# MIXING RANK-ONE ACTIONS FOR INFINITE SUMS OF FINITE GROUPS

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Abstract. Let $G$ be a countable direct sum of finite groups. We construct an uncountable family of pairwise disjoint mixing (of any order) rank-one strictly ergodic free actions of $G$ on a Cantor set. All of them possess the property of minimal self-joinings (of any order). Moreover, an example of rigid weakly mixing rank-one strictly ergodic free $G$-action is given.

## 0. Introduction and definitions

This paper was inspired by the following question of D. Rudolph:

**Question.** *Which countable discrete amenable groups $G$ have mixing (funny) rank one free actions?*

Recall that a measure preserving action $T = (T_g)_{g\in G}$ of $G$ on a standard probability space $(X, \mathfrak{B}, \mu)$ is called

— *mixing* if $\lim_{g\to\infty} \mu(A \cap T_g B) = \mu(A)\mu(B)$ for all $A, B \in \mathfrak{B}$,

— *mixing of order $l$* if for any $\epsilon > 0$ and $A_0, \dots, A_l \in \mathfrak{B}$, there exists a finite subset $K \subset G$ such that

$$|\mu(T_{g_0} A_0 \cap \dots \cap T_{g_l} A_l) - \mu(A_0) \cdots \mu(A_l)| < \epsilon$$

for each collection $g_0, \dots, g_l \in G$ with $g_i g_j^{-1} \notin K$ if $i \neq j$,

— *weakly mixing* if the diagonal action $T \times T := (T_g \times T_g)_{g\in G}$ of $G$ on the product space $(X \times X, \mathfrak{B} \otimes \mathfrak{B}, \mu \times \mu)$ is ergodic,

— *totally ergodic* if every co-finite subgroup in $G$ acts ergodically,

— *rigid* if there exists a sequence $g_n \to \infty$ in $G$ such that $\lim_{n\to\infty} \mu(A \cap T_{g_n} B) \to \mu(A \cap B)$ for all $A, B \in \mathfrak{B}$.

We say that $T$ has *funny rank one* if there exist a sequence of measurable subsets $(A_n)_{n=1}^\infty$ in $X$ and a sequence of finite subsets $(F_n)_{n=1}^\infty$ in $G$ such that the subsets $T_g F_n$, $g \in F_n$, are pairwise disjoint for any $n$ and

$$\lim_{n\to\infty} \min_{H\subset F_n} \mu\left(B \triangle \bigsqcup_{g\in H} T_g A_n\right) = 0 \quad \text{for every } B \in \mathfrak{B}.$$

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If, moreover, $(F_n)_{n=1}^\infty$ is a subsequence of some 'natural' Følner sequence in $G$, we say that $T$ has *rank one*. For instance, if $G = \mathbb{Z}^d$, this 'natural sequence' is just the sequence of cubes; if $G = \sum_{i=1}^\infty G_i$ with every $G_i$ a finite group, the sequence $\sum_{i=1}^n G_i$ is 'natural', etc.

Up to now various examples of mixing rank-one actions were constructed for

— $G = \mathbb{Z}$ in [Or], [Ru], [Ad], [CrS], etc.,
— $G = \mathbb{Z}^2$ in [AdS],
— $G = \mathbb{R}$ in [Pr], [Fa],
— $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ in [DaS].

We also mention two more constructions of rank-one actions for

— $G = \mathbb{Z} \oplus \bigoplus_{n=1}^\infty \mathbb{Z}/2\mathbb{Z}$ in [Ju], where it was claimed that the $\mathbb{Z}$-subaction is mixing but it was only shown that it is weakly mixing, and
— $G$ is a countable Abelian group with a subgroup $\mathbb{Z}^d$ such that the quotient $G/\mathbb{Z}^d$ is locally finite in [Ma], where it was proved that a $\mathbb{Z}$-subaction is mixing and it was asked whether the whole action is mixing.

Notice that in all of these examples $G$ is Abelian and has elements of infinite order. In contrast to that we provide a different class of groups for which the answer to the question of D. Rudolph is affirmative.

**Theorem 0.1.** *Let $G = \bigoplus_{i=1}^\infty G_i$, where $G_i$ is a non-trivial finite group for every $i$.*

(i) *There exist uncountably many pairwise disjoint (and hence pairwise non-isomorphic) mixing rank-one strictly ergodic actions of $G$ on a Cantor set. Moreover, these actions are mixing of any order.*
(ii) *There exists a weakly mixing rigid (and hence non-mixing) rank-one strictly ergodic action of $G$ on a Cantor set.*

Concerning (i), it is worth to note that any mixing rank-one $\mathbb{Z}$-action is mixing of any order by [Ka] and [Ry] (see also an extension of that to actions of some Abelian groups with elements of infinite order in [JuY]). We do not know whether this fact holds for all mixing rank-one action of countable sums of finite groups.

To prove the theorem, we combine the original Ornstein's idea of 'random spacer' (in the cutting-and-stacking construction process) [Or] and the more recent $(C, F)$-construction developed in [Ju], [Da1], [Da2], [DaS1], [DaS2] to produce funny rank-one actions with various dynamical properties. However, unlike all of the known examples of $(C, F)$-actions, the actions in this paper are constructed without adding any spacer (cf. with [Ju], where all the spacers relate to $\mathbb{Z}$-subaction only). Instead of that on the $n$-th step we just cut the $n$-'column' into 'subcolumns' and then rotate each 'subcolumn' in a 'random way'. In the limit we obtain a topological $G$-action on a compact Cantor space.

Our next concern is to describe all ergodic self-joinings of the $G$-actions constructed in Theorem 0.1. Recall a couple of definitions.

Given two ergodic $G$-actions $T$ and $T'$ on $(X, \mathfrak{B}, \mu)$ and $(X', \mathfrak{B}', \mu')$ respectively, we denote by $J(T, T')$ the set of *joinings* of $T$ and $T'$, i.e. the set of $(T_g \times T'_g)_{g \in G}$-invariant measures on $\mathfrak{B} \otimes \mathfrak{B}'$ whose marginals on $\mathfrak{B}$ and $\mathfrak{B}'$ are $\mu$ and $\mu'$ respectively. The corresponding dynamical system $(X \times X', \mathfrak{B} \otimes \mathfrak{B}', \mu \times \mu')$ is also called a joining of $T$ and $T'$. By $J^e(T, T') \subset J(T, T)$ we denote the subset of ergodic joinings of $T$ and $T'$ (it is never empty). In a similar way one can define the joininings $J(T_1, \ldots, T_l)$ for any finite family $T_1, \ldots, T_l$ of $G$-actions. If

$J(T_1,\dots,T_l)=\{\mu_1\times\cdots\times\mu_l\}$ then the family $T_1,\dots,T_l$ is called *disjoint.* If $T_1=\cdots=T_l$ we speak about $l$-fold self-joinings of $T_1$ and use notation $J_l(T)$ for $J(\underbrace{T,\dots,T}_{l\text{ times}})$. For $g\in G$, we denote by $g^\bullet$ the conjugacy class of $g$. We also let

$$\mathrm{FC}(G):=\{g\in G\mid g^\bullet\text{ is finite}\}.$$

Clearly, $\mathrm{FC}(G)$ is a normal subgroup of $G$. If $G$ is Abelian or $G$ is a sum of finite groups then $\mathrm{FC}(G)=G$. For any $g\in\mathrm{FC}(G)$, we define a measure $\mu_{g^\bullet}$ on $(X\times X,\mathfrak{B}\otimes\mathfrak{B})$ by setting

$$\mu_{g^\bullet}(A\times B):=\frac{1}{\#g^\bullet}\sum_{h\in g^\bullet}\mu(A\cap T_hB).$$

It is easy to verify that $\mu_{g^\bullet}$ is a self-joining of $T$. Moreover, the map $(x,T_h^{-1}x)\mapsto(x,h)$ is an isomorphism of $(X\times X,\mu_{g^\bullet},T\times T)$ onto $(X\times g^\bullet,\mu\times\nu,\widetilde{T})$, where $\nu$ is the equidistribution on $g^\bullet$ and the $G$-action $\widetilde{T}=(\widetilde{T}_t)_{t\in G}$ is given by

$$\widetilde{T}_t(x,h)=(T_tx,tht^{-1}),\ x\in X,\ h\in g^\bullet.$$

It follows that $\widetilde{T}$ (and hence the self-joining $\mu_{g^\bullet}$ of $T$) is ergodic if and only if the action $(T_t)_{t\in C(g)}$ is ergodic, where $C(g)=\{t\in G\mid tg=gt\}$ stands for the centralizer of $g$ in $G$. Notice also that $C(g)$ is a co-finite subgroup of $G$ because of $g\in\mathrm{FC}(G)$. Hence $\{\mu_{g^\bullet}\mid g\in\mathrm{FC}(G)\}\subset J_2^e(T)$ whenever $T$ is totally ergodic.

**Definition 0.2.** If $J_2^e(T)\subset\{\mu_{g^\bullet}\mid g\in\mathrm{FC}(G)\}\cup\{\mu\times\mu\}$ then we say that $T$ has *2-fold minimal self-joinings* ($\mathrm{MSJ}_2$).

This definition extends naturally to higher order self-joinings as follows. Given $l\geq 1$ and $g\in G^{l+1}$, we denote by $g^{\bullet l}$ the orbit of $g$ under the $G$-action on $G^{l+1}$ by conjugations:

$$h\cdot(g_0,\dots,g_l):=(hg_0h^{-1},\dots,hg_lh^{-1}).$$

Let $P$ be a partition of $\{0,\dots,l\}$. For an atom $p\in P$, we denote by $i_p$ the minimal element in $p$. We say that an element $g=(g_0,\dots,g_l)\in\mathrm{FC}(G)^{l+1}$ is *$P$-subordinated* if $g_{i_p}=1_G$ for all $p\in P$. For any such $g$, we define a measure $\mu_{g^{\bullet l}}$ on $(X^{l+1},\mathfrak{B}^{\otimes(l+1)})$ by setting

$$\mu_{g^{\bullet l}}(A_0\times\cdots\times A_l):=\frac{1}{\#g^{\bullet l}}\sum_{(h_0,\dots,h_l)\in g^{\bullet l}}\prod_{p\in P}\mu\Big(\bigcap_{i\in p}T_{h_i}A_i\Big).$$

It is easy to verify that $\mu_{g^{\bullet l}}$ is an $(l+1)$-fold self-joining of $T$. Reasoning as above one can check that $\mu_{g^{\bullet l}}$ is ergodic whenever $T$ is weakly mixing.

**Definition 0.3.** We say that $T$ has *$(l+1)$-fold minimal self-joinings* ($\mathrm{MSJ}_{l+1}$) if

$$J_{l+1}^e(T)\subset\{\mu_{g^{\bullet l}}\mid g\text{ is }P\text{-subordinated for a partition }P\text{ of }\{0,\dots,l\}\}.$$

If $T$ has $\mathrm{MSJ}_l$ for any $l>1$, we say that $T$ has MSJ.

In case $G$ is Abelian, these definitions agree with the—common now—definitions of $\mathrm{MSJ}_{l+1}$ and MSJ by A. del Junco and D. Rudolph [JuR] who considered self-joinings $\mu_{g^{\bullet l}}$ only when $g$ belongs to the center of $G^{l+1}$. However we find their definition somewhat restrictive for non-commutative groups since, for instance, countable sums of non-commutative finite groups can never have actions with $\mathrm{MSJ}_2$ in their sense.

Now we record the second main result of this paper.

**Theorem 0.4.** *The actions constructed in Theorem 0.1(i) all have MSJ.*

We notice that a part of the analysis from [Ru] can be carried over to the case of $G$-actions with MSJ. In this paper we only show that such actions have trivial *product centralizer*. Moreover, as follows from [Da3], every $G$-action with $\text{MSJ}_2$ is *effectively prime*, i.e. has no factors except for the obvious ones: the sub-$\sigma$-algebras of subsets fixed by finite normal subgroups in $G$. In particular, there exist no free factors.

We now briefly summarize the organization of the paper. In Section 1 we outline the $(C,F)$-construction of rank-one actions as it appeared in [Da1]. In Section 2, for any countable sum $G$ of finite groups, we construct a $(C,F)$-action $T$ of $G$ which is mixing of any order. A rigid weakly mixing action of $G$ also appears there. In Section 3 we demonstrate that $T$ has MSJ. In Section 4 we show how to perturb the construction of $T$ to obtain an uncountable family of pairwise disjoint mixing rank-one $G$-actions with MSJ. In the final Section 5 we discuss some implications of MSJ: trivial centralizer, trivial product centralizer and effective primality.

*Acknowledgement.* The author thanks the referee for the useful suggestions that improved the paper. In particular, in the present proof of Theorem 0.4 we deduce $\text{MSJ}_l$ from the $l$-fold mixing (as J. King does for $\mathbb{Z}$-actions in [Ki]). Our original proof (independent of multiple mixing) was longer and noticeably more complicated.

## 1. $(C,F)$-construction

In this section we recall the $(C,F)$-construction of rank-one actions.

From now on $G = \sum_{i=1}^{\infty} G_i$, where $G_i$ is a non-trivial finite group for each $i \geq 1$. To construct a probability preserving $(C,F)$-action of $G$ (see [Ju], [Da1], [DaS2]) we need to define two sequences $(F_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$ of finite subsets in $G$ such that the following are satisfied:

$$(F_n)_{n\geq 0} \text{ is a Folner sequence in } G,\ F_0 = \{1_G\}, \tag{1-1}$$

$$F_n C_{n+1} \subset F_{n+1},\ C_{n+1} > 1, \tag{1-2}$$

$$F_n c \cap F_n c' = \emptyset \text{ for all } c \neq c' \in C_{n+1}, \tag{1-3}$$

$$\lim_{n\to\infty} \frac{\#F_n}{\#C_1 \cdots \#C_n} < \infty. \tag{1-4}$$

Suppose that an increasing sequence of integers $0 < k_1 < k_2 < \cdots$ is given. Then we define $(F_n)_{n\geq 0}$ by setting $F_0 := \{1_G\}$ and $F_n := \sum_{i=1}^{k_n} G_i$ for $n \geq 1$. Clearly, (1-1) is satisfied. Suppose now that we are also given a sequence of maps $s_n : H_n \to F_n$, where $H_0 := \sum_{i=1}^{k_1} G_i$ and $H_n := \sum_{i=k_n+1}^{k_{n+1}} G_i$ for $n \geq 1$. Then we define two sequences of maps $c_{n+1}, \phi_n : H_n \to F_{n+1}$ by setting $\phi_n(h) := (0,h)$ and $c_{n+1}(h) := (s_n(h), h)$. Finally, we let $C_{n+1} := c_{n+1}(H_n)$ for all $n \geq 0$. It is easy to verify that (1-2)–(1-4) are all fulfilled. Moreover, a stronger version of (1-2) holds:

$$F_n C_{n+1} = F_{n+1}. \tag{1-5}$$

We now put $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$ and define a map $i_n : X_n \to X_{n+1}$ by setting

$$i_n(f_n, d_{n+1}, d_{n+2}, \dots) := (f_n d_{n+1}, d_{n+2}, \dots).$$

Clearly, $X_n$ is a compact Cantor space. It follows from (1-5) and (1-3) that $i_n$ is well defined and it is a homeomorphism of $X_n$ onto $X_{n+1}$. Denote by $X$ the topological inductive limit of the sequence $(X_n, i_n)_{n=1}^\infty$. As a topological space $X$ is canonically homeomorphic to any $X_n$ and in the sequel we will often identify $X$ with $X_n$ suppressing the canonical identification maps. We need the structure of inductive limit to define the $(C,F)$-action $T$ on $X$ as follows. Given $g \in G$, consider any $n \geq 0$ such that $g \in F_n$. Every $x \in X$ can be written as an infinite sequence $x = (f_n, d_{n+1}, d_{n+2}, \dots)$ with $f_n \in F_n$ and $d_m \in C_m$ for $m > n$ (i.e. we identify $X$ with $X_n$). Now we put

$$T_g x := (gf_n, d_{n+1}, d_{n+2}, \dots) \in X_n.$$

It is easy to verify that $T_g$ is a well defined homeomorphism of $X$. Moreover, $T_g T_{g'} = T_{gg'}$, i.e. $T := (T_g)_{g \in G}$ is a topological action of $G$ on $X$.

**Definition 1.1.** We call $T$ the $(C,F)$-action of $G$ associated with $(k_n, s_{n-1})_{n=1}^\infty$.

We list without proof several properties of $T$. They can be verified easily by the reader (see also [Da1]).

— $T$ is a minimal uniquely ergodic (i.e. strictly ergodic) free action of $G$.

— Two points $x = (f_n, d_{n+1}, d_{n+2}, \dots)$ and $x = (f'_n, d'_{n+1}, d'_{n+2}, \dots) \in X_n$ are $T$-orbit equivalent if and only if $d_i = d'_i$ eventually (i.e. for all large enough $i$). Moreover, $x' = T_g x$ if and only if

$$g = \lim_{i \to \infty} f'_n d'_{n+1} \cdots d'_{n+i} d_{n+i}^{-1} \cdots d_{n+1}^{-1} f_n^{-1}.$$

— The only $T$-invariant probability measure $\mu$ on $X$ is the product of the equidistributions on $F_n$ and $C_{n+i}$, $i \in \mathbb{N}$ (if $X$ is identified with $X_n$).

For each $A \subset F_n$, we let $[A]_n := \{x = (f_n, d_{n+1}, \dots) \in X_n \mid f_n \in A\}$ and call it an *$n$-cylinder*. The following holds:

$$\begin{aligned}
&[A]_n \cap [B]_n = [A \cap B]_n, \text{ and } [A]_n \cup [B]_n = [A \cup B]_n,\\
&[A]_n = \bigsqcup_{d \in C_{n+1}} [Ad]_{n+1},\\
&T_g[A]_n = [gA]_n \text{ if } g \in F_n,\\
&\mu([Ad]_{n+1}) = \frac{1}{\#C_{n+1}} \mu([A]_n) \text{ for any } d \in C_{n+1},\\
&\mu([A]_n) = \lambda_{F_n}(A),
\end{aligned}$$

where $\lambda_{F_n}$ is the normalized Haar measure on $F_n$. Moreover, for each measurable subset $B \subset X$,

$$\lim_{n \to \infty} \min_{A \subset F_n} \mu(B \triangle [A]_n) = 0. \tag{1-6}$$

Hence $T$ has rank one.

## 2. Mixing $(C,F)$-actions

Our purpose in this section is to construct a rank-one action of $G$ which is mixing of any order. This action will appear as a $(C,F)$-action associated with some specially selected sequence $(k_n, s_{n-1})_{n \geq 1}$. We first state several preliminary results.

Given finite sets $A$ and $B$ and a map $x \in A^B$, we denote by $\operatorname{dist} x$ or $\operatorname{dist}_{b \in B} x(b)$ the measure $(\#B)^{-1} \sum_{b \in B} \chi_{x(b)}$ on $A$. Here $\chi_{x(b)}$ stands for the probability supported at the point $x(b)$.

**Lemma 2.1.** *Let $A$ be a finite set and let $\lambda$ be the equidistribution on $A$. Then for any $\epsilon > 0$ there exist $c > 0$ and $m \in \mathbb{N}$ such that for any finite set $B$ with $\#B > m$,*

$$\lambda^B(\{x \in A^B \mid \|\operatorname{dist} x - \lambda\| > \epsilon\}) < e^{-c\#B}.$$

For the proof we refer to [Or] or [Ru]. We will also use the following combinatorial lemma.

**Lemma 2.2.** *For any $l \in \mathbb{N}$, let $N_l := 3^{l(l-1)/2}$ and $\delta_l := 5^{-l(l-1)/2}$ Let $H$ be a finite group. Then for any family $h_1, \ldots, h_l$ of mutually different elements of $H$ and any subset $B \subset H$ with $\#B > 3/\delta_l$, there exists a partition of $B$ into subsets $B_i$, $1 \le i \le N_l$, such that the subsets $h_1B_i, h_2B_i, \ldots, h_lB_i$ are mutually disjoint and $\#B_i \ge \delta_l\#B$ for any $i$.*

*Proof.* We leave to the reader the simplest case when $l = 2$. Hint: assume that $h_1 = 1_H$ and consider the partition of $H$ into the right cosets by the cyclic group generated by $h_2$.

Suppose that we already proved the assertion of the lemma for some $l$ and we want to prove it for $l + 1$. Take any $h_1 \ne h_2 \ne \cdots \ne h_{l+1} \in H$ (in such a way we denote mutually different elements of $H$). Given a subset $B \subset H$ with $\#B > 3/\delta_l$, we first partition $B$ into subsets $B_i$, $1 \le i \le N_l$, such that the subsets $h_2B_i, h_3B_i, \ldots, h_{l+1}B_i$ are mutually disjoint and $\#B_i \ge \delta_l\#B \ge 3 \cdot 5^l$. For every $i$, there exists a partition $B_i = \bigsqcup_{i_1=1}^3 B_{i,i_1}$ such that $h_1B_{i,i_1} \cap h_2B_{i,i_1} = \emptyset$ and $\#B_{i,i_1} \ge 0.2\#B_i$, $1 \le i_1 \le 3$. Next, we partition every $B_{i,i_1}$ into 3 subsets $B_{i,i_1,i_2}$ such that $h_1B_{i,i_1,i_2} \cap h_3B_{i,i_1,i_2} = \emptyset$ and $\#B_{i,i_1,i_2} \ge 0.2\#B_{i,i_1}$, $1 \le i_2 \le 3$, and so on. Finally, we obtain a partition

$$B = \bigsqcup_{i=1}^{N_l} \bigsqcup_{i_1,\ldots,i_l=1}^{3} B_{i,i_1,\ldots,i_l}$$

which is as desired. $\square$

Given a finite set $A$, a finite group $H$ and elements $h_1, \ldots, h_l \in H$, we denote by $\pi_{h_1,\ldots,h_l}$ the map $A^H \to (A^l)^H$ given by

$$(\pi_{h_1,\ldots,h_l}x)(k) = (x(h_1k), \ldots, x(h_lk)).$$

For $x \in A^H$, we define $x^* \in A^H$ by setting $x^*(h) := x(h^{-1})$, $h \in H$.

**Lemma 2.3.** *Given $l \in \mathbb{N}$ and $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that for any finite group $H$ with $\#H > m$, one can find $s \in A^H$ such that*

$$\|\operatorname{dist} \pi_{h_1,\ldots,h_l}s - \lambda^l\| < \epsilon \quad \text{and} \quad \|\operatorname{dist} \pi_{h_1,\ldots,h_l}s^* - \lambda^l\| < \epsilon \tag{2-1}$$

*for all $h_1 \ne h_2 \ne \cdots \ne h_l \in H$.*

*Proof.* Take any finite group $H$ and set

$$B_H := \bigcup_{h_1\ne\cdots\ne h_l\in H} \{x \in A^H \mid \|\operatorname{dist} \pi_{h_1,\ldots,h_l}x - \lambda^l\| > \epsilon\}.$$

To prove the left hand side inequality in (2-1) it suffices to show that $\lambda^H(B_H) < 1$ whenever $\#H$ is large enough. Moreover, since the map $A^H \ni x \mapsto x^* \in A^H$ preserves the measure $\lambda^H$, the right hand side inequality in (2-1) will follow from the left hand side one if we prove that $\lambda^H(B_H) < 0.5$.

Fix $h_1 \neq \cdots \neq h_l \in H$ and apply Lemma 2.2 to partition $H$ into subsets $H_i$, $1 \leq i \leq N_l$, such that

(2-2) $\qquad \#H_i \geq \delta_l \#H$ and

(2-3) $\qquad$ the subsets $h_1 H_i, \dots, h_l H_i$ are mutually disjoint

for every $i$. Denote by $r_i : (A^l)^H \to (A^l)^{H_i}$ the natural restriction map. Then we deduce from (2-3) that $r_i \circ \pi_{h_1,\cdots,h_l}$ maps $\lambda^H$ onto $(\lambda^l)^{H_i}$. Since $\operatorname{dist} \pi_{h_1,\dots,h_l} x = \sum_i (\#H_i/\#H) \cdot \operatorname{dist}(r_i \circ \pi_{h_1,\dots,h_l})x$, it follows that

$$\begin{aligned}\lambda^H(\{x \in A^H \mid \|\operatorname{dist} \pi_{h_1,\dots,h_l} x - \lambda^l\| > \epsilon\})&\\ \leq \sum_i \lambda^H(\{x \in A^H \mid \|\operatorname{dist}(r_i \circ \pi_{h_1,\dots,h_l})x - \lambda^l\| > \epsilon\})&\\ = \sum_i (\lambda^l)^{H_i}(\{y \in (A^l)^{H_i} \mid \|\operatorname{dist} y - \lambda^l\| > \epsilon\}).&\end{aligned}$$

By Lemma 2.2 and (2-2), there exists $c > 0$ such that if $\#H$ is large enough then the $i$-th term in the latter sum is less then $e^{-c\#H_i} < e^{-c\delta_l \#H}$. Hence

$$\lambda^H(B_H) \leq N_l \binom{\#H}{l} e^{-c\delta_l \#H}$$

and the assertion of the lemma follows. $\square$

Now we are ready to define the sequence $(k_n, s_{n-1})_{n\geq 1}$. Fix a sequence of positive reals $\epsilon_n \to 0$. On the first step one can take arbitrary $k_1$ and $s_0$. Suppose now—on the $n$-th step—we already have $k_n$ and $s_{n-1}$ and we want to define $k_{n+1}$ and $s_n$. For this, we apply Lemma 2.3 with $A := F_n$, $l := n$ and $\epsilon := \epsilon_n$ to find $k_{n+1}$ large so that there exists $s_n \in A^{H_n}$ satisfying

(2-4) $\qquad \|\operatorname{dist} \pi_{h_1,\dots,h_n} s_n - (\lambda_{F_n})^n\| < \epsilon_n$ for all $h_1 \neq \cdots \neq h_n \in H_n$.

Recall that $H_n := \sum_{i=k_n+1}^{k_{n+1}} G_i$ and $F_n := \sum_{i=1}^{k_n} G_i$ for $n \geq 1$. Without loss of generality we may also assume that $k_{n+1} - k_n \geq n$ and hence $\sum_{n=1}^{\infty} (\#H_n)^{-1} < \infty$.

Denote by $T$ the $(C, F)$-action of $G$ on $(X, \mathfrak{B}, \mu)$ associated with $(k_n, s_{n-1})_{n=1}^{\infty}$.

**Theorem 2.4.** *$T$ is mixing of any order.*

*Proof.* **(I)** We first show that $T$ is mixing (of order 1).

Recall that a sequence $g_n \to \infty$ in $G$ is called *mixing for $T$* if

$$\lim_{n\to\infty} \mu(T_{g_n} B_1 \cap B_2) = \mu(B_1)\mu(B_2) \quad \text{for all } B_1, B_2 \in \mathfrak{B}.$$

Clearly, $T$ is mixing if and only if any sequence going to infinity in $G$ contains a mixing subsequence. Since every subsequence of a mixing sequence is mixing itself, to prove (I) it suffices to show that every sequence $(g_n)_{n=1}^{\infty}$ in $G$ with $g_n \in F_{n+1} \setminus F_n$

for all $n$ is mixing. Notice first that there exist (unique) $f_n \in F_n$ and $h_n \in H_n \setminus \{1\}$ with $g_n = f_n\phi_n(h_n)$. Fix any two subsets $A, B \subset F_n$. We notice that for each $h \in H_n$,

$$g_nAc_{n+1}(h) = f_nAs_n(h)\phi_n(h_nh) = f_nAs_n(h)s_n(h_nh)^{-1}c_{n+1}(h_nh)$$

and $f_nAs_n(h)s_n(h_nh)^{-1} \subset F_n$. Hence

$$\begin{aligned} \mu(T_{g_n}[A]_n \cap [B]_n) &= \sum_{h\in H_n} \mu(T_{g_n}[Ac_{n+1}(h)]_{n+1} \cap [B]_n) \\ &= \sum_{h\in H_n} \mu([f_nAs_n(h)s_n(h_nh)^{-1}c_{n+1}(h_nh)]_{n+1} \cap [B]_n) \\ &= \sum_{h\in H_n} \mu([(f_nAs_n(h)s_n(h_nh)^{-1} \cap B)c_{n+1}(h_nh)]_{n+1}) \\ &= \frac{1}{\#H_n} \sum_{h\in H_n} \mu([f_nAs_n(h)s_n(h_nh)^{-1} \cap B]_n) \\ &= \frac{1}{\#H_n} \sum_{h\in H_n} \lambda_{F_n}(f_nAs_n(h) \cap Bs_n(h_nh)). \end{aligned} \tag{2-5}$$

We define a map $r_{A,B} : F_n \times F_n \to \mathbb{R}$ by setting

$$r_{A,B}(g, g') := \lambda_{F_n}(f_nAg \cap Bg').$$

Then it follows from (2-5) and (2-4) that

$$\begin{aligned} \mu(T_{g_n}[A]_n \cap [B]_n) &= \int_{F_n\times F_n} r_{A,B}\, d(\mathrm{dist}\, \pi_{1,h_n}s_n) \\ &= \int_{F_n\times F_n} r_{A,B}\, d\lambda_{F_n\times F_n} \pm \epsilon_n \\ &= \int_{F_n\times F_n} \lambda_{F_n}(f_nAg \cap Bg')\, d\lambda_{F_n}(g)d\lambda_{F_n}(g') \pm \epsilon_n \\ &= \lambda_{F_n}(A)\lambda_{F_n}(B) \pm \epsilon_n \\ &= \mu([A]_n)\mu([B]_n) \pm \epsilon_n. \end{aligned}$$

Hence we have

$$\max_{A,B\subset F_n} |\mu(T_{g_n}[A]_n \cap [B]_n) - \mu([A]_n)\mu([B]_n)| < \epsilon_n. \tag{2-6}$$

This and (1-6) imply that the sequence $(g_n)_{n=1}^{\infty}$ is mixing.

**(II)** Now we fix $l > 1$ and prove that $T$ is mixing of order $l$. To this end it is sufficient to show the following: given $l + 1$ sequences $(g_{0,n})_{n=1}^{\infty}, \dots, (g_{l,n})_{n=1}^{\infty}$ in $G$ such that $g_{i,n} \in F_{n+1}$ and $g_{i,n}g_{j,n}^{-1} \notin F_n$ whenever $i \neq j$,

$$\max_{A_0,\dots,A_l} |\mu(T_{g_{0,n}}[A_0]_n \cap \cdots \cap T_{g_{l,n}}[A_l]_n) - \mu([A_0]_n)\cdots\mu([A_l]_n)| < \epsilon_n$$

for all $n > l$. Notice that for every $n \in \mathbb{N}$ and $0 \le j \le l$, there exist unique $f_{j,n} \in F_n$ and $h_{j,n} \in H_n$ with $g_{j,n} = f_{j,n}\phi_n(h_{j,n})$. Moreover, $h_{0,n} \ne h_{2,n} \cdots \ne h_{1,n}$. Then slightly modifying the argument in (I), we compute

$$\begin{aligned}&\mu(T_{g_{0,n}}[A_0]_n \cap \cdots \cap T_{g_{l,n}}[A_l]_n)\\ &\qquad= \int_{F_n^l} \lambda_{F_n}(f_{0,n}A_0g_0 \cap \cdots \cap f_{l,n}A_lg_l)\, d(\lambda_{F_n})^{l+1}(g_0,\dots,g_l) \pm \epsilon_n\\ &\qquad= \lambda_{F_n}(A_0)\cdots\lambda_{F_n}(A_l) \pm \epsilon_n = \mu([A_0]_n)\cdots\mu([A_l]_n) \pm \epsilon_n.\end{aligned} \tag{2-7}$$

□

To construct a weakly mixing rigid action of $G$ we define another sequence $(\widetilde{k}_n, \widetilde{s}_{n-1})_{n\ge 1}$. When $n$ is odd, we choose $\widetilde{k}_n$ and $\widetilde{s}_{n-1}$ to satisfy the following weaker version of (2-4):

$$\max_{1\ne h\in H_n} \|\mathrm{dist}\,\pi_{1,h}s_n - \lambda_{F_n}\times\lambda_{F_n}\| < \epsilon_n. \tag{2-8}$$

When $n$ is even, we just set $\widetilde{k}_n := \widetilde{k}_{n-1} + 1$ and $\widetilde{s}_n \equiv 1_G$. Denote by $\widetilde{T}$ the $(C,F)$-action of $G$ on $(\widetilde{X}, \widetilde{\mathfrak{B}}, \widetilde{\mu})$ associated with $(\widetilde{k}_n, \widetilde{s}_{n-1})_{n=1}^\infty$.

**Theorem 2.5.** *$\widetilde{T}$ is weakly mixing and rigid.*

*Proof.* Take any sequence $h_n \in H_{2n} \setminus \{1\}$. It follows from the part (I) of the proof of Theorem 2.4 and (2-8) that the sequence $(\phi_{2n}(h_n))_{n=1}^\infty$ is mixing for $\widetilde{T}$. Clearly, it is also mixing for $\widetilde{T}\times\widetilde{T}$. Hence $\widetilde{T}\times\widetilde{T}$ is ergodic, i.e. $\widetilde{T}$ is weakly mixing.

Now take any sequence $h_n \in H_{2n+1}\setminus\{1\}$. Notice that (2-5) holds for any choice of $(k_n, s_{n-1})_{n\ge1}$. Hence we deduce from (2-5) and the definition of $\widetilde{s}_{2n+1}$ that

$$\mu(\widetilde{T}_{\phi_{2n+1}(h_n)}[A]_{2n+1} \cap [B]_{2n+1}) = \lambda_{F_{2n+1}}(A\cap B) = \mu([A\cap B]_{2n+1})$$

for all subsets $A, B \subset F_{2n+1}$. This plus (1-6) yield

$$\lim_{n\to\infty} \mu(\widetilde{T}_{\phi_{2n+1}(h_n)}\widetilde{A}\cap\widetilde{B}) = \mu(\widetilde{A}\cap\widetilde{B})$$

for all $\widetilde{A}, \widetilde{B} \in \widetilde{\mathfrak{B}}$. This means that $\widetilde{T}$ is rigid. □

## 3. Self-joinings of $T$

This section is devoted entirely to the proof of the following theorem.

**Theorem 3.1.** *The action $T$ constructed in the previous section has MSJ.*

*Proof.* **(I)** We first show that $T$ has $\mathrm{MSJ}_2$. Since $T$ is weakly mixing, we need to establish that

$$J_2^e(T) = \{\mu_{g^\bullet} \mid g \in G\} \cup \{\mu\times\mu\}.$$

Take any $\nu \in J_2^e(T)$. Let $\mathfrak{F}_n$ denote the sub-$\sigma$-algebra of $(T_g\times T_g)_{g\in F_n}$-invariant subsets. Then $\mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \cdots$ and $\bigcap_n \mathfrak{F}_n = \{\emptyset, X\times X\}$ (mod $\nu$). Since there are only countably many cylinders, we deduce from the martingale convergence theorem that for $\nu$-a.a. $(x,x')$,

$$E(\chi_{B\times B'}|\mathfrak{F}_{n-1})(x,x') = \frac{1}{\#F_{n-1}}\sum_{g\in F_{n-1}} \chi_{B\times B'}(T_gx, T_gx') \to \nu(B\times B') \tag{3-1}$$

as $n \to \infty$ for any pair of cylinders $B, B' \subset X$. Fix such a point $(x, x')$. It is called *generic* for $(T \times T, \nu)$. Given any $n > 0$, we can write $x$ and $x'$ as infinite sequences

$$x = (f_n, d_{n+1}, d_{n+2}, \dots) \ \text{ and } \ x' = (f'_n, d'_{n+1}, d'_{n+2}, \dots)$$

with $f_n, f'_n \in F_n$ and $d_i, d'_i \in C_i$ for all $i > n$. Recall that $f_n := f_0 d_1 \cdots d_n$ and $f'_n := f'_0 d'_1 \cdots d'_n$. We set $t_n := f'_n f_n^{-1}$, $n > 0$. Fix a pair of cylinders, say $m$-cylinders, $B$ and $B'$. If $n > m$ and $g \in F_n$ then $T_g x' = (g f'_n, d'_{n+1}, d'_{n+2}, \dots)$. Hence $T_g x' \in B'$ if and only if $T_g T_{t_n} x \in B'$. Therefore

$$\chi_{B \times B'}(T_g x, T_g x') = \chi_{T_g^{-1} B \cap T_{t_n}^{-1} T_g^{-1} B'}(x).$$

Since $x$ is generic for $(T, \mu)$, it follows that

$$\lim_{l \to \infty} \frac{1}{\# F_l} \sum_{a \in F_l} \chi_{T_g^{-1} B \cap T_{t_n}^{-1} T_g^{-1} B'}(T_a x) = \mu(T_g^{-1} B \cap T_{t_n}^{-1} T_g^{-1} B').$$

Therefore (3-1) yields

$$\text{(3-2)} \qquad \lim_{n \to \infty} \frac{1}{\# F_{n-1}} \sum_{g \in F_{n-1}} \mu(T_g^{-1} B \cap T_{t_n}^{-1} T_g^{-1} B') = \nu(B \times B').$$

Consider now two cases. If $t_n \notin F_{n-1}$ for infinitely many $n$ then passing to the limit in (3-2) along this subsequence and making use of (2-6) we obtain that $\mu(B)\mu(B') = \nu(B \times B')$. Hence $\mu \times \mu = \nu$. If, otherwise, there exists $N > 0$ such that $t_n \in F_{n-1}$, i.e. $d_n = d'_n$, for all $n > N$ then $x$ and $x'$ are $T$-orbit equivalent, $t_n = t_N$ and

$$\begin{aligned} \frac{1}{\# F_{n-1}} \sum_{g \in F_{n-1}} \mu(T_g^{-1} B \cap T_{t_n}^{-1} T_g^{-1} B') &= \frac{1}{\# F_N} \sum_{g \in F_N} \mu(B \cap T_g T_{t_N}^{-1} T_g^{-1} B') \\ &= \mu_{(t_N^{-1})^\bullet}(B \times B') \end{aligned}$$

Passing to the limit in (3-1) we obtain that $\nu = \mu_{(t_N^{-1})^\bullet}$.

**(II)** Now we fix $l > 1$ and show that $T$ has MSJ$_{l+1}$. Take any joining $\nu \in J^e_{l+1}(T)$ and fix a generic point $(x_0, \dots, x_l)$ for $(T \times \cdots \times T, \nu)$. Define a partition $P$ of $\{0, \dots, l\}$ by setting: $i_1$ and $i_2$ are in the same atom of $P$ if $x_{i_1}$ and $x_{i_2}$ are $T$-orbit equivalent. As in (I), for any $n$, we can write

$$x_j = (f_{j,n-1}, d_{j,n}, d_{j,n+1}, \dots) \in X_{n-1}, \quad j = 0, \dots, l.$$

Suppose first that $\# P = l + 1$, i.e. $P$ is the finest possible. Then by the proof of (I), each 2-dimensional marginal of $\nu$ is $\mu \times \mu$. Since $\sum_{i=1}^\infty (\# C_i)^{-1} < \infty$ and $\mu = \lambda_{F_0} \times \lambda_{C_1} \times \lambda_{C_2} \times \cdots$, it follows from the Borel-Cantelli lemma that for $\nu$-a.a. $(y_0, \dots, y_l) \in X^{l+1}$,

$$\exists N > 0 \ \text{ such that } y_{0,i} \neq y_{1,i} \neq \cdots \neq y_{l,i} \ \text{ whenever } i > N,$$

where $y_{j,i} \in C_i$ is the $i$-th coordinate of $y_j \in F_0 \times C_1 \times C_2 \times \cdots$. Hence without loss of generality we may assume that this condition is satisfied for $(x_0, \dots, x_l)$. Thus, if we set $t_{j,n} := f_{j,n} {f_{0,n}}^{-1} = f_{j,n-1} d_{j,n} {d_{0,n}}^{-1} {f_{0,n-1}}^{-1}$ then $t_{j,n} t_{i,n}^{-1} \notin F_{n-1}$

whenever $i \neq j$. Slightly modifying our reasoning in (I) and making use of (2-7) instead of (2-6) we now obtain

$$\begin{aligned}
\nu(B_0 \times \cdots \times B_l) &= \lim_{n\to\infty} \sum_{g\in F_{n-1}} \chi_{B_0\times\cdots\times B_l}(T_g x_0, \dots . T_g x_l) \\
&= \lim_{n\to\infty} \sum_{g\in F_{n-1}} \chi_{B_0\times\cdots\times B_l}(T_g x_0, T_g T_{t_{1,n}} x_0, \dots, T_g T_{t_{l,n}} x_0) \\
&= \lim_{n\to\infty} \sum_{g\in F_{n-1}} \mu(T_g B_0 \cap T_{t_{1,n}}^{-1} T_g B_1 \cap \cdots \cap T_{t_{l,n}}^{-1} T_g B_l) \\
&= \mu(B_0)\cdots\mu(B_l)
\end{aligned}$$

for any $(l+1)$-tuple of cylinders $B_0, \dots, B_l$. Hence $\nu = \mu \times \cdots \times \mu$.

Consider now the general case and put $t_{j,n} := f_{j,n} f_{i_p,n}^{-1}$ for each $j \in p$, $p \in P$. Recall that $i_p = \min_{j\in p} j$. Then

$$\chi_{B_0\times\cdots\times B_l}(T_g x_0, \dots, T_g x_l) = \prod_{p\in P} \chi_{A_p}(x_{i_p}),$$

where $A_p := \bigcap_{j\in p} T_{t_{j,n}}^{-1} T_g^{-1} B_j$. Notice that the point $(x_{i_p})_{p\in P} \in X^{\{i_p \mid p\in P\}}$ is generic for $(T \times \cdots \times T\ (\#P \text{ times}), \kappa)$, where $\kappa$ stands for the projection of $\nu$ onto $X^{\{i_p|p\in P\}}$. By the first part of (II), $\kappa = \mu \times \cdots \times \mu$ ($\#P$ times). Hence

$$\begin{aligned}
\nu(B_0 \times \cdots \times B_l) &= \lim_{n\to\infty} \frac{1}{\#F_{n-1}} \sum_{g\in F_{n-1}} \chi_{B_0\times\cdots\times B_l}(T_g x_0, \dots . T_g x_l) \\
&= \lim_{n\to\infty} \frac{1}{\#F_{n-1}} \sum_{g\in F_{n-1}} \prod_{p\in P} \mu(A_p).
\end{aligned}$$

As in (I), a 'stabilization' property holds: there exists $M > 0$ such that $t_{j,n} = t_{j,M}$ for all $n > M$. We now set $g := (t_{0,M}^{-1}, \dots, t_{l,M}^{-1})$. Clearly, $g$ is $P$-subordinated. Hence

$$\nu(B_0 \times \cdots \times B_l) = \frac{1}{\#F_M} \sum_{g\in F_M} \prod_{p\in P} \mu\Bigg(\bigcap_{j\in p} T_g T_{t_{j,M}} T_g^{-1} B_j\Bigg) = \mu_{g^{\bullet l}}(B_0 \times \cdots \times B_l).$$

□

## 4. Uncountably many mixing actions with MSJ

In this section the proof of Theorems 0.1(i) and 0.4 will be completed. We first apply Lemma 2.3 to construct $k_{n+1}$ and $s_n, \widehat{s}_n \in F_n^{H_n}$ in such a way that (2-4) is satisfied for both $s_n$ and $\widehat{s}_n$ and, in addition,

$$\|\text{dist}_{h\in H_n}(s_n(hk), \widehat{s}_n(hk')) - \lambda_{F_n} \times \lambda_{F_n}\| < \epsilon_n \tag{4-1}$$

for all $k, k' \in H_n$. Next, given $\sigma \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, we define $s_n^\sigma : H_n \to F_n$ by setting

$$s_n^\sigma = \begin{cases} s_n & \text{if } \sigma(n) = 0, \\ \widehat{s}_n & \text{if } \sigma(n) = 1. \end{cases}$$

Now we denote by $T^\sigma$ the $(C,F)$-action of $G$ associated with $(k_n, s_{n-1}^\sigma)_{n=1}^\infty$. Let $\Sigma$ be an uncountable subset of $\{0,1\}^{\mathbb{N}}$ such that for any $\sigma, \sigma' \in \Sigma$, the subset $\{n \in \mathbb{N} \mid \sigma(n) \neq \sigma'(n)\}$ is infinite.

**Theorem 4.1.**

(i) *For any $\sigma \in \{0,1\}^{\mathbb{N}}$, the action $T^\sigma$ is mixing and has MSJ.*

(ii) *If $\sigma, \sigma' \in \Sigma$ and $\sigma \neq \sigma'$ then $T^\sigma$ and $T^{\sigma'}$ are disjoint.*

*Proof.* (i) follows from the proof of Theorem 3.1, since (2-4) is satisfied for $s_n^\sigma$ for all $\sigma \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

(ii) Let $\nu \in J^e(T^\sigma, T^{\sigma'})$. Take a generic point $(x, x')$ for $(T^\sigma \times T^{\sigma'}, \nu)$. Consider any $n$ such that $\sigma(n) \neq \sigma'(n)$. Then we can write $x$ and $x'$ as infinite sequences $x = (f_n, d_{n+1}, d_{n+2}, \dots)$ and $x' = (f'_n, d'_{n+1}, d'_{n+2}, \dots)$ with $f_n, f'_n \in F_n$ and $d_m, d'_m \in C_m$ for all $m > n$. Take any $g \in F_{n+1}$. Then we have the following expansions

$$g = a\phi_n(h),\ d_{n+1} = s_n^\sigma(h_n)\phi_n(h_n) \text{ and } d'_{n+1} = s_n^{\sigma'}(h'_n)\phi_n(h'_n)$$

for some uniquely determined $a \in F_n$ and $h, h_n, h'_n \in H_n$. Since

$$\begin{aligned} gf_nd_{n+1} &= af_ns_n^\sigma(h_n)s_n^\sigma(hh_n)^{-1}c_{n+1}(hh_n) \quad \text{and} \\ gf'_nd'_{n+1} &= af'_ns_n^{\sigma'}(h'_n)s_n^{\sigma'}(hh'_n)^{-1}c_{n+1}(hh'_n), \end{aligned}$$

the following holds for any pair of subsets $A, A' \subset F_n$:

$$\begin{aligned} &\frac{\#\{g \in F_{n+1} \mid (T_g^\sigma x, T_g^{\sigma'} x') \in [A]_n \times [A']_n\}}{\#F_{n+1}} \\ &= \frac{1}{\#F_n} \sum_{a \in F_n} \frac{\#\{h \in H_n \mid af_ns_n^\sigma(h_n)s_n^\sigma(hh_n)^{-1} \in A,\ af'_ns_n^{\sigma'}(h'_n)s_n^{\sigma'}(hh'_n)^{-1} \in A'\}}{\#H_n} \\ &= \frac{1}{\#F_n} \sum_{a \in F_n} \xi_n(A^{-1}af_ns_n^\sigma(h_n) \times A'^{-1}af'_ns_n^{\sigma'}(h'_n)), \end{aligned}$$

where $\xi_n := \mathrm{dist}_{h \in H_n}(s_n^\sigma(hh_n), s_n^{\sigma'}(hh'_n))$. This and (4-1) yield

$$\begin{aligned} \frac{\#\{g \in F_{n+1} \mid (T_g^\sigma x, T_g^{\sigma'} x') \in [A]_n \times [A']_n\}}{\#F_{n+1}} &= \lambda_{F_n}(A)\lambda_{F_n}(A') \pm \epsilon_n \\ &= \mu([A]_n)\mu([A']_n) \pm \epsilon_n. \end{aligned} \tag{4-2}$$

Since $(x, x')$ is generic for $(T^\sigma \times T^{\sigma'}, \nu)$ and (4-2) holds for infinitely many $n$, we deduce that $\nu = \mu \times \mu$. □

By refining the above argument the reader can strengthen Theorem 0.1(i) as follows: there exists an uncountable family of mixing (of any order) rank-one $G$-actions with MSJ such that any finite subfamily of it is disjoint.

## 5. On $G$-actions with MSJ

It follows immediately from Definition 0.2 that if $T$ has $\mathrm{MSJ}_2$ then the centralizer $C(T)$ of $T$ is 'trivial', i.e. $C(T) = \{T_g \mid g \in C(G)\}$, where $C(G)$ denotes the center of $G$. Moreover, we will show that $T$ has *trivial product centralizer* (as D. Rudolph did in [Ru] for $\mathbb{Z}$-actions).

Let $(X^l, \mathfrak{B}^{\otimes l}, \mu^l, T^{(l)})$ denote the $l$-fold Cartesian product of $(X, \mathfrak{B}, \mu, T)$. Given a permutation $\sigma$ of $\{1, \dots, l\}$ and $g_1, \dots, g_n \in C(T)$, we define a transformation $U_{\sigma, g_1, \dots, g_l}$ of $(X^l, \mathfrak{B}^{\otimes l}, \mu^l, T^{(l)})$ by setting

$$U_{\sigma, g_1, \dots, g_l}(x_1, \dots, x_l) := (T_{g_1}x_{\sigma(1)}, \dots, T_{g_l}x_{\sigma(l)}).$$

Of course, $U_{\sigma, g_1, \dots, g_l} \in C(T^{(l)})$. We show that for the actions with MSJ, the converse also holds.

**Proposition 5.1.** *If $T$ has MSJ then for any $l \in \mathbb{N}$, each element of $C(T^{(l)})$ equals to $U_{\sigma,g_1,\dots,g_l}$ for some permutation $\sigma$ and elements $g_1,\dots,g_l \in C(G)$.*

*Proof.* Let $S \in C(T^{(l)})$. We define an ergodic 2-fold self-joining $\nu$ of $T^{(l)}$ by setting $\nu(A \times B) := \mu^l(A \cap S^{-1}B)$ for all $A, B \in \mathfrak{B}^{\otimes l}$. Notice that $\nu \in J^e_{2l}(T)$. Since $T$ has $\mathrm{MSJ}_{2l}$, there exists a partition $P$ of $\{1,\dots,2l\}$ and a $P$-subordinated element $g = (g_1,\dots,g_{2l}) \in \mathrm{FC}(G)^{2l}$ such that

$$\nu(A_1 \times \cdots \times A_{2l}) = \frac{1}{\#g^{\bullet 2l}} \sum_{(h_1,\dots,h_{2l}) \in g^{\bullet 2l}} \prod_{p \in P} \mu\Bigg(\bigcap_{i \in p} T_{h_i} A_i\Bigg). \tag{5-1}$$

for all subsets $A_1,\dots,A_{2l} \in \mathfrak{B}$. Substituting at first $A_1 = \cdots = A_l = X$ and then $A_{l+1} = \cdots = A_{2l} = X$ in (5-1) we derive that $\#P = l$, $\#p = 2$ for all $p \in P$ and $\#g^{\bullet 2l} = 1$. Hence $g_1,\dots,g_{2l} \in C(G)$ and there exists a bijection $\sigma$ of $\{1,\dots,l\}$ such that $P = \{\{i, \sigma(i) + l\} \mid i = 1,\dots,l\}$. Therefore in follows from (5-1) that

$$S^{-1}(A_{l+1} \times \cdots \times A_{2l}) = T_{g_{l+1}} A_{l+\sigma(1)} \times \cdots \times T_{g_{2l}} A_{l+\sigma(l)}.$$

□

As a simple corollary we derive that if $T$ has MSJ then the $G$-actions $T, T^{(2)}, \dots$ and $T \times T \times \cdots$ are pairwise non-isomorphic.

After this paper was submitted the author introduced a companion to MSJ concept of *simplicity* for actions of locally compact second countable groups [Da3]. As appeared, this concept is more general that the simplicity in the sense of A. del Junco and D. Rudolph [JuR] even for $\mathbb{Z}$-actions. For instance, there exist simple transformations which are disjoint from all 2-fold del Junco-Rudolph's-simple ones. It is shown in [Da3] that an analogue of Veech theorem on the structure of factors holds for this extended class of simple actions. In particular, if $T$ has $\mathrm{MSJ}_2$ then for every non-trivial factor $\mathfrak{F}$ of $T$ there exists a compact normal subgroup $K$ of $G$ such that

$$\mathfrak{F} = \mathrm{Fix}\, K := \{A \in \mathfrak{B} \mid \mu(T_k A \triangle A) = 0 \text{ for all } k \in K\}.$$

Thus if $T$ has $\mathrm{MSJ}_2$ then $T$ is *effectively prime*, i.e. $T$ has no effective factors. (Recall that a $G$-action $Q$ is called effective if $Q_g \neq \mathrm{Id}$ for each $g \neq 1_G$.)

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