many other works.

The issue of understanding if a function belongs to the Laguerre-Pólya class is rather subtle.

The *partial theta-function*, $g_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{a^{j^2}}$, $a > 1$, was studied in [5] and [6]. The main question of [5] is whether the partial theta-function and its Taylor sections belong to the Laguerre-Pólya class. In [5] it is proved that there exists a constant $q_\infty, q_\infty \approx 3.23363666$, such that the partial theta-function (and all its odd Taylor sections) belongs to the Laguerre-Pólya class if and only if $a^2 \geq q_\infty$. There is also a series of works dedicated to the properties of the partial theta-function (see, for example [7], [8]). The paper [9] explains the importance of the constant q_∞ in the class of entire functions with positive coefficients having Taylor sections with only real zeros.

In our research, the class of entire functions $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with positive coefficients having decreasing second quotients of Taylor coefficients $q_n = q_n(f) := \frac{a_{n-1}^2}{a_{n-2}a_n}$ was studied. We have found the minimal constant such that if the limit of second quotients is not less than this constant, then the entire function belongs to the Laguerre-Pólya class (or, in other words, it has only real zeros).

Theorem 1 (T. H. Nguyen, A. Vishnyakova, [11]). *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ for all k, be an entire function. Suppose that $q_n(f)$ are decreasing in n, i.e. $q_2 \geq q_3 \geq q_4 \geq \ldots$, and $\lim_{n\to\infty} q_n(f) = b \geq q_\infty$. Then all the zeros of f are real and negative, in other words $f \in \mathcal{L} - \mathcal{P}$.*

Further, we also investigated the class of the entire functions with positive coefficients having increasing second quotients of Taylor coefficients. In [2] and [1], some important special functions with increasing sequence of second quotients of Taylor coefficients are studied.

Theorem 2 (T. H. Nguyen, A. Vishnyakova). *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$, *be an entire function. Suppose that the quotients* $q_n(f)$ *are increasing in* n, *and* $\lim_{n\to\infty} q_n(f) = c < q_\infty$. *Then the function* f *does not belong to the Laguerre-Pólya class.*

References

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