MULTIPLE AND POLYNOMIAL RECURRENCE FOR ABELIAN ACTIONS IN INFINITE MEASURE

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Abstract. We apply the (C,F)-construction from a previous paper of the first author to produce a number of funny rank one infinite measure preserving actions of discrete countable Abelian groups G with 'unusual' multiple recurrence properties. In particular, we construct the following for each $p \in \mathbb{N} \cup \{\infty\}$:

(i) a p-recurrent action $T = (T_g)_{g \in G}$ such that (if $p \neq \infty$) no one transformation T_g is $(p+1)$-recurrent for every element g of infinite order,
(ii) an action $T = (T_g)_{g \in G}$ such that for every finite sequence $g_1, \dots, g_r \in G$ without torsion the transformation $T_{g_1} \times \dots \times T_{g_r}$ is ergodic, p-recurrent but (if $p \neq \infty$) not $(p+1)$-recurrent,
(iii) a p-polynomially recurrent (C,F)-transformation which (if $p \neq \infty$) is not $(p+1)$-recurrent.

∞-recurrence here means multiple recurrence. Moreover, we show that there exists a (C,F)-transformation which is rigid (and hence multiply recurrent) but not polynomially recurrent. Nevertheless, the subset of polynomially recurrent transformations is generic in the group of infinite measure preserving transformations endowed with the weak topology.

0. Introduction

The first named author introduced in [**8**] a (C,F)-construction of funny rank one infinite measure preserving actions of countable discrete Abelian groups (cf. [**14**]). It was used to provide examples of such actions with 'unusual' i.e. impossible in the classical probability preserving setting) weak mixing properties. The goal of the present work is to exhibit new (C,F)-actions with various 'unusual' properties of multiple or polynomial recurrence.

Recall that while elaborating a new proof of Szemerédi's theorem on arithmetic progressions, Furnstenberg showed [**10**] that every probability preserving transformation is multiply recurrent. This fact was refined later as follows: every such a transformation is polynomially recurrent [**6**]. Furthermore, these results were extended successively to actions of more general groups like Abelian [**11**] or solvable ones [**15**].

Now we review briefly the known results related to multiple recurrence in infinite measure. For consistency of notation ∞-recurrence below denotes the multiple recurrence. Eigen, Hajian and Halverson constructed in [**9**] infinite measure preserving odometers—i.e. infinite counterparts of transformations with pure discrete

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rational spectrum—with every possible index of recurrence. This means that for any $p \in \mathbb{N} \cup \{\infty\}$ there exists an odometer which is p-recurrent but (if $p \neq \infty$) not $(p+1)$-recurrent. Recall that in the probability preserving case, Furnstenberg's multiply recurrence theorem is proved differently for the transformations with pure discrete spectrum and weakly mixing ones. Thus it is of interest to ask if this theorem is still true for weakly mixing infinite measure preserving transformations. However, it is not quite clear what is the proper definition of weak mixing in infinite measure (see a discussion in [**8**]). We recall the 'scale' of weak mixing in infinite measure where every next notion is strictly stronger than the previous one:

— a transformation T has trivial L^∞-spectrum.
— $T \times T$ is ergodic.
— T is of infinite ergodic index, i.e. $\underbrace{T \times \cdots \times T}_{k \text{ times}}$ is ergodic for every k.
— T is power weakly mixing, i.e. $T^{n_1} \times \cdots \times T^{n_k}$ is ergodic for all $n_1, \ldots, n_k \neq 0$.

Aaronson and Nakada showed in [**2**] that an infinite measure preserving Markov shift T is p-recurrent if and only if $\underbrace{T \times \cdots \times T}_{p \text{ times}}$ is conservative. It follows from this and [**1**] that in the class of ergodic Markov shifts infinite ergodic index implies multiple recurrence. However, in general this is not true. Two counterexamples were constructed in [**4**] and [**12**]. The first one is an infinite ergodic index transformation which is not 2-recurrent. The second one is a power weakly mixing transformation which is 3-recurrent but not 16-recurrent. We also mention a 'positive' result. It was shown in [**5**] that the set of multiply recurrent transformations is generic in the group of infinite measure preserving ones.

The transformations in [**4**] and [**12**] were constructed via the well known geometrical cutting-and-stacking procedure. In the present paper we utilize the more universal algebraic (C, F)-construction to produce a richer and finer family of counterexamples. Notice that unlike the probability preserving setting neither polynomial recurrence nor multiple recurrence for actions of more general groups than $\mathbb{Z}$ have been considered earlier in the literature.

Now we record the main results of this work together with some comments. Throughout this paper G is a countable discrete Abelian group. We assume that the subset G_∞ of elements of infinite order is not empty.

— Given $p \in \mathbb{N} \cup \{\infty\}$, there exists a (C, F)-action $T = (T_g)_{g \in G}$ such that the transformation T_g is p-recurrent but (if $p \neq \infty$) not $(p+1)$-recurrent for every $g \in G_\infty$ (see Theorems 2.3–2.6).
— Given $p \in \mathbb{N} \cup \{\infty\}$, there exists a p-recurrent (C, F)-action $T = (T_g)_{g \in G}$ such that (if $p \neq \infty$) no one transformation T_g is $(p+1)$-recurrent for all $g \in G_\infty$ (see Theorems 3.3).

We notice that in infinite measure there is a distinction between p-recurrence of T as a whole (Definition 3.2) and p-recurrence of every transformation T_g, $g \in G$ (Definition 2.1). We call the latter property *individual* p-recurrence of T. If the free rank of G is one and $p = \infty$ then the both concepts are equivalent. However

— If the free rank of G is more than one, then there exists a (C, F)-action $T = (T_g)_{g \in G}$ which is not 2-recurrent but is individually multiply recurrent. It is even individually rigid (see Theorem 3.4(i)).

Next, we are concerned with polynomial recurrence of infinite measure preserving transformations.

— Given $p \in \mathbb{N} \cup \{\infty\}$, there exists a p-polynomially recurrent (C, F)-transformation which (if $p \neq \infty$) is not $(p+1)$-recurrent (see Theorem 4.3).

— There exists a (C, F)-transformation which is rigid (and hence multiply recurrent) but not 2-polynomially recurrent (see Theorem 4.2).

— The subset of polynomially recurrent transformations is generic in the group of infinite measure preserving transformations endowed with the weak topology (see Theorem 4.4).

Since polynomial recurrence implies multiple recurrence, we get a new proof of the fact that the set of multiply recurrent transformations is generic (cf. [**5**]). The following our assertion extends and refines a theorem of [**12**].

— Given $p \in \mathbb{N} \cup \{\infty\}$, there exists a (C, F)-action $T = (T_g)_{g \in G}$ such that for every finite sequence $g_1, \dots, g_r \in G_\infty$, the transformation $T_{g_1} \times \cdots \times T_{g_r}$ is ergodic, p-recurrent but (if $p \neq \infty$) not $(p+1)$-recurrent (see Theorem 5.5).

Note that every (C, F)-action is a minimal G-action on a locally compact Cantor set. It is worthwhile to observe that while proving the main results of this paper we obtain as byproducts topological counterparts of them (see the final § 6 for details).

1. (C, F)-actions and rigidity

In this section we recall the (C, F)-construction of funny rank one Abelian actions as it appeared in [**8**] (cf. [**14**]). All the examples of actions that will be presented in this paper are of that kind. We find necessary and sufficient conditions for a (C, F)-transformation to be rigid or partially rigid (see Theorem 1.5, Corollary 1.6).

Let T stand for an invertible measure preserving transformation of a σ-finite standard measure space $(X, \mathfrak{B}, \mu)$.

Definition 1.1.

(i) If there is a sequence $n_i \to \infty$ such that $T^{n_i} \to \mathrm{Id}$ weakly then T is called *rigid.*

(ii) Let $0 < \delta \leq 1$. Then T is called *at least δ-partially rigid* if there exists a sequence $n_i \to \infty$ such that

$$\liminf_{i \to \infty} \mu(T^{n_i} A \cap A) \geq \delta \mu(A) \tag{1-1}$$

for every subset $A \in \mathfrak{B}$ of finite measure.

(iii) If T is at least δ-partially rigid but not at least ϵ-partially rigid for any $\epsilon > \delta$ then T is called *δ-partially rigid.*

Clearly, T is rigid if and only if it is 1-partially rigid. It was shown recently in [**5**] that a generic (nonsingular) transformation of $(X, \mathfrak{B}, \mu)$ is rigid both in the weak and uniform topologies on the transformation group. Observe also that it suffices to check (1-1) only on a dense subfamily in $\mathfrak{B}$. The following statement is easy and we omit its proof (cf. [**3**]).

Lemma 1.2.

(i) *If T is at least δ-partially rigid then $\underbrace{T \times \cdots \times T}_{l \text{ times}}$ is at least δ^l-partially rigid for every $l \in \mathbb{N}$.*

(ii) *If T is at least δ-partially rigid then T (and hence every Cartesian power of T) is conservative.*

(iii) *T is at least δ-partially rigid if and only if it is at least ϵ-partially rigid for all $\epsilon \in (0, \delta)$.*

Now we recall the construction of (C, F)-actions. For an element $h \in G$ and a finite subset $F \subset G$, we set $F(h) := F \cap (F - h)$. Two finite subsets C_1 and C_2 of G are called *independent* if

$$(C_1 - C_1) \cap (C_2 - C_2) = \{0\}.$$

Let $(C_n)_{n=1}^{\infty}$ and $(F_n)_{n=0}^{\infty}$ be two sequences of finite G-subsets such that $F_0 = \{0\}$ and for each $n > 0$ the following properties are satisfied:

$$F_{n-1} + C_n \subset F_n, \ \#C_n > 1, \tag{1-2}$$

$$F_{n-1} \text{ and } C_n \quad \text{are independent.} \tag{1-3}$$

We put $X_n := F_n \times \prod_{k>n} C_k$, endow X_n with the (compact) product topology and define a continuous embedding $X_n \to X_{n+1}$ by setting

$$(f_n, c_{n+1}, c_{n+2}, \dots) \mapsto (f_n + c_{n+1}, c_{n+2}, \dots).$$

Then $X_1 \subset X_2 \subset \dots$. Let $X := \bigcup_n X_n$ stand for the topological inductive limit of the sequence X_n. Clearly, X is a locally compact non-compact totally disconnected metrizable space without isolated points and X_n is clopen in X. Assume in addition that

(1-4) given $g \in G$, there is $m \in \mathbb{N}$ with $g + F_{n-1} + C_n \subset F_n$ for all $n > m$.

Given $g \in G$ and $n \in \mathbb{N}$, we set

$$D_g^{(n)} := F_n(g) \times \prod_{k>n} C_k \qquad \text{and} \qquad R_g^{(n)} := D_{-g}^{(n)}.$$

Clearly, $D_g^{(n)}$ and $R_g^{(n)}$ are clopen subsets of X_n. Moreover, $D_g^{(n)} \subset D_g^{(n+1)}$ and $R_g^{(n)} \subset R_g^{(n+1)}$. Define a map $T_g^{(n)} \colon D_g^{(n)} \to R_g^{(n)}$ by setting

$$T_g^{(n)}(f_n, c_{n+1}, \dots) := (f_n + g, c_{n+1}, \dots).$$

Clearly, it is a homeomorphism. Put

$$D_g := \bigcup_{n=1}^{\infty} D_g^{(n)} \text{ and } R_g := \bigcup_{n=1}^{\infty} R_g^{(n)}.$$

Then a homeomorphism $T_g \colon D_g \to R_g$ is well defined by $T_g \restriction D_g^{(n)} = T_g^{(n)}$. It follows from (1-4) that $D_g = R_g = X$ for each $g \in G$.

Proposition 1.3 [8].

(i) $T = \{T_g\}_{g\in G}$ *is a minimal free action of* G *on* X,

(ii) *Two points* $x, y \in X$ *are* T*-orbit equivalent if and only if there are* $n \le m$ *with* $x = (x_i)_{i\ge n}, y = (y_i)_{i\ge n} \in X_n$ *and* $x_i = y_i$ *for all* $i \ge m$. *Furthermore,* $y = T_g x$ *for* $g = \sum_{i\ge n}(y_i - x_i)$.

(iii) *there is a unique (ergodic)* σ*-finite* T*-invariant measure on* X *such that* $\mu(X_0) = 1$,

(iv) μ *is finite if and only if*

$$\lim_{n\to\infty} \frac{\#F_n}{\#C_1 \cdots \#C_n} < \infty,$$

(v) T *has funny rank one.*

Given $f \in F_n$, we set $[f]_n := \{x = (x_i)_{i\ge n} \in X_n \mid x_n = f\}$ and call it a cylinder. Clearly, $[f]_n = \bigsqcup_{c\in C_{n+1}}[f + c]_{n+1}$. The sign $\bigsqcup$ means the union of disjoint subsets. Denote by $\mathcal{K}$ the family of compact open subsets of X. Then $A \in \mathcal{K}$ if and only if A is a finite union of cylinders.

We will use often the following additional property

$$\sum_{i<n}(C_i - C_i) \text{ and } C_n - C_n \text{ are independent.} \tag{1-5}$$

Remark 1.4. In this paper, the subsets F_n, C_n will be constructed inductively. On the n-th step we will only define C_n explicitly. As for F_n, one can take any finite subset in G which is sufficiently large to satisfy (1-2) and (1-4) and make infinite the limit in Proposition 1.3(iv).

Theorem 1.5. *Let* T *be the* (C, F)*-action associated with* (C_n, F_n) *satisfying* (1-2)–(1-5). *For* $g \in G$, *the transformation* T_g *is at least* δ*-partially rigid if and only if there exists a sequence* $m_n \to \infty$ *such that* $m_n g = \sum_{i>n} g_i^{(n)}$ *with* $g_i^{(n)} \in C_i - C_i$ *and*

$$\prod_{i>n} \frac{\#C_i(g_i^{(n)})}{\#C_i} \ge (1 - \frac{1}{n})\delta.$$

Proof. ($\Longrightarrow$) Since T_g is at least δ-partially rigid, for any n there exists $m_n > n$ with

$$\mu(T_g^{m_n}[0]_n \cap [0]_n) \ge \delta(1 - \frac{1}{n})\mu([0]_n). \tag{1-6}$$

It follows from the definition of T that $m_n g \in \sum_{i>n}(C_i - C_i)$. Hence there is a finite expansion $m_n g = \sum_{i=n+1}^{l_n} g_i^{(n)}$ with $g_i^{(n)} \in C_i - C_i$ for all i. Since $[0]_n = \bigsqcup_{c_{n+1}\in C_{n+1},\dots,c_{l_n}\in C_{l_n}}[c_{n+1} + \cdots + c_{l_n}]_{l_n}$ and (1-5) holds, we have

$$T_g^{m_n}[0]_n \cap [0]_n = \bigsqcup_{c_{n+1}\in C_{n+1}(g_{n+1}^{(n)}),\dots,c_{l_n}\in C_{l_n}(g_{l_n}^{(n)})} [c_{n+1} + \cdots + c_{l_n}]_{l_n}.$$

Hence

$$\prod_{i=n+1}^{l_n} \frac{\#C_i(g_i^{(n)})}{\#C_i} = \frac{\mu(T_g^{m_n}[0]_n \cap [0]_n)}{\mu([0]_n)} \ge \delta(1 - \frac{1}{n}).$$

($\Longleftarrow$) Repeat the same argument in the opposite direction to get (1-6). Since G is Abelian, we have

$$\mu(T_g^{m_n}[f]_n \cap [f]_n) \geq \delta(1-\frac{1}{n})\mu([f]_n)$$

for every $f \in F_n$. It is easy to deduce from this inequality that

$$\liminf_{i\to\infty} \mu(T_g^{m_i}A \cap A) \geq \delta(1-\frac{1}{n})\mu(A)$$

for all $A \in \mathfrak{B}$ of finite measure. This means that T_g is at least $\delta(1-\frac{1}{n})$-partially rigid for every n. By Lemma 1.2(iii), T_g is at least δ-partially rigid. $\square$

Corollary 1.6. *Let $g \in G_\infty$.*

(i) *If*

$$\limsup_{i\to\infty} \max_{0\neq m\in\mathbb{Z}} \frac{\#C_i(mg)}{\#C_i} \geq \delta$$

then T_g is at least δ-partially rigid.

(ii) *If $\#C_n \leq M$ for all n and a transformation T_g is δ-partially rigid then $\delta \leq \frac{M-1}{M}$.*

Proof. We observe that every $g \in G$ belongs to $F_n - F_n$ eventually. This follows from (1-2) and (1-4). Then (1-3) implies that $g \notin C_n - C_n$ eventually for every $g \neq 0$. Thus if $C_{i_k}(m_k g) \neq \emptyset$ for a sequence $i_k \to \infty$ then $m_k \to \infty$. The statement of (i) follows.

(ii) is obvious. $\square$

2. On "individual" multiple recurrence of (C,F)-actions

We begin this section by recalling the basic concepts of p-recurrence and multiple recurrence. Then we construct (C,F)-actions $T = (T_g)_{g\in G}$ of Abelian groups such that the transformations T_g (with g of infinite order) have any prescribed "index" of recurrency (Theorems 2.3, 2.5, 2.6). Some necessary and sufficient conditions for p-recurrence are presented in Remark 2.4. They are given in terms of C_n and are very easy to check. The word "individual" from the name of the section means that we examine the recurrence properties of every T_g separately. On the contrary, in the next section we are concerned with the (multiple) recurrence of T as a whole.

Definition 2.1.

(i) Let p be a positive integer. A transformation T of a σ-finite measure space $(X, \mathfrak{B}, \mu)$ is called *p-recurrent* if for every subset $B \in \mathfrak{B}$ of positive measure there exists a positive integer k such that $\mu(B \cap T^{-k}B \cap \cdots \cap T^{-kp}B) > 0$.

(ii) If T is p-recurrent for any $p > 0$, then it is called *multiply recurrent*.

It is easy to see that T is 1-recurrent if and only if it is conservative. Clearly, if T is rigid then it is multiply recurrent. By the Furnstenberg theorem [**10**], every finite measure preserving transformation is multiply recurrent. The situation is different in infinite measure (see [**9**], [**2**], [**12**]). We note that only $\mathbb{Z}$-actions are considered in these papers. We produce here new examples of infinite measure preserving actions of arbitrary countable discrete Abelian groups.

The following lemma is rather standard and therefore we omit its proof.

Lemma 2.2. *Let $\mathfrak{F}$ be a dense subfamily in $\mathfrak{B}$ and $0 < \delta < 1$. If for every $B \in \mathfrak{F}$ of finite measure there exists $k > 0$ such that $\mu(B \cap T^{-k}B \cap \cdots \cap T^{-kp}B) \geq \delta\mu(B)$ then T is p-recurrent.*

Denote by $G_\infty = \{g_n \mid n = 1, 2, \dots\}$ the subset of G-elements of infinite order.

Theorem 2.3. *Let $p > 0$ and $C_n := \{0, k_n g_n, 2k_n g_n, \dots, pk_n g_n\}$ with k_n being large enough to satisfy* (1-3) *and* (1-5)*. Denote by $T = (T_g)_{g\in G}$ the associated (C, F)-action* (*see Remark* 1.4)*. Then for every $g \in G_\infty$, the transformation T_g is p-recurrent but not $(p+1)$-recurrent.* (*It is easy to see that T_g is multiply recurrent for every $g \notin G_\infty$.*)

Proof. Take $g \in G_\infty$. We first prove that T_g is p-recurrent. Notice that the ring $\mathcal{K}$ of compact open subsets of X is dense in $\mathfrak{B}$. Every subset $B \in \mathcal{K}$ can be represented as $B = \bigsqcup_{[f]_n \subset B} [f]_n$ for some n. Take $m > n$ such that g_m is a multiple of g. Clearly,

$$[f]_n = \bigsqcup_{c_{n+1}\in C_{n+1},\dots,c_m\in C_m} [f + c_{n+1} + \cdots + c_m]_m$$

for each $f \in F_n$. Put

$$B' := \bigsqcup_{[f]_n\subset B, c_{n+1}\in C_{n+1},\dots,c_{m-1}\in C_{m-1}} [f + c_{n+1} + \cdots + c_{m-1}]_m$$

Then $B' \subset B$ and $\mu(B') = \frac{1}{p+1}\mu(B)$. Moreover,

$$T_{ik_m g_m}[f + c_{n+1} + \cdots + c_{m-1}]_m = [f + c_{n+1} + \cdots + c_{m-1} + ik_m g_m]_m$$

and hence $T_{ik_m g_m} B' \subset B$ for all $i = 0, \dots, p$. Since g_m is a multiple of g, it follows from Lemma 2.2 that T_g is p-recurrent.

Now let us prove that T_g is not $(p + 1)$-recurrent. Actually, suppose that the contrary holds. Then there exists $x^{(0)} \in X_0$ and $m > 0$ such that $T_{jmg}x^{(0)} =: x^{(j)} \in X_0$ for all $j = 1, \dots, p + 1$. Let $x^{(j)} = (x_i^{(j)})_{i=1}^\infty$ with $x_i^{(j)} \in C_i$ for all $i > 0$ and $j = 0, \dots, p+1$. It follows from the definition of T that $\sum_{i=1}^\infty (x_i^{(j+1)} - x_i^{(j)}) = mg$, $j = 0, \dots, p$. Let n be the smallest integer such that $\sum_{i=1}^n (x_i^{(1)} - x_i^{(0)}) = mg$. We deduce from (1-5) that

$$0 \neq x_n^{(1)} - x_n^{(0)} = x_n^{(2)} - x_n^{(1)} = \cdots = x_n^{(p+1)} - x_n^{(p)}.$$

Hence $x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(p+1)}$ is an arithmetic progression in C_n of length $p + 2$, a contradiction. □

It is easy to deduce from Corollary 1.6 that T_g is $\frac{p}{p+1}$-partially rigid for every $g \in G_\infty$.

Remark 2.4. Slightly modifying the above proof we can establish the following more general facts:

(i) Let T be a (C, F)-action and $g \in G$. If

$$\limsup_{n\to\infty} \max_{0\neq m\in\mathbb{Z}} \frac{\#(C_n(mg) \cap C_n(2mg) \cap \cdots \cap C_n(pmg))}{\#C_n} > 0$$

then T_g is p-recurrent.

(ii) Let T be a (C, F)-action and (1-5) hold. If C_n does not contain any arithmetic progression of length $p+1$ then T_g is not p-recurrent for any $g \in G_\infty$.

The following statement is an analogue of Theorem 2.3 for multiple recurrence.

Theorem 2.5. *Let $(g_n)_{n>0}$ be as above and $C_n := \{0, k_n g_n, 2k_n g_n, \dots, nk_n g_n\}$ with k_n large so that* (1-3) *is satisfied. Denote by $(T_g)_{g\in G}$ the associated (C,F)-action. Then for every $g \in G$, the transformation T_g is rigid and hence multiply recurrent.*

Proof. Apply Corollary 1.6(i). □

We can also produce non-rigid multiply recurrent transformations.

Theorem 2.6. *Let*

$$C_n := \{0, k_n g_n, 2k_n g_n, \dots, nk_n g_n, (nk_n)^2 g_n, \dots, (nk_n)^n g_n\},$$

with k_n chosen exactly as in Theorem 2.5*. Denote by $(T_g)_{g\in G}$ the corresponding (C,F)-action. Then for every $g \in G_\infty$, the transformation T_g is multiply recurrent but not rigid.*

Proof. The multiple recurrence follows from Remark 2.4(i). To show that T_g is not rigid we apply Theorem 1.5. Actually,

$$\max_{0\neq g\in C_n - C_n} \frac{\#C_n(g)}{\#C_n} = \frac{\#C_n(k_n g_n)}{\#C_n} = \frac{n}{2n} = \frac{1}{2}.$$

It follows from Theorem 1.5 that if T_g is at least δ-partially rigid then $\delta \le \frac{1}{2}$. By Lemma 1.2(iii), T_g is not rigid. □

It is easy to deduce from Corollary 1.6(i) that T_g is indeed 0.5-partially rigid (provided that k_n is chosen so that (1-5) holds in addition to (1-3)).

3. Multiple recurrence of Abelian actions

We investigate here multiple recurrence of actions of countable discrete Abelian groups G. For every $p > 0$, we construct an infinite measure preserving (C,F)-action of G which is p-recurrent but not $(p+1)$-recurrent. Examples of multiply recurrent actions are also given. Next, we demonstrate a difference between the p-recurrence and the "individual" p-recurrence (see § 2) of infinite actions. We also show that a similar gap between the multiple recurrence and the "individual" multiple recurrence is a specific property of actions of the higher free rank groups.

Recall that a family $e_1, \dots, e_k \in G$ is *$\mathbb{Z}$-independent* if the homomorphism $\mathbb{Z}^k \ni (n_1, \dots, n_k) \mapsto n_1 e_1 + \dots + n_k e_k \in G$ is one-to-one. If such a family exists we say that the *free rank* of G is greater than $k-1$.

Lemma 3.1. *Let $e_1, \dots, e_n$ be a $\mathbb{Z}$-independent family in G. Then there exists a map (pseudonorm) $G \ni g \mapsto \|g\| \in \mathbb{R}_+$ such that the following properties are satisfied:*

(i) $\|g_1 + g_2\| \le \|g_1\| + \|g_2\|$ *for all* $g_1, g_2 \in G$,
(ii) $\|lg\| = l\|g\|$ *for all* $g \in G$ *and* $l \in \mathbb{Z}$.
(iii) $\|g\| = 0$ *if and only if g is a torsion,*
(iv) $\|le_i + ke_j\| = \max(|l|, |k|)$ *for all* $i, j = 1, \dots, n$ *and* $k, l \in \mathbb{Z}$.

Proof. Let H stand for the periodic part of G. Then the quotient group G/H is torsion free. Hence there is a group embedding α of G/H into $\bigoplus_{n=1}^\infty \mathbb{Q}$ (see [**13**]).

It follows from the assumptions of the lemma that $\alpha(e_1+H),\dots,\alpha(e_n+H)$ are independent vectors in $\bigoplus_{n=1}^\infty \mathbb{Q}$. Take a norm $|.|$ on $\bigoplus_{n=1}^\infty \mathbb{Q}$ such that

$$|l\alpha(e_i+H)+k\alpha(e_j+H)|=\max(|l|,|k|)$$

for all $i,j=1,\dots,n$ and $k,l\in\mathbb{Z}$ and define $\|.\|$ on G by setting

$$\|g\|:=|\alpha(g+H)|.$$

□

Definition 2.1 extends naturally to actions of Abelian groups as follows.

Definition 3.2. Let G be a countable discrete infinite Abelian group and $T=(T_g)_{g\in G}$ a measure preserving action of G on a σ-finite measure space $(X,\mathfrak{B},\mu)$.

(i) Given a positive integer $p>0$, the action T is called *p-recurrent* if for every subset $B\in\mathfrak{B}$ of positive measure and every $g_1,\dots,g_p\in G$, there exists a positive integer k such that $\mu(B\cap T_{kg_1}B\cap\cdots\cap T_{kg_p}B)>0$.

(ii) If T is p-recurrent for any $p>0$, then it is called *multiply recurrent.*

Clearly, T is 1-recurrent if and only if it is conservative. Every finite measure preserving G-action is multiply recurrent [**11**]. However in infinite measure we demonstrate the following

Theorem 3.3.

(i) *Given $p>0$, there exists a p-recurrent (C,F)-action T such that no one transformation T_g is $(p+1)$-recurrent, $g\in G_\infty$. (Hence T is not $(p+1)$-recurrent.)*

(ii) *There exists a multiply recurrent (C,F)-action.*

Proof. (i) We call a finite sequence $g=(g_1,\dots,g_p)\in(G_\infty)^p$ *admissible* if $g_i-g_j\in G_\infty$ for all $i\neq j$. Let $\{g^{(n)}\mid n\in\mathbb{N}\}$ stand for the (enumerated) family of admissible elements $g^{(n)}=(g_1^{(n)},\dots,g_p^{(n)})$. Now we set

$$C_n:=\{0,k_ng_1^{(n)},\dots,k_ng_p^{(n)}\}$$

with k_n being large enough to satisfy (1-3) and (1-5). Let $T=\{T_g\}_{g\in G}$ stand for the associated (C,F)-action (see Remark 1.4).

It is clear how to modify the proof of Theorem 2.3 to establish that T is p-recurrent. Notice that it suffices to check the inequality from Definition 3.2 on the admissible sequences only.

By Remark 2.4(ii), T_g is not $(p+1)$-recurrent for every $g\in G_\infty$.

(ii) can be proved in a similar way. □

Notice that Theorem 3.3(i) is stronger than Theorem 2.3. Actually, if T is p-recurrent then so is T_g for every $g\in G$. We will show in Theorem 3.4 that the converse does not hold. Now let us compare Theorem 3.3(ii) with Theorem 2.5. It is easy to see that in case $G=\mathbb{Z}$, T is multiply recurrent if and only if so is T_1. Moreover, T_1 is multiply recurrent if and only if so is T_n for every $0\neq n\in\mathbb{Z}$. Thus the multiple recurrence is equivalent to the "individual" multiply recurrence. The same holds for $G=\mathbb{Q}$ or any other group of free rank one. Hence for such groups the statements of Theorems 3.3(ii) and 2.5 are equivalent. However we will demonstrate that this is no longer true for the groups of higher free rank.

Theorem 3.4.

(i) *Let the free rank of G be greater than one, $\{e_1, e_2\}$ a $\mathbb{Z}$-independent family in G and $\|.\|$ the corresponding pseudonorm on G from Lemma* 3.1*. Let C_n be like that in Theorem* 2.5 *but k_n chosen in such a way that the following condition*

(3-1) $$\|f\| \le 0.1\|g_n\| \text{ for all } f \in \sum_{i<n}(C_i - C_i)$$

holds (in addition to (1-3)*). Denote by $T = \{T_g\}_{g\in G}$ the associated (C,F)-action. Then for every $g \in G$, the transformation T_g is rigid (and hence multiple recurrent) but T is not* 2*-recurrent.*

(ii) *Let G be arbitrary, $e_1 \in G_\infty$ and $\|.\|$ the corresponding pseudonorm on G from Lemma* 3.1*. Given $p > 0$, let C_n be like that in Theorem* 2.3 *but k_n chosen in such a way that the following condition*

$$\|f\| \le \frac{0.1\|g_n\|}{p} \text{ for all } f \in \sum_{i<n}(C_i - C_i)$$

holds (in addition to (1-3)*). Denote by $T = \{T_g\}_{g\in G}$ the associated (C,F)-action. Then for every $g \in G$, the transformation T_g is p-recurrent and not $(p+1)$-recurrent while T is not* 2*-recurrent.*

Proof. (i) We only need to check that T is not 2-recurrent. Actually, if T is 2-recurrent, then there exist $x^{(0)} \in X_0$ and $m > 0$ such that $x^{(j)} := T_{me_j}x^{(0)} \in X_0$ for $j = 1, 2$. Let $x^{(j)} = (x_i^{(j)})_{i=1}^\infty$ with $x_i^{(j)} \in C_i$. Then there is $l > 0$ with

$$\sum_{i=1}^{l}(x_i^{(j)} - x_i^{(0)}) = me_j, \ j = 1, 2.$$

Without loss of generality we may assume that $x_l^{(1)} - x_l^{(0)} \neq 0$. We denote $f_j := \sum_{i=1}^{l-1}(x_i^{(j)} - x_i^{(0)})$. Then

(3-2) $$x_l^{(j)} - x_l^{(0)} = me_j + f_j, \ j = 1, 2.$$

Notice that $C_l - x_l^{(0)}$ is an arithmetic progression. Every element of it is a multiple of g_l. Hence we deduce from Lemma 3.1(ii) that

$$\frac{\|x_l^{(j)} - x_l^{(0)}\|}{\|g_l\|} \in \mathbb{Z} \text{ for } j = 1, 2.$$

It follows from (3-1), (3-2) and Lemma 3.1 that

$$\begin{cases} q_1 := \frac{\|x_l^{(1)} - x_l^{(0)}\|}{\|g_l\|} & = \frac{m}{\|g_l\|} \pm 0.1, \\ q_2 := \frac{\|x_l^{(2)} - x_l^{(0)}\|}{\|g_l\|} & = \frac{m}{\|g_l\|} \pm 0.1 \end{cases}$$

for some $0 \le q_1, q_2 \in \mathbb{N}$, $q_1 \neq 0$. Hence $q_1 = q_2$. This implies $x_l^{(2)} - x_l^{(0)} = \pm(x_l^{(1)} - x_l^{(0)})$. It follows from this and (3-2) that $me_1 \pm me_2 = f_2 \pm f_1$. By Lemma 3.1(iv), $m \le \|f_1\| + \|f_2\|$. Applying (3-1) we obtain

$$0.9 \le q_1 - 0.1 \le \frac{m}{\|g_l\|} \le 0.1 + 0.1,$$

a contradiction.

(ii) Use a similar idea to show that $X_0 \cap T_{me_1}X_0 \cap T_{2pme_1}X_0 = \emptyset$ for any $m > 0$. $\square$

4. Polynomial recurrence

In this section we are concerned with the polynomial recurrence of (C, F)-transformations. For simplicity, we only consider the case $G = \mathbb{Z}$. However, the interested reader may extend the results to general Abelian groups. Clearly, polynomial recurrence implies multiple recurrence. We show however that the converse is not true. We also construct, for any $p > 0$, a transformation which is p-polynomially recurrent but not $(p+1)$-recurrent. Finally, we prove that the set of polynomially recurrent transformations is generic in the group $\mathrm{Aut}_0(X, \mu)$ of μ-preserving transformations furnished with the weak topology.

Let $\mathcal{P} := \{q \in \mathbb{Q}[t] \mid q(\mathbb{Z}) \subset \mathbb{Z} \text{ and } q(0) = 0\}$.

Definition 4.1. Let T be a measure preserving transformation of $(X, \mathfrak{B}, \mu)$.

(i) T is called *p-polynomially recurrent* if for every $q_1, \dots, q_p \in \mathcal{P}$ and $B \in \mathfrak{B}$ of positive measure there exists $n \in \mathbb{N}$ with

$$\mu(B \cap T^{q_1(n)} B \cap \dots \cap T^{q_p(n)} B) > 0.$$

(ii) If T is *p-polynomially recurrent* for every $p \in \mathbb{N}$ then it is called polynomially recurrent.

We first show that in infinite measure there are multiply recurrent (even rigid!) transformations which are not polynomially recurrent.

Theorem 4.2. *Let $C_n := \{0, k_n, 2k_n, \dots, k_n^2\}$, where k_n is large so that* (1-3) *is satisfied and, in addition,*

$$\sum_{i<n} k_i^6 < 0.1 k_n. \tag{4-1}$$

Denote by $T = (T_n)_{n \in \mathbb{Z}}$ the associated (C, F)-action. Then T_1 is rigid but not 2-polynomially recurrent.

Proof. If $X_0 \cap T_1^{-m} X_0 \cap T_1^{-m^3} X_0 \neq \emptyset$ for some $m > 0$ then there exist l and l' such that

$$\begin{cases} \sum_{i=1}^{l} (x_i^{(1)} - x_i^{(0)}) = m, \\ \sum_{i=1}^{l'} (x_i^{(2)} - x_i^{(0)}) = m^3 \end{cases} \tag{4-2}$$

with $x_i^{(j)} \in C_i$, $x_l^{(1)} \neq x_l^{(0)}$ and $x_{l'}^{(2)} \neq x_{l'}^{(0)}$. It follows from (4-1), (4-2) and the definition of C_n that

$$\begin{cases} 0.9 k_l < k_l - \sum_{i<l} k_i^2 \leq m \leq \sum_{i \leq l} k_i^2 < 1.1 k_l^2, \\ 0.9 k_{l'} < m^3 < 1.1 k_{l'}^2. \end{cases}$$

This implies

$$\begin{cases} 0.9^3 k_l^3 < 1.1 k_{l'}^2, \\ 0.9 k_{l'} < 1.1^3 k_l^6 \end{cases}$$

which is incompatible with (4-1). Hence T_1 is not 2-polynomially recurrent. By Corollary 1.6(i), T_1 is rigid. □

Now we provide examples of ergodic transformations with all possible "indices" of polynomial recurrence.

Theorem 4.3.

(i) *For every $p > 0$, there exists a (C, F)-transformation which is p-polynomially recurrent but not $(p+1)$-recurrent.*

(ii) *There exists a (C, F)-transformation which is polynomially recurrent.*

Proof. We will prove only the second claim. The first one can be demonstrated in a similar way.

We consider pairs $(p, \overline{q})$ with $p \in \mathbb{N}$, $\overline{q} = (q_1, \dots, q_p) \in \mathcal{P}^p$, $q_i \neq 0$ and $q_i \neq q_j$ whenever $i \neq j$. Let $(p_n, \overline{q}_n)$ be a sequence of such pairs where every possible pair occurs infinitely often. Let $\overline{q}_n = (q_1^{(n)}, \dots, q_{p_n}^{(n)})$. We put

$$C_n := \{0, q_1^{(n)}(k_n), \dots, q_{p_n}^{(n)}(k_n)\},$$

where k_n is large enough to satisfy (1-3). Denote by T the associated (C, F)-action of $\mathbb{Z}$. A slight modification of the proof of Theorem 2.3 is only needed to show that the transformation T_1 is polynomially recurrent. We leave this to the reader.

Remark that T_1 enjoys the following property (cf. Lemma 2.2) which is stronger than polynomial recurrence: given $p > 0$, a subset $B \in \mathfrak{B}$ of finite measure and polynomials $q_1, \dots, q_p \in \mathcal{P}$ then there exist infinitely many $k > 0$ with

$$\mu(B \cap T_1^{q_1(k)} B \cap \dots \cap T_1^{q_p(k)} B) \geq \frac{\mu(B)}{2(p+1)}.$$

□

We endow the group $\mathrm{Aut}_0(X, \mu)$ with the weak topology. Recall that a sequence Q_n of μ-preserving transformations converges *weakly* to $Q \in \mathrm{Aut}_0(X, \mu)$ if and only if $\mu(Q_n B \triangle QB) \to 0$ as $n \to \infty$ for every subset $B \in \mathfrak{B}$ of finite measure.

Theorem 4.4. *The subset $\mathcal{H}$ of polynomially recurrent transformations is generic in $Aut_0(X, \mu)$, i.e. contains a dense G_δ.*

Proof. For $(q_1, \dots, q_p) \in \mathcal{P}^p$, we set

$$\begin{aligned}\mathcal{A}(q_1, \dots, q_p) := \{T \in \mathrm{Aut}_0(X, \mu) \mid\ & \text{there exists } n_i \to \infty \text{ such that} \\ & \liminf_{i \to \infty} \mu(B \cap T^{q_1(n_i)} B \cap \dots \cap T^{q_p(n_i)} B) \geq \frac{\mu(B)}{2(p+1)} \\ & \text{for every } B \in \mathfrak{B},\ \mu(B) < \infty\}.\end{aligned}$$

We also set $\mathcal{A} = \bigcap_{p=1}^{\infty} \bigcap_{(q_1, \dots, q_p) \in \mathcal{P}^p} \mathcal{A}(q_1, \dots, q_p)$. Clearly, $\mathcal{A} \subset \mathcal{H}$. Fix a dense countable family $(B_i)_{i=1}^{\infty}$ in $\mathfrak{B}$ with $\mu(B_i) < \infty$ for each i. Since

$$\begin{aligned}\mathcal{A}(q_1, \dots, q_p) = \bigcap_{t=1}^{\infty} \bigcap_{k > 3p} \bigcap_{M=1}^{\infty} \bigcup_{n > M} \bigcap_{i=1}^{t} \Big\{ T \in \mathrm{Aut}_0(X, \mu) \,\Big|\, \\ \mu(B_i \cap T^{q_1(n)} B_i \cap \dots \cap T^{q_p(n)} B_i) > \left(\frac{1}{2(p+1)} - \frac{1}{k} \right) \mu(B_i) \Big\}\end{aligned}$$

and the map $\mathrm{Aut}_0(X, \mu) \ni T \mapsto \mu(B_i \cap T^{q_1(n)} B_i \cap \dots \cap T^{q_p(n)} B_i) \in \mathbb{R}$ is continuous, it follows that $\mathcal{A}(q_1, \dots, q_p)$ and hence $\mathcal{A}$ is a G_δ. Next, we observe that $\mathcal{A}$ is not empty. Actually, the ergodic transformation constructed in Theorem 4.3(ii) belongs to $\mathcal{A}$ if we choose $(B_i)_i = \mathcal{K}$. Observe also that if a transformation $T \in \mathcal{A}$ then the entire conjugacy class of T is contained in $\mathcal{A}$. Now it suffices to apply [**7**, Theorem 7] which states that the conjugacy class of every ergodic transformation is dense in $\mathrm{Aut}_0(X, \mu)$. □

5. Power weak mixing and multiple (and polynomial) recurrence

In this section we construct new examples of (C,F)-actions. Unlike those that have already appeared in this paper these actions have "strong" weak mixing properties. Namely, for each group G and $p>0$, we provide a power weakly mixing (C,F)-action of G such that every transformation T_g, $g\in G_\infty$, is p-recurrent but not $(p+1)$-recurrent (Theorem 5.5).

We first observe that the (C,F)-construction is well suited to control the Cartesian powers of the actions.

Observation 5.1. For a (C,F)-action $T=(T_g)_{g\in G}$ associated with $(C_n)_{n>0}$ and $(F_n)_{n\geq 0}$, consider its r-fold Cartesian product $T^{(r)}=(T^{(r)}_{(g_1,\dots,g_r)})_{(g_1,\dots,g_r)\in G^r}$, where

$$T^{(r)}_{(g_1,\dots,g_r)}:=T_{g_1}\times\cdots\times T_{g_r}.$$

It is easy to see that $T^{(r)}$ is just the (C,F)-action associated with $(C_n^r)_{n>0}$ and $(F_n^r)_{n\geq 0}$. The upper indices mean the r-fold Cartesian product. Notice that $(C_n^r)_{n>0}$ and $(F_n^r)_{n\geq 0}$ enjoy (1-2)–(1-4) since $(C_n)_{n>0}$ and $(F_n)_{n\geq 0}$ do. Moreover, if $(C_n)_{n>0}$ satisfies (1-5) then so does $(C_n^r)_{n>0}$.

The following lemma follows easily from [**8**, Lemma 2.4]

Lemma 5.2. *Let T be a (C,F)-action and $\delta:G\to\mathbb{R}_+^*$ a map with $\sum_{g\in G}\delta(g)<\frac{1}{2}$. Take $g\in G_\infty$. If for any $n>0$ and $f,f'\in F_n$ there exist $N\in\mathbb{Z}$ and $A\subset[f]_n$ such that $\mu(A)\geq\delta(f'-f)\mu([f]_n)$ and $T_g^NA\subset[f']_n$ then T_g is ergodic.*

For the reader's convenience we first illustrate the idea incorporated in the proof of the main result of § 5 on the following—somewhat simpler—statement.

Proposition 5.3.

(i) *Given $p>0$, there exists a (C,F)-action T such that for each $g\in G_\infty$, the transformation T_g is of infinite ergodic index, p-recurrent but not $(p+1)$-recurrent.*

(ii) *There exists a (C,F)-action T such that for each $g\in G_\infty$, the transformation T_g is of infinite ergodic index and multiply recurrent.*

Proof. (i) According to Remark 1.4, we are only to define C_n, $n\in\mathbb{N}$. Suppose that on the n-th step we already have F_{n-1}. Let $F_{n-1}-F_{n-1}=\{f_i^{(n-1)}\mid i=1,\dots,k\}$ with $f_1^{(n-1)}=0$. We select integers $d_0,d_1,\dots,d_k$ in such a way that $d_0=0$ and $\frac{d_i}{d}\geq\delta(f_i)$, $i=1,\dots,k$, where $d:=d_1+\cdots+d_k$ and $\delta:G\to\mathbb{R}_+^*$ a map from Lemma 5.2. Recall that G_∞ is enumerated as $\{g_n\mid n\in\mathbb{N}\}$. Take any pseudonorm $\|.\|$ on G satisfying (i)–(iii) from Lemma 3.1. Now we put

$$\begin{aligned}A_i&:=\{j(q_ng_n+f_i^{(n-1)})\mid j=0,\dots,p\},\ i=1,\dots,k\text{ and}\\ C_n&:=\bigsqcup_{i=1}^{k}\ \bigsqcup_{s=d_0+\cdots+d_{i-1}}^{d_1+\cdots+d_i-1}(h_{s,n}g_n+A_i),\end{aligned}$$

where q_n and $h_{s,n}$ are integers such that the following properties are satisfied

$$\#C_n = (p+1)d,$$

$$\max\left\{\frac{\|f\|}{\|c\|}\ \middle|\ f \in F_{n-1} - F_{n-1}, 0 \neq c \in C_n - C_n\right\} \leq \frac{0.1}{p}, \tag{5-1}$$

$$\bigcup_{i=1}^{k}\left\{\frac{\|c'\|}{\|c\|}\ \middle|\ c, c' \in (C_n - C_n) \setminus (A_i - A_i)\right\} \cap \left(\frac{1}{3p}, 3p\right) = \emptyset, \tag{5-2}$$

$$\max_{1\leq i\leq k}\left\{\frac{\|a'\|}{\|a\|}\ \middle|\ a, a' \in A_i - A_i, a \neq 0\right\} \leq p + 0.1. \tag{5-3}$$

Notice that (5-1) implies (1-3). Hence the corresponding (C,F)-action T of G is well defined.

It is easy to calculate that

$$\limsup_{n\to\infty}\max_{0\neq m\in\mathbb{Z}}\frac{\#(C_n(mg)\cap C_n(2mg)\cap\cdots\cap C_n(pmg))}{\#C_n} \geq \frac{\delta(0)}{p+1}.$$

Hence T_g is p-recurrent by Remark 2.4(i). Let us prove that it is not $(p+1)$-recurrent. To achieve this we can not utilize Remark 2.4(ii) any longer, since (1-5) does not hold for C_n. Thus we have to argue in a different way. If T_g were $(p+1)$-recurrent, then there exist $x^{(0)} \in X_0$ and $m > 0$ with $T_{mg}x^{(0)} =: x^{(1)} \in X_0$ and $T_{(p+1)mg}x^{(0)} =: x^{(p+1)} \in X_0$. Let $x^{(s)} = (x_i^{(s)})_{i=1}^{\infty}$ with $x_i^{(s)} \in C_i$, $s = 0, 1, p+1$. Denote by n the smallest integer such that

$$\sum_{i<n}(x_i^{(s)} - x_i^{(0)}) + (x_n^{(s)} - x_n^{(0)}) = smg,\ \ s = 1, p+1.$$

It follows from (5-1) that $x_n^{(s)} - x_n^{(0)} \neq 0$ and

$$\|x_n^{(s)} - x_n^{(0)}\|\left(1 \pm \frac{0.1}{p}\right) = sm\|g\|$$

for $s = 1, p+1$. Hence

$$\frac{\|x_n^{(p+1)} - x_n^{(0)}\|}{\|x_n^{(1)} - x_n^{(0)}\|} = (p+1)\left(1 \pm \frac{0.3}{p}\right) = p + 1 \pm 0.6. \tag{5-4}$$

From (5-2) we deduce that $x_n^{(p+1)} - x_n^{(0)}, x_n^{(1)} - x_n^{(0)} \in A_i - A_i$ for some $i \in \{1, \ldots, k\}$. But then (5-4) contradicts (5-3).

Now let us demonstrate that T_g is ergodic. For $n > 0$ and $f, f' \in F_n$, let $m > n$ be a positive integer such that g_m is a power of g. Since $0 \in \bigcap_{n>0} C_n$, it follows that $F_n \subset F_{m-1}$ and hence $F_n - F_n \subset F_{m-1} - F_{m-1}$. Therefore $f' - f = f_i^{(m-1)}$ for some i. We set

$$A := \bigsqcup_{c_{n+1}\in C_{n+1},\ldots,c_{m-1}\in C_{m-1}}\ \bigsqcup_{s=d_0+\cdots+d_{i-1}}^{d_1+\cdots+d_i-1}[f + c_{n+1} + \cdots$$
$$+ c_{m-1} + h_{s,m}g_m + q_m g_m + f_i^{(m-1)}]_m.$$

Clearly, $A \subset [f]_n$ and

$$\frac{\mu(A)}{\mu([f]_n)} = \frac{1}{p+1}\frac{d_i}{d} > \frac{1}{p+1}\delta(f'-f).$$

Moreover,

$$\begin{aligned}&T_{-q_m g_m}[f + c_{n+1} + \cdots + c_{m-1} + h_{s,m}g_m + q_m g_m + f_i^{(m-1)}]_m\\ &= [f' + c_{n+1} + \cdots + c_{m-1} + h_{s,m}g_m]_m \subset [f']_m.\end{aligned}$$

Thus we deduce from Lemma 5.2 that T_g is ergodic.

In view of Observation 5.1, only a slight modification of the above argument is needed to show that the transformation $\underbrace{T_g \times \cdots \times T_g}_{r \text{ times}}$ is ergodic for every $r > 1$.

(ii) is easier than (i). To define C_n now we need only to satisfy (1-3) instead of (5-1)–(5-3). However p is no longer fixed. It runs over $\mathbb{N}$. $\square$

Remark 5.4. Let T be any (C, F)-action of G. Slightly modifying the above proof of the ergodicity we can establish a more general fact: if

$$\limsup_{n\to\infty} \max_{m\in\mathbb{Z}} \min_{f\in F_{n-1}-F_{n-1}} \frac{\#C_n(f+mg)}{\#C_n \cdot \delta(f)} \geq 0$$

then T_g is ergodic.

Now we are ready to prove

Theorem 5.5.

(i) *Given $p > 0$, there exists a (C, F)-action T of G such that for every finite sequence $g_1, \dots, g_r \in G_\infty$, the transformation $T_{g_1} \times \cdots \times T_{g_r}$ is ergodic, p-recurrent but not $(p+1)$-recurrent.*

(ii) *There exists a (C, F)-action T of G such that for every finite sequence $g_1, \dots, g_r \in G_\infty$, the transformation $T_{g_1} \times \cdots \times T_{g_r}$ is ergodic and multiply recurrent.*

Proof. Part (i) is similar to that of Proposition 5.3(i). We only define $(C_n)_{n>0}$. Let us enumerate all finite sequences of elements from G_∞: $g^{(1)}, g^{(2)}, \dots$. Let $g^{(n)} = (g_1^{(n)}, \dots, g_{l_n}^{(n)})$ and $F_{n-1} - F_{n-1} = \{f_i^{(n-1)} \mid i = 1, \dots, k\}$ with $f_1^{(n-1)} = 0$. Select $d_0, d_1, \dots, d_k$ exactly as in the proof of Proposition 5.3. Now we put

$$\begin{aligned}A_{i,t} &:= \{j(q_n g_t^{(n)} + f_i^{(n-1)}) \mid j = 0, \dots, p\},\\ C_n &:= \bigsqcup_{t=1}^{l_n} \bigsqcup_{i=1}^{k} \bigsqcup_{s=d_0+\cdots+d_{i-1}}^{d_1+\cdots+d_i-1} (h_{s,t,n} g_t^{(n)} + A_{i,t}),\end{aligned}$$

where q_n and $h_{s,t,n}$ are integers such that the following properties are satisfied

$$\#C_n = (p+1)l_n d,$$

$$\max\left\{\frac{\|f\|}{\|c\|} \,\middle|\, f \in F_{n-1} - F_{n-1}, 0 \neq c \in C_n - C_n\right\} \leq \frac{0.1}{p},$$

$$\bigcup_{i=1}^{k}\bigcup_{t=1}^{l_n}\left\{\frac{\|c'\|}{\|c\|} \,\middle|\, c, c' \in (C_n - C_n) \setminus (A_{i,t} - A_{i,t})\right\} \cap \left(\frac{1}{3p}, 3p\right) = \emptyset,$$

$$\max_{1\leq i\leq k, 1\leq t\leq l_n} \left\{\frac{\|a'\|}{\|a\|} \,\middle|\, a, a' \in A_{i,t} - A_{i,t}, a \neq 0\right\} \leq p + 0.1.$$

Denote by T the associated (C,F)-action of G. For $f=(f_1,\dots,f_r)\in F_{n-1}^r$, we let $\delta_r(f):=\delta(f_1)\cdots\delta(f_r)$. Then we have

$$\limsup_{n\to\infty}\max_{0\neq m\in\mathbb{Z}}\frac{\#C_n^r((mg_1,\dots,mg_r))\cap\cdots\cap C_n^r((pmg_1,\dots,pmg_r))}{\#C_n^r}\geq\frac{\delta(0)^r}{(p+1)^r r^r}$$

$$\limsup_{n\to\infty}\max_{m\in\mathbb{Z}}\min_{f\in F_{n-1}-F_{n-1}}\frac{\#C_n^r(f+(mg_1,\dots,mg_r)}{\#C_n^r\delta_r(f)}\geq\frac{1}{(p+1)^r r^r}.$$

It follows from Remark 2.4(i) and Remark 5.4 that $T_{g_1}\times\cdots\times T_{g_r}$ is p-recurrent and ergodic respectively.

Since the Cartesian product of a non-$(p+1)$-recurrent transformation with any transformation is not $(p+1)$-recurrent, it only remains to show that T_g is not $(p+1)$-recurrent for every $g\in G_\infty$. To this end just repeat the corresponding part of the proof of Proposition 5.3 almost verbatim.

(ii) It is easy to verify that for the (C,F)-actions T from [**8**, Theorem 2.13], the ergodic transformation $T_{g_1}\times\cdots\times T_{g_r}$ is multiply recurrent. □

Remark 5.6. The interested reader may refine Theorem 5.5 by replacing the p-recurrence and multiply recurrence properties with p-polynomial recurrence and polynomial recurrence respectively. Another way to improve this theorem is to arrange p-recurrence and multiply recurrence for the "whole action" $T^{(r)}$of G^r, $r>1$.

6. On topological recurrence

In this section we discuss topological counterparts of the results from § 2–5.

Definition 6.1 (cf. 2.1).

(i) Let p be a positive integer. A homeomorphism T of a topological space X is called *topologically p-recurrent* if for every nonempty open subset $O\subset X$ there exists a positive integer k such that

$$O\cap T^{-k}O\cap\cdots\cap T^{-kp}O\neq\emptyset. \tag{6-1}$$

(ii) If T is topologically p-recurrent for any $p>0$, then it is called *topologically multiply recurrent.*

Clearly, it suffices to check (6-1) only on a base of the topology. In a similar way one can state topological analogues of Definitions 3.2 and 4.1, i.e. the concepts of topological p-polynomial recurrence and p-recurrence of a topological action of G.

Recall that every (C,F)-action is a minimal free action of G on a locally compact Cantor set. Analyzing the proofs of the results from § 2–5 we can get immediately topological counterparts for most of them. Actually, when establishing (measure theoretical) p-recurrence and ergodicity we worked only with the cylinders. Every cylinder is of positive measure and the ring of cylinders form a base of the topology. Next, while checking the lack of $(p+1)$-recurrence we were only to show that some intersections of subsets are of measure zero. However we proved indeed the stronger fact that these intersections are empty. That is exactly what we need to establish the lack of topological $(p+1)$-recurrence.

Thus, as a byproduct of the proofs of Theorems 2.3 and 2.5 we may record the following statement.

Theorem 6.2.

(i) *Let T be the (C,F)-action from Theorem* 2.3. *Then for every $g \in G_\infty$, the transformation T_g is topologically p-recurrent but not topologically $(p+1)$-recurrent.*

(ii) *Let T be the (C,F)-action from Theorem* 2.5. *Then for every $g \in G_\infty$, the transformation T_g is topologically multiply recurrent.*

In a similar way we can also 'topologize' Theorems 3.3, 3.4, 4.2, 4.3, 5.5. In doing so, we have to replace the terms "rigid" and "ergodic" in the statements of these theorems with "topologically multiply recurrent" and "topologically transitive" respectively. Recall that a homeomorphism is *topologically transitive* if the orbit of every nonempty open set is dense.

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