EXPLICIT SOLUTION OF ROKHLIN'S PROBLEM ON HOMOGENEOUS SPECTRUM AND APPLICATIONS

Alexandre I. Danilenko

Abstract. For each $n > 1$, we construct explicitly a rigid weakly mixing rank n transformation with homogeneous spectrum of multiplicity n. The fact of existence of such transformations was established recently by O. Ageev via Baire category arguments (a new short category proof is also given here). As an application, for any subset $M \subset \mathbb{N}$ containing 1, a weakly mixing transformation whose essential range for the spectral multiplicity equals $n \cdot M$ is constructed.

0. Introduction

Let $(X, \mathfrak{B}, \mu)$ be a standard non-atomic probability space and let $\mathrm{Aut}_0(X, \mu)$ stand for the group of μ-preserving transformations of X. For each $S \in \mathrm{Aut}_0(X, \mu)$, we denote by $\mathcal{M}(S)$ the set of essential values for the multiplicity function of the unitary operator $f \mapsto f \circ S^{-1}$ on the Hilbert space $L^2_0(X, \mu) := L^2(X, \mu) \ominus \mathbb{C}$. We consider the problem of

whether each subset of $\mathbb{N} \cup \{\infty\}$ can be realized as $\mathcal{M}(S)$ for an ergodic S?

Recall that the first example of S with non-trivial $\mathcal{M}(S)$, i.e. $\mathcal{M}(S) \neq \{1\}, \{\infty\}$ or $\{1, \infty\}$, appeared in [Os]. It was shown there that $2 \leq \sup \mathcal{M}(S) \leq 30$. A real breakthrough was made by E. Robinson in [R1], where for a given $n \in \mathbb{N}$, an ergodic S with $\mathcal{M}(S) = \{1, n\}$ was constructed. In his example, S is a compact group extension of a transformation admitting a good cyclic approximation. This approach was further elaborated by various authors in [R2], [G–L], [KL] to obtain finally the following result: for each subset $M \subset \mathbb{N} \cup \{\infty\}$ such that $1 \in M$, there exists an ergodic S with $\mathcal{M}(S) = M$. (Later Ageev reproved this result in [A3] via Baire catedory arguments.) However the case $1 \notin M$ is considerably less studied. For instance, the following Rokhlin's problem on homogeneous spectrum was open for several decades:

given $n > 1$, is there an ergodic transformation S with $\mathcal{M}(S) = \{n\}$?

The affirmative answer to this problem was given for $n = 2$ in [Ry1] and independently in [A1] by showing that $\mathcal{M}(S \times S) = \{2\}$ for a generic $S \in \mathrm{Aut}_0(X, \mu)$. Also, a conjecture of A. Katok [Ka] was proved in [A1]:

$$\mathcal{M}(\underbrace{S \times \cdots \times S}_{n \text{ times}}) = \{n, n(n-1), \ldots, n!\}$$

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for a generic $S \in \mathrm{Aut}_0(X, \mu)$. Recently, O. Ageev in [A4] applied a different, so-called *group action approach* to solve affirmatively Rokhlin's problem for any n. The idea is to select, for each n, a non-commutative group G_n and a point $g_n \in G_n$ in such a way that for a generic (in a natural Polish topology) G_n-action T, the transformation T_{g_n} is ergodic and has homogeneous spectrum of multiplicity n.

In Section 1 we give another—shorter and simpler—proof of Ageev's result (see Theorem 1.4). We show how to bypass the *analytic approximation technique* which is crucial in [A4]. In contrast, our approach (based on Proposition 1.1) is *algebraic* without any use of the ϵ-δ-argument. We note also that our choice of G_n is different from Ageev's one.

In Section 2 we—following [A4]—investigate the 'generic' spectral multiplicity of transformations T_h for some other points $h \in G_n$, $h \neq g_n$.

Notice that Ageev's solution of Rokhlin's problem is based on Baire category arguments and it is not constructive. Thus, except for the case $n = 2$ (see [Ry2] and [A2]) no explicit ergodic transformations with homogeneous spectrum have been known so far. Our purpose in Section 3 is to apply the algorithmic (C, F)-*construction* to produce such transformations for any n (see Theorem 2.4). We recall that this construction appeared first in [Ju] (and, independently, in [Da1]) as an algebraic counterpart for the cutting-and-stacking to produce (funny) rank-one actions for a wide class of groups.

In Section 4 we use the explicit construction of Section 3 to contribute to the general spectral multiplicity problem. Combining this construction with the *compact group extension method* from [G–L] and [KL] we construct for each $n \in \mathbb{N}$ and a subset $M \subset \mathbb{N}$ containing 1, a weakly mixing S with $\mathcal{M}(S) = n \cdot M$ (Theorem 4.1). For this, we introduce a concept of (C, F)-*cocycles*. They are defined on the orbit equivalence relation of the corresponding (C, F)-action. Since this equivalence relation is an *inductive* tail equivalence relation on an infinite product space, a (C, F)-cocycle is determined by a sequence of maps defined on finite sets. Notice that (C, F)-cocycles are a generalization of Morse cocycles studied by many authors (e.g. [Ke], [Ma], [G–L], [Go, Section 5]).

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1. Short proof of Ageev's theorem

From now on we fix $n > 1$ and a family $e_1, \dots, e_n$ of generators for $\mathbb{Z}^n$. Define a 'cyclic' group automorphism $A : \mathbb{Z}^n \to \mathbb{Z}^n$ by setting $Ae_1 := e_2, \dots, Ae_{n-1} := e_n$ and $Ae_n := e_1$. Let G denote the semidirect product $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ with the multiplication law as follows

$$(v, m)(w, l) := (v + A^m w, m + l), \quad v, w \in \mathbb{Z}^n, \; m, l \in \mathbb{Z}.$$

Then we have a natural embedding $v \mapsto (v, 0)$ of $\mathbb{Z}^n$ into G. We also let $e_0 := (0, 1) \in G$ and $e_{n+1} := e_0^n$. Notice that G is generated by e_1 and e_0. Moreover, $e_0 e_i e_0^{-1} = Ae_i$ for all $i = 1, \dots, n$. Let H be the subgroup of G generated by $e_1, \dots, e_{n+1}$. Then H is a free Abelian group with $n+1$ generators. It is normal in G and the quotient G/H is a cyclic group of order n. Moreover, A extends naturally to H via the conjugation by e_0. We denote this extension by the same symbol A. While multiplying elements of H we will often utilize the symbol $+$.

By an action T of G we mean a group homomorphism $G \ni g \mapsto T_g \in \mathrm{Aut}_0(X, \mu)$.

Proposition 1.1. *There exists a free action T of G such that the transformation T_{e_0} is ergodic and has a pure point spectrum.*

Proof. Let $\Lambda_1 \supset \Lambda_2 \supset \cdots$ be a nested sequence of lattices (i.e. co-finite subgroups) in G such that $\bigcap_{i=1}^{\infty} \Lambda_i = \{1_G\}$. Then the corresponding homogeneous G-spaces G/Λ_i are intertwined by the canonical G-equivariant projection maps:

$$G/\Lambda_1 \leftarrow G/\Lambda_2 \leftarrow \cdots .$$

Let (X, μ) stand for the projective limit of the sequence $(G/\Lambda_i, \lambda_i)_{i=1}^{\infty}$, where λ_i is the equi-distribution on G/Λ_i. Denote by $\pi_i : (X, \mu) \to (G/\Lambda_i, \lambda_i)$ the canonical projections. Then there exists a unique μ-preserving action T of G on X such that $\pi_i \circ T_g = g \cdot \pi_i$ for all $g \in G$ and $i \geq 1$. Since $\bigcap_{i=1}^{\infty} \Lambda_i = \{1_G\}$, it follows that T is free. To complete the proof it suffices to select the sequence $(\Lambda_i)_{i=1}^{\infty}$ in such a way that the element $e_0 \in G$ acts transitively on G/Λ_i for each $i \geq 1$. Let Λ_i be generated by $k_{1,i}e_1, \ldots, k_{n,i}e_n$ and $k_{n+1,i}(e_1 + \cdots + e_{n+1})$, where

(a) the positive integers $k_{1,i}, \ldots, k_{n+1,i}$ are pairwise coprime and
(b) $k_{j,i-1} \mid k_{j,i}$ and $k_{j,i-1} \neq k_{j,i}$ for all $j = 1, \ldots, n+1$.

It follows from (b) that $\Lambda_1 \supset \Lambda_2 \supset \cdots$ and $\bigcap_{i=1}^{\infty} \Lambda_i = \{1_G\}$. Since $\Lambda_i \subset H$ and $e_0^j \notin H$ for any $1 \leq j < n$, it remains to show that $\Lambda_i + \mathbb{Z}e_{n+1} = H$. Denote by l the index of the lattice $\Lambda_i + \mathbb{Z}e_{n+1}$ in H. Then l divides the index (in H) of the sublattice generated by $k_{2,i}e_2, \ldots, k_{n,i}e_n, k_{n+1,i}(e_1 + \cdots + e_{n+1})$ and e_{n+1}. The latter index is $k_{2,i} \cdots k_{n+1,i}$. In a similar way, $l \mid (k_{s,i}^{-1} \cdot k_{1,i} \cdots k_{n+1,i})$ for each $s = 1, \ldots, n+1$. Now it follows from (a) that $l = 1$. □

It is well known that $\mathrm{Aut}_0(X, \mu)$ is a Polish group when endowed with the weak topology [Ha] defined by: $S_m \to S$ if $\mu(S_m A \triangle SA) \to 0$ for each $A \in \mathfrak{B}$. Furnish the product space $\mathrm{Aut}_0(X, \nu)^G$ with the product topology. Denote by $\mathcal{A}_G \subset \mathrm{Aut}_0(X, \nu)^G$ the subset of all measure preserving actions of G on $(X, \mathfrak{B}, \mu)$. It is clear that $\mathcal{A}_G$ is closed and hence Polish in the induced topology. There exists a natural continuous action of $\mathrm{Aut}_0(X, \mu)$ on $\mathcal{A}_G$ by conjugation:

$$(R \bullet T)_g := RT_gR^{-1} \quad \text{for } R \in \mathrm{Aut}_0(X, \mu) \text{ and } T \in \mathcal{A}_G.$$

Proposition 1.2. *The following two subsets are residual in $\mathcal{A}_G$:*

$$\mathcal{S} := \{T \in \mathcal{A}_G \mid T_{e_0} \textit{ has a simple spectrum}\} \textit{ and}$$
$$\mathcal{W} := \{T \in \mathcal{A}_G \mid T_g \textit{ is weakly mixing for each } g \in G, g \neq 1_G\}.$$

Proof. In view of the following facts:

(i) the subsets of weakly mixing transformations and transformations with simple spectra are both G_δ in $\mathrm{Aut}_0(X, \mu)$ and invariant under conjugation;
(ii) the map $\mathcal{A}_G \ni T \mapsto T_g \in \mathrm{Aut}_0(X, \mu)$ is continuous and $\mathrm{Aut}_0(X, \mu)$-equivariant for each $g \in G$;
(iii) the $\mathrm{Aut}_0(X, \mu)$-orbit of any free G-action is dense in $\mathcal{A}_G$ by [FW, Claim 18],

it remains to show that $\mathcal{S}$ and $\mathcal{W}$ contain at least one free G-action. Each Bernoullian G-action is free and belongs to $\mathcal{W}$. Since each ergodic transformation with pure point spectrum has a simple spectrum, $\mathcal{S}$ contains a free G-action by Proposition 1.1. □

According to an advice of the referee, we now briefly outline the proof of (iii) in order to make our exposition self-contained. By [GlK], the set of free actions is dense in $\mathcal{A}_G$. Next, we note that G is a monotilable group. This means that there exist a Følner sequence $(F_n)_{n=1}^{\infty}$ and a sequence $(C_n)_{n=1}^{\infty}$ such that $\{F_n c \mid c \in C_n\}$ is a partition of G for each n. Take two free G-actions T and T'. Fix $\epsilon > 0$ and finite sequences $g_1, \dots, g_k \in G$ and $B_1, \dots, B_k \in \mathfrak{B}$. Find N large so that $\#(g_i F_N \triangle F_n)/\#F_N < \epsilon$ for all $1 \le i \le k$. By Rokhlin's lemma for monotilable groups [OW], there is a subset $A \in \mathfrak{B}$ such that the sets $T_g A$, $g \in F_N$, are pairwise disjoint and $\mu(\bigsqcup_{g\in F_N} T_g A) > 1 - \epsilon$. Denote by A' a subset with the similar properties corresponding to T'. Without loss of generality we may assume that $\mu(A) = \mu(A')$. Let $R : A \to A'$ be a μ-preserving bijection. Then we can extend it to a μ-preserving one-to-one transformation R of X in such a way that $RT_g x = T'_g Rx$ for all $x \in A$ and $g \in F_N$. It follows that $\mu(RT_{g_i} R^{-1} B_j \triangle T'_{g_i} B_j) < 3\epsilon$ for all $1 \le i, j \le k$. This implies that T' belongs to the closure of the $\mathrm{Aut}_0(X, \mu)$-orbit of T.

Lemma 1.3. *Let $\mathcal{H}$ be a separable Hilbert space and let $U : H \ni h \mapsto U_h \in \mathcal{U}(\mathcal{H})$ be a unitary representation of H in $\mathcal{H}$. If U is unitarily equivalent to $U \circ A$ and for each $1 \le l < n$ with $l \mid n$, the operator $U_{e_{l+1}-e_1}$ has no non-trivial fixed vector then $\mathcal{M}(U_{e_{n+1}}) \subset \{n, 2n, \dots\} \cup \{\infty\}$.*

Proof. By the spectral theorem for U, there exist a probability measure σ on the dual group $\widehat{H}$ and a Borel map $k : \widehat{H} \to \mathbb{N} \cup \{\infty\}$ such that the following decomposition holds (up to unitary equivalence):

$$L_0^2(X, \mu) = \int_{\widehat{H}} \mathcal{H}_w \, d\sigma(w) \quad \text{and} \quad U_h = \int_{\widehat{H}} w(h) I_w \, d\sigma(w), \tag{1-1}$$

for each $h \in H$, where $w \mapsto \mathcal{H}_w$ is a Borel field of Hilbert spaces, $\dim \mathcal{H}_w = k(w)$ and I_w the identity operator on $\mathcal{H}_w$. The inclusion $\mathbb{Z} \ni m \mapsto m e_{n+1} \in H$ induces a projection $\pi : \widehat{H} \to \mathbb{T}$. Let $\sigma = \int_{\mathbb{T}} \sigma_z \, d\widehat{\sigma}(z)$ denote the desintegration of σ relative to this projection. Then we derive from (1-1) that

$$L_0^2(X, \mu) = \int_{\mathbb{T}} \mathcal{H}'_z \, d\widehat{\sigma}(z) \quad \text{and} \quad U_{e_{n+1}} = \int_{\mathbb{T}} z I_z \, d\widehat{\sigma}(z),$$

where $\mathcal{H}'_z := \int_{\widehat{H}} \mathcal{H}_w \, d\sigma_z(w)$. Let $l(z) := \dim \mathcal{H}'_z$, $z \in \mathbb{T}$. Then

$$l(z) = \begin{cases} \infty, & \text{if } \sigma_z \text{ is not purely atomic} \\ \sum_{\sigma_z(w)>0} k(w), & \text{otherwise.} \end{cases} \tag{1-2}$$

Since U is unitarily equivalent to $U \circ A$ and $A^n = \mathrm{Id}$, we may assume without loss of generality that k and σ are both invariant under the dual (to A) automorphism $A^\star$ of $\widehat{H}$. We claim that

$$\text{for } \sigma\text{-a.a. } w \in \widehat{H}, \text{ the } A^\star\text{-orbit of } w \text{ has length } n. \tag{1-3}$$

Indeed, otherwise there exists $1 \le l < n$ such that $\sigma(\{w \in \widehat{H} \mid (A^\star)^l w = w\}) > 0$ and $l \mid n$. Then (1-1) implies that the unitary $U_{e_1 - A^l e_1}$ has a non-trivial fixed vector. However this contradicts to a condition of the lemma.

Since $Ae_{n+1} = e_{n+1}$, we have $\pi \circ A^\star = \pi$. Therefore it follows from the invariance of σ under $A^\star$ that $\sigma_z \circ A^\star = \sigma_z$ for $\widehat{\sigma}$-a.a. $z \in \mathbb{T}$. Hence (1-2), (1-3) and the fact $k \circ A^\star = k$ imply that $n \mid l(z)$ for $\widehat{\sigma}$-a.a. z, i.e. $\mathcal{M}(T_{e_{n+1}}) \subset \{n, 2n, \dots\} \cup \{\infty\}$. □

Now we state and prove a modified version of the main result from [A2].

Theorem 1.4. *For each $T \in \mathcal{S} \cap \mathcal{W}$, i.e. for a generic action from $\mathcal{A}_G$, the transformation $T_{e_{n+1}}$ is weakly mixing and $\mathcal{M}(T_{e_{n+1}}) = \{n\}$.*

Proof. Denote by $U : G \ni g \mapsto U_g \in \mathcal{U}(L^2_0(X,\mu))$ the unitary representation of G associated with T. Since $T \in \mathcal{W}$ and $U_{e_0} U_h U_{e_0}^{-1} = U_{Ah}$ for each $h \in H$, it follows from Lemma 1.3 that $\mathcal{M}(T_{e_{n+1}}) \subset \{n, 2n, \dots\} \cup \{\infty\}$. On the other hand, $\mathcal{M}(T_{e_0}) = \{1\}$ since $T \in \mathcal{S}$. Hence $\mathcal{M}(T_{e_{n+1}})$ is bounded by n from above (we recall that $e_{n+1} = e_0^n$). Therefore $\mathcal{M}(T_{e_{n+1}}) = \{n\}$. □

Remark 1.5. Indeed we established more: if $U = (U_g)_{g\in G}$ is a unitary representation of G such that the operator U_{e_0} has a simple spectrum and for each $1 \le l < n$ with $l \mid n$, the operator $U_{e_{l+1}-e_1}$ has no non-trivial fixed vectors then the operator $U_{e_{n+1}}$ has a homogeneous spectrum of multiplicity n.

Remark 1.6. Define a group automorphism A' of $\mathbb{Z}^{n-1}$ by setting $A'e_1 := e_2$,..., $A'e_{n-2} := e_{n-1}$ and $A'e_{n-1} := -e_1 - \dots - e_{n-1}$. Ageev in [A2] considers the group $G^* := \mathbb{Z}^{n-1} \rtimes_{A'} \mathbb{Z}$ instead of G. Notice that the results of Section 1 (and their proofs) hold as well for G^* with obvious minor modifications. The advantage of G will become apparent in Section 2, where we investigate the generic multiplicity function for T_{e_1} and in Sections 3 and 4, where we construct explicit actions of G.

2. T_{e_1} has a simple spectrum for a generic $T \in \mathcal{A}_G$

Recall that G is generated by e_0 and e_1. We studied the spectral multiplicity of the transformation T_{e_0} for a generic G-action T in Section 1. Now we are going to investigate the spectral multiplicity of T_{e_1} (as Ageev did in [A2] for the group G^*, see Remark 1.5). We let

$$\mathcal{E} := \{T \in \mathcal{A}_G \mid \mathcal{M}(T_{e_1}) = \{1\}\}.$$

The following statement is the main result of this section.

Proposition 2.1. *$\mathcal{E}$ is a dense G_δ in $\mathcal{A}_G$.*

Arguing in the same way as in the proof of Proposition 1.1 we see that $\mathcal{E}$ is a dense G_δ whenever $\mathcal{E}$ contains a free G-action. Thus to prove Proposition 2.1 it is enough to construct such an action. For this, we will exploit the concept of co-induced action introduced by Dooley, Golodets, Rudolph and Sinelshchikov.

Definition 2.2 ([D–S], [GS]**).** Let Γ be a countable group and Λ a subgroup of Γ. Let $\widehat{T} = (\widehat{T}_h)_{h\in\Lambda}$ be a measure preserving action of Λ on a standard probability space $(Y, \mathfrak{C}, \nu)$. Select a cross-section $\sigma : \Lambda\backslash\Gamma \to \Gamma$ of the quotient map $\Gamma \to \Lambda\backslash\Gamma$ with $\sigma(\Lambda) = 1_\Gamma$. Define an action $T = (T_g)_{g\in\Gamma}$ of Γ on the product space $(X, \mathfrak{B}, \mu) := (Y, \mathfrak{C}, \nu)^{\Lambda\backslash\Gamma}$ by setting

$$(T_g x)(\Lambda g') := \widehat{T}_{\sigma(\Lambda g') g \sigma(\Lambda g' g)^{-1}} x(\Lambda g' g)$$

for all maps $x : \Lambda\backslash\Gamma \to Y$ and $g \in \Gamma$. Then T is said to be *co-induced from* $\widehat{T}$.

It is easy to see that T does not depend (up to conjugacy) on the choice of σ. Moreover, if $\widehat{T}$ is free or ergodic then so is T.

Now we are going to apply the co-inducing procedure to the pair $H \subset G$. Take a family $z_1, \dots, z_{n+1}$ of 'rationally independent' elements of the circle $\mathbb{T}$. This means that if $z_1^{t_1} \cdots z_{n+1}^{t_{n+1}} = 1$ for some $t_1, \dots, t_{n+1} \in \mathbb{Z}$ then $t_1 = \cdots = t_{n+1} = 0$. Define an action $\widehat{T}$ of H on the circle $\mathbb{T}$ equipped with Haar measure $\lambda_{\mathbb{T}}$ by setting

$$\widehat{T}_{e_1^{t_1} \cdots e_{n+1}^{t_{n+1}}} z := z_1^{t_1} \cdots z_{n+1}^{t_{n+1}} z, \quad t_1, \dots, t_{n+1} \in \mathbb{Z}.$$

It is obvious that $\widehat{T}$ is free and ergodic. Let $\sigma(He_0^j) := e_0^j$ for all $j = 0, \dots, n-1$. Then the G-action T co-induced from $\widehat{T}$ via the cross-section σ is defined on the n-torus $(\mathbb{T}^n, \lambda_{\mathbb{T}^n})$. The generators of G act as follows

$$T_{e_1}(z_1, \dots, z_n) = (\widehat{T}_{e_1} z_1, \widehat{T}_{e_2} z_2 \dots, \widehat{T}_{e_n} z_n)$$
$$T_{e_0}(z_1, \dots, z_n) = (z_2, \dots, z_n, \widehat{T}_{e_{n+1}} z_1).$$

We see that the transformation T_{e_1} is ergodic and has a pure point spectrum. Hence $\mathcal{M}(T_{e_1}) = \{1\}$, i.e. $T \in \mathcal{E}$. It remains to note that T is free (and ergodic) since so is $\widehat{T}$.

3. (C, F)-actions, rank and homogeneous spectrum

We start this section by reminding the (C, F)-construction (see [Da1]–[Da3], [DS1] and [DS2] for details). Let $(C_m)_{m=1}^{\infty}$ and $(F_m)_{m=0}^{\infty}$ be two sequences of finite G-subsets such that for each $m \geq 0$ the following properties are satisfied:

$$\text{(3-1)} \qquad \begin{gathered} F_m C_{m+1} \subset F_{m+1}, \ \#C_{m+1} > 1 \ \text{ and} \\ \text{the sets } F_m c, \ c \in C_{m+1}, \text{ are pairwise disjoint.} \end{gathered}$$

We put $X_m := F_m \times C_{m+1} \times C_{m+2} \times \cdots$, endow X_m with the (compact) product topology and define a continuous embedding $X_m \to X_{m+1}$ by setting

$$(f_m, c_{m+1}, c_{m+2}, \dots) \mapsto (f_m c_{m+1}, c_{m+2}, \dots).$$

Then we have $X_1 \subset X_2 \subset \cdots$. Let $X := \bigcup_m X_m$ stand for the topological inductive limit of the sequence X_m. Clearly, X is a locally compact totally disconnected metrizable space without isolated points and X_m is clopen in X. Hence the corresponding Borel σ-algebra $\mathfrak{B}$ is standard. Assume in addition that

$$\text{(3-2)} \qquad \prod_{m=1}^{\infty} \frac{\#F_{m+1}}{\#F_m \#C_{m+1}} < \infty.$$

Then it is easy to see that there exists a unique probability measure μ on $(X, \mathfrak{B})$ such that the restriction of μ onto each X_m is the infinite product measure

$$\tau_m \times \lambda_{m+1} \times \lambda_{m+2} \times \cdots,$$

where λ_j is the equidistribution on C_j and τ_m is a finite measure on F_m with $\tau_m(f) = \tau_m(f')$ for all $f, f' \in F_m$. Thus $(X, \mathfrak{B}, \mu)$ is a standard probability space. Given $g \in G$ and $m > 0$, we set

$$D_g^{(m)} := (F_m \cap g^{-1} F_m) \times C_{m+1} \times C_{m+2} \cdots \quad \text{and} \quad R_g^{(m)} := D_{g^{-1}}^{(m)}.$$

Clearly, $D_g^{(m)}$ and $R_g^{(m)}$ are clopen subsets of X_n. Moreover, $D_g^{(m)} \subset D_g^{(m+1)}$ and $R_g^{(m)} \subset R_g^{(m+1)}$. Define a map $T_g^{(m)} \colon D_g^{(m)} \to R_g^{(m)}$ by setting

$$T_g^{(m)}(f_m, c_{m+1}, \dots) := (gf_m, c_{m+1}, \dots).$$

Clearly, it is a homeomorphism. Put

$$D_g := \bigcup_{m=1}^{\infty} D_g^{(m)} \quad \text{and} \quad R_g := \bigcup_{m=1}^{\infty} R_g^{(m)} = D_{g^{-1}}.$$

Then D_g and R_g are open subsets of X. Moreover, a homeomorphism $T_g \colon D_g \to R_g$ is well defined by $T_g \restriction D_g^{(m)} = T_g^{(m)}$ for all m. Suppose now that

(3-3) $$(F_m)_{m\geq 0} \text{ is a left Følner sequence in } G.$$

This implies $\mu(D_g^{(m)}) \to 1$ as $m \to \infty$. Hence $\mu(D_g) = \mu(R_g) = 1$. Since $\mu(O) > 0$ for each open subset $O \subset X$, it follows that the subset $D := \bigcap_{g\in G} D_g = \bigcap_{g\in G} R_g$ is a dense G_δ of full μ-measure. It is easy to see that $T_{g_2 g_1} = T_{g_2} T_{g_1}$ on D for all $g_1, g_2 \in G$. Thus $T := (T_g)_{g\in G}$ is a continuous G-action on the Polish (in the induced topology) space D. This action is minimal. (To see this, just notice that the T-orbit equivalence relation restricted to any X_m is just the tail equivalence relation on X_m.) Moreover, T preserves μ and T is free and ergodic.

Definition 3.1. We call T *the (C, F)-action of G associated to* $(C_{m+1}, F_m)_{m=0}^{\infty}$.

In the sequel we will not distinguish between sets, maps, transformations which agree a.e. For each subset $A \subset F_m$, we let

$$[A]_m := \{x = (f_m, c_{m+1}, \dots) \in X_m \mid f_m \in A\}$$

and call it an *m-cylinder*. The following holds

$$\begin{aligned}
&[A \cap B]_m = [A]_m \cap [B]_m \text{ and } [A \cup B]_m = [A]_m \cup [B]_m,\\
&[A]_m = [AC_{m+1}]_{m+1} = \bigsqcup_{c\in C_{m+1}} [Ac]_{m+1},\\
&T_g[A]_m = [gA]_m \text{ if } gA \subset F_m,\\
&\mu([A]_m) = \#C_{m+1} \cdot \mu([Ac]_{m+1}) \text{ for every } c \in C_{m+1},\\
&\mu([A]_m) = \mu(X_m)\frac{\#A}{\#F_m},
\end{aligned}$$

where the sign $\sqcup$ means the union of mutually disjoint sets. Moreover, given $Y \in \mathfrak{B}$,

(3-4) $$\min_{A\subset F_m} \mu(Y \triangle [A]_m) \to 0 \quad \text{as } m \to \infty.$$

Now we remind the definition of rank. Let S be an ergodic transformation of a standard probability space $(Y, \mathfrak{F}, \nu)$.

Definition 3.2. The *rank* of S—we will denote it by $\mathrm{rk}(S)$—is the smallest $r \in \mathbb{N}$ (or infinity) such that there exist measurable subsets $B_j^{(m)}$ and positive integers $h_j^{(m)}$ such that the subsets $S^i B_j^{(m)}$, $j = 1, \dots, r$, $i = 1, \dots, h_j^{(m)}$, are pairwise disjoint and approximate the entire σ-algebra $\mathfrak{F}$ as $m \to \infty$. The latter means that given $B \in \mathfrak{F}$, there are subsets $A^{(m)}$ such that $\nu(B \triangle A^{(m)}) \to 0$ and every $A^{(m)}$ is a union of several subsets $S^i B_j^{(m)}$ with $0 \le j \le r$ and $1 \le i \le h_j^{(m)}$. If, moreover, $h_1^{(m)} = \dots = h_r^{(m)}$ for each m, we say that S has *uniform rank* r.

We now reproduce a simple but useful statement from [A4].

Lemma 3.3.

(i) *If* $\mathrm{rk}(S) = 1$ *and* $\sup \mathcal{M}(S^n) = n$ *then* S^n *has uniform rank* n.

(ii) *If* $\mathrm{rk}(S) = 1$ *and* $\mathcal{M}(S^n) = \{n\}$ *then* S^n *is weakly mixing.*

Proof. (i) Of course, $\mathrm{rk}(S^n) \le n \cdot \mathrm{rk}(S)$. By [Ch], $\sup \mathcal{M}(S) \le \mathrm{rk}(S)$. This implies that if $\mathrm{rk}(S) = 1$ and $\sup \mathcal{M}(S^n) = n$ then $\mathrm{rk}(S^n) = n$. Moreover, it is straightforward that in this case the rank of S^n is uniform.

(ii) Since S has rank one, S is ergodic. Hence the dimension of the subspace $\mathcal{H} := \{f \in L_0^2(X, \mu) \mid f \circ S^n = f\}$ is at most $n - 1$. However $\mathcal{M}(S^n) = \{n\}$ and therefore $\mathcal{H}$ is trivial, i.e. S^n is ergodic. Then the multiplicity of each eigenvalue of S^n is 1. Hence if S^n were not weakly mixing then $\mathcal{M}(S^n) \ni 1$, a contradiction. □

Our purpose now is to construct a (C, F)-action $T \in \mathcal{A}_G$ satisfying the conditions of Remark 1.4. To this end we will determine the sequence $(C_{m+1}, F_m)_{m=0}^{\infty}$ via an inductive process. We need some notation. Given $a > 0$ and $l_1, \dots, l_{n+1} > 0$, define a cube and a parallelepiped in H by setting

$$I(a) := \left\{ \sum_{i=1}^{n+1} t_i e_i \mid t_i \in \mathbb{Z}, 0 \le t_i < a \text{ for all } i \right\} \quad \text{and}$$

$$J(l_1, \dots, l_{n+1}) := \left\{ \sum_{i=1}^{n} t_i e_i + t_{n+1}(e_1 + \dots + e_{n+1}) \mid t_i \in \mathbb{Z}, 0 \le t_i < l_i \text{ for all } i \right\}.$$

Fix an increasing sequence of integers $r_m \to \infty$. Notice that $G = \bigsqcup_{0 \le j < n} e_0^j H$. Suppose that for some $k > 0$, there exists $a_k > 0$ such that

$$F_{5k} = \bigsqcup_{0 \le j < n} e_0^j I(a_k). \tag{3-5}$$

We are going to construct $C_{5k+1}, F_{5k+1}, C_{5k+2}, \dots, F_{5k+5}$ in five consecutive steps.

Step 1. We define first three maps $\phi_{5k} : H \to H$, $\sigma_{5k} : H \to \{0, \dots, n-1\}$ and $c_{5k+1} : H \to G$ by setting

$$\begin{aligned} \phi_{5k}(t) &:= (a_k + 1)t, \\ \sigma_{5k}(t) &\equiv t_1 + 2t_2 + \dots + n t_n \pmod{n} \quad \text{and} \\ c_{5k+1}(t) &:= e_0^{\sigma_{5k}(t)} \phi_{5k}(t), \end{aligned}$$

where $t = t_1 e_1 + \dots + t_{n+1} e_{n+1} \in H$ with $t_1, \dots, t_{n+1} \in \mathbb{Z}$. Then ϕ_{5k} is a group homomorphism with the following tiling property:

$$H := \bigsqcup_{t \in H} (\phi_{5k}(t) + I(a_k + 1)). \tag{3-6}$$

The ‘perturbation’ σ_{5k} satisfies

$$\sigma_{5k}(t) - \sigma_{5k}(t + A^j(e_1 - e_2)) \in \{1, 1-n\} \tag{3-7}$$

for all $t \in H$ and $j \in \mathbb{Z}$. Since $e_0 I(a_k) = I(a_k)e_0$ and $e_0^n I(a_k) \subset I(a_k+1)$, we obtain

$$F_{5k}c_{5k+1}(t) = \bigsqcup_{0\le j<n} e_0^{j+\sigma_{5k}(t)}(\phi_{5k}(t) + I(a_k)) \subset \bigsqcup_{0\le j<n} e_0^j(\phi_{5k}(t) + I(a_k+1)).$$

This and (3-6) yield $F_{5k}c_{5k+1}(t) \cap F_{5k}c_{5k+1}(t') = \emptyset$ whenever $t \neq t'$. We now set

$$C_{5k+1} := c_{5k+1}(I(r_k)) \quad \text{and} \quad F_{5k+1} := \bigsqcup_{0\le j<n} e_0^j I(r_k(a_k+1)).$$

It is straightforward that

$$\frac{\#F_{5k+1}}{\#F_{5k}\#C_{5k+1}} = \left(1 + \frac{1}{a_k}\right)^{n+1}. \tag{3-8}$$

Consider a map $\alpha : H \ni t \mapsto \alpha(t) := t + A^{-\sigma_{5k}(t)}(e_1 - e_2) \in H$. It is easy to derive from (3-7) that

$$\alpha \text{ is one-to-one and hence} \quad \frac{\#(\alpha(I(r_k)) \triangle I(r_k))}{\#I(r_k)} \le \left(\frac{4}{r_k}\right)^n. \tag{3-9}$$

Next, we have

$$\begin{aligned}\phi_{5k}(e_1 - e_2)I(a_k)C_{5k+1} &= \bigsqcup_{t\in I(r_k)} I(a_k)e_0^{\sigma_{5k}(t)}\phi_{5k}(A^{-\sigma_{5k}(t)}(e_1-e_2))\phi_{5k}(t)\\ &= \bigsqcup_{t\in I(r_k)} I(a_k)e_0^{\sigma_{5k}(t)-\sigma_{5k}(\alpha(t))}c_{5k+1}(\alpha(t)).\end{aligned}$$

It follows from this, (3-7) and (3-9) that

$$\frac{\#(\phi_{5k}(e_1-e_2)I(a_k)C_{5k+1} \triangle e_0 I(a_k)C_{5k+1})}{\#I(a_k)\#C_{5k+1}} \le \frac{2}{a_k} + \left(\frac{4}{r_k}\right)^n.$$

This inequality holds also—with the same proof—if we replace $e_1 - e_2$ in the left-hand side with $A^j(e_1 - e_2)$ for each $j = 1, \ldots, n-1$. This, in turn, yields

$$\frac{\#(\phi_{5k}(e_1-e_2)^j I(a_k)C_{5k+1} \triangle e_0^j I(a_k)C_{5k+1})}{\#I(a_k)\#C_{5k+1}} \le n\left(\frac{2}{a_k} + \left(\frac{4}{r_k}\right)^n\right). \tag{3-10}$$

Step 2. We now let $b_k := (a_k+1)r_k + r_k^2$. Define three maps $\phi_{5k+1}, s_{5k+1}, c_{5k} : H \to H$ by setting

$$\begin{aligned}&\phi_{5k+1}(t) := b_k t,\\ &s_{5k+1}(t) := \frac{t_1^2 + t_1}{2}e_1 + t_1\sum_{i=2}^{n+1} t_i e_i \quad \text{and}\\ &c_{5k+2}(t) := s_{5k+1}(t) + \phi_{5k+1}(t).\end{aligned}$$

Then ϕ_{5k+1} is a group homomorphism with the following tiling property:

$$H := \bigsqcup_{t\in H} (\phi_{5k+1}(t) + I(b_k)). \tag{3-11}$$

The 'perturbation' s_{5k+1} satisfies

$$s_{5k+1}(t+e_j-e_1) - s_{5k+1}(t) = (t_1-1)e_j - \sum_{i=1}^{n+1} t_i e_i, \; j \neq 1 \tag{3-12}$$

for all $t \in H$. Since

$$\begin{aligned} F_{5k+1}c_{5k+2}(t) &= \bigsqcup_{0\le j<n} e_0^j(s_{5k+1}(t) + \phi_{5k+1}(t) + I((a_k+1)r_k)) \\ &\subset \bigsqcup_{0\le j<n} e_0^j(\phi_{5k+1}(t) + I(b_k)), \end{aligned}$$

it follows from (3-11) that $F_{5k+1}c_{5k+2}(t) \cap F_{5k+1}c_{5k+2}(t') = \emptyset$ if $t \neq t'$. We now set

$$C_{5k+2} := c_{5k+2}(I(r_k)) \text{ and } F_{5k+2} := \bigsqcup_{0\le j<n} e_0^j I(r_k b_k).$$

It is straightforward that

$$\frac{\#F_{5k+2}}{\#F_{5k+1}\#C_{5k+2}} = \left(1 + \frac{r_k}{a_k+1}\right)^{n+1}. \tag{3-13}$$

Step 3. We now define a map $\phi_{5k+2} : H \to H$ by setting

$$\phi_{5k+2}(t) := b_k r_k \sum_{i=1}^{n+1} t_i e_i.$$

Then ϕ_{5k+2} is a group homomorphism with the following tiling property:

$$H := \bigsqcup_{t\in H} (\phi_{5k+2}(t) + I(r_k b_k)).$$

This implies that $F_{5k+2}\phi_{5k+2}(t) \cap F_{5k+2}\phi_{5k+2}(t') = \emptyset$ whenever $t \neq t'$. Select pairwise coprime integers $l_1^{(k)}, \dots, l_{n+1}^{(k)}$ and a finite subset $L_k \subset H$ in such a way

$$\begin{aligned} &\phi_{5k+2}(L_k) + I(r_k b_k) \subset J(l_1^{(k)}, \dots, l_{n+1}^{(k)}) \text{ and} \\ &\frac{\#J(l_1^{(k)}, \dots, l_{n+1}^{(k)}) \setminus (\phi_{5k+2}(L_k) + I(r_k b_k))}{\#J(l_1^{(k)}, \dots, l_{n+1}^{(k)})} < \frac{1}{a_k}. \end{aligned} \tag{3-14}$$

We now set

$$C_{5k+3} := \phi_{5k+2}(L_k) \text{ and } F_{5k+3} := \bigsqcup_{0\le j<n} e_0^j J(l_1^{(k)}, \dots, l_{n+1}^{(k)}).$$

It follows from (3-14) that

$$\frac{\#F_{5k+3}}{\#F_{5k+2}\#C_{5k+3}} < 1 + \frac{1}{a_k}. \tag{3-15}$$

Step 4. We now define a group homomorphism $\phi_{5k+3} : H \to H$ by setting

$$\phi_{5k+3}(t) := \sum_{i=1}^{n} t_i l_i^{(k)} e_i + t_{n+1} l_{n+1}^{(k)} (e_1 + \cdots + e_{n+1}).$$

Then we have $H = \bigsqcup_{t \in H} (\phi_{5k+3}(t) + J(l_1^{(k)}, \dots, l_{n+1}^{(k)}))$. This implies

$$F_{5k+3}\phi_{5k+3}(t) \cap F_{5k+3}\phi_{5k+3}(t) = \emptyset \text{ if } t \neq t'.$$

It follows from the proof of Proposition 1.1 that if we write e_0^i as $e_0^i = f_i \phi_{5k+3}(t^{(i)})$ with $f_i \in F_{5k+3}$ and $t^{(i)} \in H$ then $\{f_i \mid i = 0, \dots, \#F_{5k+3} - 1\} = F_{5k+3}$ (we make use of the fact that $l_1^{(k)}, \dots, l_{n+1}^{(k)}$ are pairwise coprime). Put

$$Z_k := \{t^{(i)} \mid i = 0, \dots, \#F_{5k+3} - 1\}$$

and take m_k large so that

(3-16) $$\frac{\#((Z_k + I(m_k)) \cap I(m_k))}{\#I(m_k)} > 1 - \frac{1}{a_k}.$$

Now we set

$$C_{5k+4} := \phi_{5k+3}(I(m_k)) \text{ and } F_{5k+4} := \bigsqcup_{0 \le j < n} e_0^j J(m_k l_1^{(k)}, \dots, m_k l_{n+1}^{(k)}).$$

Notice that

(3-17) $$F_{5k+4} = F_{5k+3} C_{5k+4}.$$

Step 5. Now we define a group homomorphism $\phi_{5k+4} : H \to H$ by setting

$$\phi_{5k+4}(t) := m_k \phi_{5k+3}(t).$$

Then we have $H = \bigsqcup_{t \in H} (\phi_{5k+4}(t) + J(m_k l_1^{(k)}, \dots, m_k l_{n+1}^{(k)}))$. This implies

$$F_{5k+4}\phi_{5k+4}(t) \cap F_{5k+4}\phi_{5k+4}(t) = \emptyset \text{ if } t \neq t'.$$

Select now a positive integer a_{k+1} and a finite subset $M_k \subset H$ in such a way that

(3-18) $$\begin{gathered} \phi_{5k+4}(M_k) + J(m_k l_1^{(k)}, \dots, m_k l_{n+1}^{(k)}) \subset I(a_{k+1})) \quad \text{and} \\ \frac{\#I(a_{k+1})) \setminus (\phi_{5k+4}(M_k) + J(m_k l_1^{(k)}, \dots, m_k l_{n+1}^{(k)}))}{\#I(a_{k+1})} < \frac{1}{a_k}. \end{gathered}$$

Now we set

$$C_{5k+5} := \phi_{5k+4}(M_k) \quad \text{and} \quad F_{5k+5} := \bigsqcup_{0 \le j < n} e_0^j I(a_{k+1}).$$

It follows from (3-18) that

(3-19) $$\frac{\#F_{5k+5}}{\#F_{5k+4}\#C_{5k+5}} < 1 + \frac{1}{a_k}.$$

Now starting with arbitrary $(C_{m+1}, F_m)_{m=0}^4$ and iterating the above 5-steps-procedure infinitely many times we obtain two sequences $(C_m)_{m=1}^\infty$ and $(F_m)_{m=0}^\infty$. Notice that (3-1) is satisfied automatically by the construction. Moreover, it is easy to see the sequence $(a_m)_{m=1}^\infty$ appearing as a byproduct of the construction grows fast so that $\sum_{m=1}^\infty r_m/a_m < \infty$. Therefore (3-8), (3-13), (3-15), (3-17), (3-19) imply (3-2). Of course, (3-3) holds. Hence the (C, F)-action T of G associated with $(C_{m+1}, F_m)_{m=1}^\infty$ is well defined on a standard probability space $(X, \mathfrak{B}, \mu)$. Now we state one of the main results of this paper.

Theorem 3.4. *The transformation $T_{e_{n+1}}$ is weakly mixing and rigid. It has uniform rank n and $\mathcal{M}(T_{e_{n+1}}) = \{n\}$.*

We preface the proof of this theorem with several auxiliary lemmata.

Lemma 3.5. *The subaction $(T_h)_{h\in H}$ is ergodic.*

Proof. We will use the notation introduced on Step 1. Suppose that $(T_h)_{h\in H}$ is not ergodic, i.e. there exists a subset $B \in \mathfrak{B}$ such that $0 < \mu(B) < 1$ and $T_h B = B$ for all $h \in H$. Since $(T_g)_{g\in G}$ is ergodic, we can find $l > 0$ such that $l \mid n$ and

$$T^j_{e_0} B \cap B = \emptyset \text{ if } 0 < j < l,\ T^l_{e_0} B = B \text{ and } \mu\Bigg(\bigsqcup_{j=0}^{l-1} T^j_0 B\Bigg) = 1. \tag{3-20}$$

In what follows we will assume that $l = n$. (The general case is considered in a similar way.) Let $\epsilon := 10^{-6}n^{-4}$. In view of (3-4) and (3-5) there exist $k > 0$ and subsets $B_0, \dots, B_{n-1} \subset I(a_k - 1)$ such that $\epsilon > n(2/a_k + (4/r_k)^n)$ and

$$\epsilon > \mu\Bigg(B\triangle\Bigg[\bigsqcup_{0\le j<n} e^j_0 B_j\Bigg]_{5k}\Bigg) = \mu\Bigg(B\triangle \bigsqcup_{0\le j<n} T^j_{e_0}[B_j]_{5k}\Bigg) \tag{3-21}$$

Since $B = T^n_{e_0} B$ and $e^n_0 B_j = e_{n+1} + B_j \subset I(a_k)$, it follows from (3-21) that

$$\sum_{0\le j<n} \mu([B_j]_{5k} \triangle T^n_{e_0}[B_j]_{5k}) < 2\epsilon.$$

This inequality plus (3-20) and (3-21) imply that

$$\mu([B_j]_{5k} \cap [B_{j'}]_{5k}) < 6\epsilon \text{ for all } 0 \le j \ne j' < n \text{ and}$$
$$\mu\Bigg(\bigcup_{j=0}^{n-1} [B_j]_{5k}\Bigg) > \frac{1}{n} - 2\epsilon. \tag{3-22}$$

On the other hand, it follows from (3-10) that

$$\mu(T_{\phi_{5k}(e_1-e_2)^j}[I(a_k)]_{5k} \triangle [e^j_0 I(a_k)]_{5k}) < \epsilon.$$

Since $T_{\phi_{5k}(e_1-e_2)} B = B$, we obtain

$$\mu(T_{\phi_{5k}(e_1-e_2)^j}(B \cap [I(a_k)]_{5k}) \triangle (B \cap [e^j_0 I(a_k)]_{5k})) < \epsilon.$$

Hence $\mu(T_{\phi_{5k}(e_1-e_2)^j}[B_0]_{5k} \triangle [e^j_0 B_j]_{5k}) < 3\epsilon$ and therefore

$$|\mu([B_0]_{5k}) - \mu([B_j]_{5k})| < 3\epsilon \text{ for all } j.$$

Comparing this with (3-22) we deduce that

$$\mu([B_j]_{5k}) = \frac{1}{n^2} \pm 15n\epsilon \quad \text{for } j = 0, \dots, n-1. \tag{3-23}$$

Now, given a permutation τ of $I(a_k)$, we define a transformation $R_\tau \in \mathrm{Aut}_0(X, \mu)$ by setting

$$R_\tau x := \begin{cases} T_{\tau(b)-b} x, & \text{if } x \in [b]_{5k} \text{ for some } b \in I(a_k) \\ x, & \text{otherwise.} \end{cases}$$

Since $\tau(b) - b \in H$ for all $b \in I(a_k)$, it follows that B is invariant under R_τ. Then we deduce from (3-21) that $\mu([B_0]_{5k} \triangle [\tau(B_0)]_{5k}) < 2\epsilon$. Since τ is arbitrary, it follows that either $\mu([B_0]_{5k}) < \epsilon$ or $\mu([B_0]_{5k}) > n^{-1} - \epsilon$. However neither is compatible with (3-23). □

Lemma 3.6. *For each $l = 2, \dots, n+1$, the transformation $T_{e_1 - e_l}$ is ergodic.*

Proof. We will use the notation introduced on Steps 1 and 2. Since the proof of the ergodicity is similar for all of these transformations, we consider only the case $l = 2$. Let B, D be two measurable subsets of X with $\mu(B) = \mu(D) > 0$. Let $\epsilon := \mu(B)^2 n^{-n} 10^{-6}$. By (3-4) we can approximate D with a cylinder D' such that $\mu(D \triangle D') < \epsilon$. For $k > 0$ and $t \in H$, we let

$$\beta_k(t) := s_{5k+1}(t + e_2 - e_1) - s_{5k+1}(t).$$

Then $\phi_{5k+1}(e_1 - e_2) + c_{5k+2}(t) = \beta_k(t + e_1 - e_2) + c_{5k+2}(t + e_1 - e_2)$ for all $t \in H$. Denote by $\mathcal{A}_k$ the following bounded linear operator in $L^2(X, \mu)$:

$$\mathcal{A}_k f := \frac{1}{\#I(r_k)} \sum_{h \in \beta_k(I(r_k))} f \circ T_h.$$

Since $r_k \nearrow +\infty$, it follows from (3-12) that $\beta_k(I(r_k))$ is an increasing Følner sequence in H. Since $(T_h)_{h \in H}$ is ergodic by Lemma 3.5, we derive from the mean ergodic theorem that $M_k f \to \int f \, d\mu$ strongly for each $f \in L^2(X, \mu)$. Select k so large that the following are satisfied:

(i) $\|\mathcal{A}_k \chi_{D'} - \mu(D')\|_2 < \epsilon$,
(ii) $r_k^{-2} < \epsilon$,
(iii) D' is a $(5k+1)$-cylinder, i.e. $D' = [D^*]_{5k+1}$ for a finite subset $D^* \subset F_{5k+1}$,
(iv) there exists a finite subset $B^* \subset F_{5k+1} \cap H$ such that $\mu([B^*]_{5k+1} \setminus B) < \epsilon$ and $\mu([B^*]_{5k+1}) > \frac{\mu(B)}{2n^n}$.

Only (iv) needs to be explained. Let $t - t' = \sum_{i=1}^{n+1} t_i e_i$ for some $t_1, \dots, t_{n+1} \in \mathbb{Z}$. If $t_1, \dots, t_n$ are multiples of n then $\sigma_{5k}(t) = \sigma_{5k}(t')$. Since $r_k \nearrow +\infty$, we obtain

$$\frac{\#\{t \in I(r_k) \mid \sigma_{5k}(t) = j\}}{\#I(r_k)} > \frac{2}{3n^n}$$

for every $j \in \{0, \dots, n-1\}$ whenever k is large enough. This implies, in turn, that

$$\frac{\#(fC_{5k+1} \cap H)}{\#C_{5k+1}} > \frac{2}{3n^n} \quad \text{for every } f \in F_{5k}.$$

Hence if $\widetilde{B}$ is any subsets of F_{5k} then $\mu([\widetilde{B}C_{5k+1} \cap H]_{5k+1}) \geq \frac{1}{2n^n} \mu([\widetilde{B}]_{5k})$. It remains to put $B^* := \widetilde{B}C_{5k+1} \cap H$ for a subset $\widetilde{B} \subset F_{5k}$ such that the cylinder $[\widetilde{B}]_{5k}$ approximates B up to ϵ in μ.

In view of (iv) we may assume without loss of generality that $B^* + I(r_k) - I(r_k) \subset$

F_{5k+1}. Let $I' := I(r_k) \cap (I(r_k) + e_2 - e_1)$. Then we have

$$\begin{aligned}
\mu(T_{\phi_{5k+1}(e_1-e_2)}B \cap D) &> \mu(T_{\phi_{5k+1}(e_1-e_2)}[B^*]_{5k+1} \cap D') - 2\epsilon \\
&= \sum_{t \in I(r_k)} \mu(T_{\phi_{5k+1}(e_1-e_2)}[B^* + c_{5k+2}(t)]_{5k+2} \cap D') - 2\epsilon \\
&> \sum_{t \in I'} \mu([\phi_{5k+1}(e_1 - e_2) + B^* + c_{5k+2}(t)]_{5k+2} \cap D') - 3\epsilon \\
&= \sum_{t \in I'} \mu([\beta_k(t + e_1 - e_2) + B^* + c_{5k+2}(t + e_1 - e_2)]_{5k+2} \cap D') - 3\epsilon \\
&= \sum_{t \in I' + e_1 - e_2} \mu([((\beta_k(t) + B^*) \cap D^*) + c_{5k+2}(t)]_{5k+2}) - 3\epsilon \\
&= \frac{1}{\#I(r_k)} \sum_{t \in I' + e_1 - e_2} \mu([(\beta_k(t) + B^*) \cap D^*)]_{5k+1}) - 3\epsilon \\
&> \frac{1}{\#I(r_k)} \sum_{t \in I(r_k)} \mu(T_{\beta_k(t)}[B^*]_{5k+1} \cap D') - 4\epsilon \\
&= \langle \chi_{[B^*]_{5k+1}}, \mathcal{A}_k \chi_{D'} \rangle - 4\epsilon,
\end{aligned}$$

where $\langle .,. \rangle$ denotes the inner product in $L^2(X, \mu)$. Applying (i) and (iv) we obtain

$$\mu(T_{\phi_{5k+1}(e_1-e_2)}B \cap D) > \mu([B^*]_{5k+1})\mu(D') - 5\epsilon > \frac{\mu(B)^2}{2n^n} - 6\epsilon > 0.$$

It remains to notice that $\phi_{5k+1}(e_1 - e_2)$ is a multiple of $e_1 - e_2$. □

Lemma 3.7. *The transformation T_{e_0} has rank one.*

Proof. We will use the notation introduced on Step 4. Let

$$B_k := (Z_k + I(m_k)) \cap I(m_k) \text{ and } S_k := [\phi_{5k+3}(B_k)]_{5k+4} \subset [0]_{5k+3}.$$

Then we have

$$T_{e_0}^i S_k = [f_i \phi_{5k+3}(t^{(i)} + B_k)]_{5k+4} \subset [f_i]_{5k+3}.$$

Hence $T_{e_0}^i S_k \cap T_{e_0}^{i'} S_k = \emptyset$ if $0 \le i \ne i' < \#F_{5k+3}$. We claim that the sequence of T_{e_0}-towers $\{T_{e_0}^i S_k \mid i = 0, \dots, \#F_{5k+3} - 1\}$ approximates the whole σ-algebra $\mathfrak{B}$ as $k \to \infty$. Indeed (3-16) implies that

$$\mu([f_i]_{5k+3} \setminus T_{e_0}^i S_k) \le \frac{[f_i]_{5k+3}}{a_k}$$

for all $i = 0, \dots, \#F_{5k+3} - 1$. It remains to make use of (3-4). □

Remark 3.8. Also, it is easy to see that $T_{e_0}^{\#F_{5k+3}} S_k \subset [0]_{5k+3}$. This implies that $T_{e_0}^{\#F_{5k+3}} \to \mathrm{Id}$ as $k \to \infty$.

Proof of Theorem 3.4. It follows from Remark 1.5, Lemmata 3.6 and 3.7 and the fact that each rank-one transformation has a simple spectrum ([Ba], [Ch]) that $\mathcal{M}(T_{e_{n+1}}) = \{n\}$. Then Lemma 3.3 yields that $T_{e_{n+1}}$ is weakly mixing and has uniform rank n. Since $\#F_{5k+3}$ is divisible by n and $e_{n+1} = e_0^n$, we derive from Remark 3.8 that $T_{e_{n+1}}$ is rigid. □

4. (C,F)-cocycles and the spectral multiplicity for compact group extensions of (C,F)-transformations

Our purpose in this section is to prove the following assertion.

Theorem 4.1. *Given a subset $M \subset \mathbb{N}$ with $1 \in M$, there exists a weakly mixing transformation S such that $\mathcal{M}(S) = n \cdot M$.*

To prove this theorem we incorporate the main ideas from [G–S] and [KL] into the construction outlined in Section 3.

Let $T = (T_g)_{g\in G}$ be a free measure preserving action of G on a standard probability space $(X, \mathfrak{B}, \mu)$. Denote by $\mathcal{R}$ the T-orbit equivalence relation on X.

Definition 4.2 [FM]**.** Let K be a compact metric Abelian group.

(i) A Borel map $\alpha : \mathcal{R} \to K$ is called a *cocycle* of $\mathcal{R}$ if there exists a μ-conull subset $X^* \subset X$ such that

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z) \text{ whenever } x \sim_{\mathcal{R}} y \sim_{\mathcal{R}} z \text{ and } x, y, z \in X^*.$$

(ii) Two cocycles $\alpha, \beta : \mathcal{R} \to K$ are *cohomologous* if there exist a Borel map $\phi : X \to K$ and a conull subset $X^* \subset X$ such that

$$\alpha(x, y) = \phi(x)\beta(x, y)\phi(y)^{-1} \text{ for all } (x, y) \in \mathcal{R} \cap (X^* \times X^*).$$

Denote by λ_K Haar measure on K. Then for each cocycle $\alpha : \mathcal{R} \to K$, we can define a G-action $T^\alpha = (T^\alpha_g)_{g\in G}$ on the product space $(X \times K, \mu \times \lambda_K)$ by setting

$$T^\alpha_g(x, k) := (T_g x, \alpha(T_g x, x)k).$$

This action is called a *compact group extension of* T. Denote by $\widehat{U} : G \ni g \mapsto \widehat{U}_g$ the associated (with T^α) unitary representation of G on the space $L^2(X \times K, \mu \times \lambda_K)$. Then we have a natural decomposition of this space into a countable sum of $\widehat{U}$-invariant subspaces:

$$L^2(X \times K, \mu \times \lambda_K) = \bigoplus_{\eta\in\widehat{K}} L^2(X, \mu) \otimes \eta.$$

Notice that the restriction of $\widehat{U}$ to $L^2(X, \mu)\otimes\eta$ is unitarily equivalent to the unitary representation $U_\eta : G \ni g \mapsto U_\eta(g)$ on $L^2(X, \mu)$ given by

$$(U_\eta(g)f)(x) := \eta(\alpha(T_g^{-1}x, x))f(T_g^{-1}x), \ x \in X.$$

We will denote by U the restriction of $U_{1_{\widehat{K}}}$ to $L^2_0(X, \mu)$. If K_0 is a closed subgroup of K then it determines a factor G-action $T^{\alpha,K_0} = (T^{\alpha,K_0}_g)_{g\in G}$ of T^α on the quotient space $(X \times K/K_0, \mu \times \lambda_{K/K_0})$ by

$$T^{\alpha,K_0}_g(x, kK_0) = (T_g x, \alpha(T_g x, x)kK_0).$$

Let $\mathcal{K} := \{\eta \in \widehat{K} \mid \eta(h) = 1 \text{ for all } h \in K_0\}$. Then (up to a natural identification)

$$L^2(X \times K/K_0, \mu \times \lambda_{K/K_0}) = \bigoplus_{\eta\in\mathcal{K}} L^2(X, \mu) \otimes \eta$$

and $L^2(X, \mu) \otimes \eta$ is now invariant under the unitary representation of G associated with T^{α,K_0}. Given a continuous group automorphism v of K, we set

$$\mathcal{O}(v, K_0) := \{\#(\{\eta, \eta \circ v, \eta \circ v^2, \dots\} \cap \mathcal{K}) \mid \eta \in \mathcal{K}\}.$$

The following algebraic lemma was proved in [KL].

Lemma 4.3. *Given a finite subset $M \subset \mathbb{N}$ with $1 \in M$, there exist a compact metric Abelian group K, a subgroup $K_0 \subset K$ and a continuous group automorphism $v : G \to G$ such that $\mathcal{O}(v, K_0) = M$.*

Theorem 4.4. *Let $\alpha : \mathcal{R} \to K$ be a cocycle, K_0 a closed subgroup of K and v a continuous group automorphism of K. If the following are satisfied:*

(i) *the unitary operators $U_\eta(e_{n+1})$ and $U_{\eta\circ v}(e_{n+1})$ are unitarily equivalent for each $\eta \in \widehat{K}$;*

(ii) *the measures of maximal spectral type of $U_\eta(e_{n+1})$ and $U_{\eta'}(e_{n+1})$ are disjoint whenever $\eta' \neq \eta \circ v^i$ for any $i \in \mathbb{Z}$;*

(iii) *the unitary operator $U_\eta(e_0)$ has a simple spectrum for each $\eta \in \widehat{K}$;*

(iv) *the unitary operators $U(e_{l+1} - e_1)$ and $U_\eta(e_{l+1} - e_1)$ have no non-trivial fixed vectors for any $1 \le l < n$ with $l \mid n$ and non-trivial $\eta \in \widehat{K}$*

then $\mathcal{M}(T^{\alpha,K_0}_{e_{n+1}}) = n \cdot \mathcal{O}(v, K_0)$.

Proof. It follows from (iii), (iv) and Remark 1.5 that the operator $U(e_{n+1})$ and the operators $U_\eta(e_{n+1})$, $\eta \in \widehat{K} \setminus \{1_{\widehat{K}}\}$, have homogeneous spectra of multiplicity n. It remains to apply (i) and (ii). □

Fix an arbitrary K and an automorphism $v : K \to K$ with $\#\{\eta \circ v^i \mid i \in \mathbb{Z}\} < \infty$ for each $\eta \in \widehat{K}$. To derive Theorem 4.1 from Theorem 4.4 and Lemma 4.3 it remains to construct a dynamical system $(X, \mathfrak{B}, \mu, T)$ and a cocycle $\alpha : \mathcal{R} \to K$ satisfying (i)–(iv) and such that the transformation $T^\alpha_{e_{n+1}}$ is weakly mixing.

Let $C(G)$ stand for the center of G and let $C(T)$ denote the centralizer of T, i.e.

$$C(T) := \{S \in \mathrm{Aut}_0(X, \mu) \mid ST_g = T_gS \text{ for all } g \in G\}.$$

If $S \in C(T)$ then $(S\times S)\mathcal{R} = \mathcal{R}$. Hence the map $\alpha\circ S : \mathcal{R} \ni (x, y) \mapsto \alpha(Sx, Sy) \in K$ is a well defined cocycle of $\mathcal{R}$.

To satisfy Theorem 4.4(i) we will use the following simple statement whose proof we leave to the reader (cf. [G–M, Proposition 1]).

Lemma 4.5. *Let $S \in C(T)$. If the cocycles $v \circ \alpha$ and $\alpha \circ S$ are cohomologous then for each $\eta \in \widehat{K}$, the unitary representations U_η and $U_{\eta\circ v}$ of G are unitarily equivalent.*

To satisfy Theorem 4.4(ii) we—following [G–L]—will exploit the concept of θ-weak mixing.

Definition 4.6. Let θ be a complex number, $|\theta| \le 1$. A unitary operator W on a Hilbert space $\mathcal{H}$ is called *θ-weakly mixing* if there exists a sequence $m_1 < m_2 < \cdots$ such that $\langle W^{m_i} h, h\rangle \to \theta\|h\|^2$ as $i \to \infty$ for every $h \in \mathcal{H}$.

If $\theta \neq 1$ then W has no non-trivial fixed vectors. If $|\theta| \neq 1$ then W has no non-trivial eigenvectors (e.g. [G–L]).

Lemma 4.7 [G–L]**.** *If two unitary operators W_i, $i = 1, 2$, are θ_i-weakly mixing (along the same sequence) and $\theta_1 \neq \theta_2$ then the measures of maximal spectral types of W_1 and W_2 are disjoint.*

Next, we recall that a sequence $m_1 < m_2 < \cdots$ is called *rigid* for a transformation $R \in \mathrm{Aut}_0(X, \mu)$ if $R^{m_i} \to \mathrm{Id}$ in $\mathrm{Aut}_0(X, \mu)$. We will use the following criterion for θ-weak mixing.

Lemma 4.8 ([G–L, Proposition 5] and [KL, Proposition 1])**.** *Let $g \in G$ and let $m_1 < m_2 < \cdots$ be a rigid sequence for T_g. If $\int_X \eta(\alpha(T_g^{-m_i}x, x))\,d\mu(x) \to \theta$ as $i \to \infty$ then the unitary operator $V_\eta(g)$ is θ-weakly mixing along $(m_i)_{i=1}^\infty$.*

From now on $(X, \mathfrak{B}, \mu, T)$ is the dynamical system constructed in Section 3. This means that T is the (C, F)-action of G associated with the sequence $(C_{m+1}, F_m)_{m=0}^\infty$ that was explicitly determined there. Below we will use extensively the notation introduced in Section 3. As we already noted the T-orbit equivalence relation $\mathcal{R}$ coincides with the tail equivalence relation on (a μ-conull subset of) X. We say that two points $x, x' \in X$ are *tail equivalent* if there exists $m > 0$ such that

$$x = (f_m, c_{m+1}, c_{m+2}, \dots) \in X_m \text{ and } x' = (f'_m, c_{m+1}, c_{m+2}, \dots) \in X_m. \tag{4-1}$$

Now we distinguish a special class of cocycles. Suppose that there are maps $\beta_m : F_m \to K$ and $\alpha_m : C_m \to K$ such that

$$\beta_{m+1}(fc) = \beta_m(f)\alpha_{m+1}(c) \text{ for all } f \in F_m, c \in C_{m+1}. \tag{4-2}$$

Take a point $(x, x') \in \mathcal{R}$. Then we can find $m > 0$ such that (4-1) holds. Now we put

$$\alpha(x, x') := \beta_m(f_m)\beta_m(f'_m)^{-1}.$$

It is easy to derive from (3-1) and (4-2) that α is a well defined cocycle of $\mathcal{R}$ with values in K.

Definition 4.9. We call α the (C, F)-*cocycle associated with* $(\alpha_{m+1}, \beta_m)_{m=0}^\infty$.

It is an easy exercise for the reader to verify that the cohomology class of α is determined completely by the sequence $(\alpha_m)_{m\geq 0}$ alone. Hint: consider the restriction of α to $\mathcal{R} \cap (X_0 \times X_0)$. Moreover, given any sequence of maps $\alpha_m : C_m \to K$, we can define the second sequence $\beta_m : F_m \to K$ recurrently by setting

$$\beta_m(d) := \begin{cases} \beta_{m-1}(f)\alpha_m(c), & \text{if } d = fc \text{ with } f \in F_{m-1} \text{ and } c \in C_m \\ 1_K, & \text{if } d \in F_m \setminus (F_{m-1}C_m). \end{cases}$$

Then (4-2) is obviously satisfied. In this case we say that the corresponding (C, F)-cocycle *is associated with* $(\alpha_m)_{m=1}^\infty$.

From now on α is the (C, F)-cocycle associated with a sequence $(\alpha_m)_{m=1}^\infty$. Our purpose is to select this sequence in such a way to satisfy the conditions (i)–(iv) from the statement of Theorem 4.4. We will do this step by step. The following lemma shows how to satisfy (iii).

Lemma 4.10. *If $\alpha_{5k+4}(c) = 1_K$ for all $c \in C_{5k+4}$ and $k \in \mathbb{N}$ then the operator $U_\eta(e_0)$ has a simple spectrum for each $\eta \in \widehat{K}$.*

Proof. We first refer the reader to the proof of Lemma 3.7, where the subset S_k was defined. Recall also that for any k, there exists an enumeration $\{f_i \mid i = 0, \dots, \#F_{5k+3} - 1\}$ of F_{5k+3} such that if $y \in S_k$ and $y = (1_G, c_{5k+4}, c_{5k+5}, \dots) \in X_{5k+3}$ then

$$T_{e_0^{-i}}y = (f_i, c'_{5k+4}, c_{5k+5}, \dots)$$

for some $c'_{5k+4} \in C_{5k+4}$ depending on y and i. This and the condition of the lemma imply that

$$\alpha(T_{e_0^{-i}} y, y) = \beta_{5k+5}(f_i)\alpha_{5k+4}(c'_{5k+4})\alpha_{5k+4}(c_{5k+4})^{-1}\beta_{5k+3}(1_G)^{-1}$$
$$= \beta_{5k+5}(f_i)\beta_{5k+3}(1_G)^{-1}.$$

Hence for each $i = 0, \ldots, \#F_{5k+3} - 1$ and $\eta \in \widehat{K}$, there exists a complex number $d_i \in \mathbb{T}$ such that $U_\eta(e_0^i)\chi_{S_k} = d_i \chi_{T_{e_0^i} S_k}$. Therefore the linear span of the family of vectors $\{U_\eta(e_0^i)\chi_{S_k} \mid 0 \le i < \#F_{5k+3}\}$ in $L^2(X, \mu)$ equals to the linear span of $\{\chi_{T^i_{e_0} S_k} \mid 0 \le i < \#F_{5k+3} - 1\}$. It remains to note that the T_{e_0}-towers $\{T^i_{e_0} S_k \mid 0 \le i < \#F_{5k+3}\}$ approximate $\mathfrak{B}$ as $k \to \infty$ by Lemma 3.7. $\square$

Now we pass to (i) from the statement of Theorem 4.4. Without loss of generality we may assume that the sequence $(F_m)_{m=0}^\infty$ satisfies the following condition

$$\text{(4-3)} \qquad \frac{\#(F_m F_m^{-1} F_{m+1} \cap F_{m+1})}{\#F_{m+1}} \to 1.$$

Now let $\bar{z} = (z_m)_{m=1}^\infty$ be a sequence of elements from $C(G)$ such that

$$\text{(4-4)} \qquad \sum_{m=1}^\infty \frac{\#(C_m \triangle z_m C_m)}{\#C_m} < \infty.$$

For a positive integer m, we let

$$X_m^{\bar{z}} := (F_m \cap z_1^{-1} \cdots z_m^{-1} F_m) \times (C_{m+1} \cap z_{m+1}^{-1} C_{m+1}) \times \cdots \subset X_m.$$

It follows from (4-3) and (4-4) that $\#(z_1^{-1} \cdots z_m^{-1} F_m \cap F_m)/\#F_m \to 1$ and hence

$$\text{(4-5)} \qquad X_1^{\bar{z}} \subset X_2^{\bar{z}} \subset \cdots \quad \text{and} \quad \mu(X_m^{\bar{z}}) \to 1 \text{ as } m \to \infty.$$

For each $x = (f_m, c_{m+1}, c_{m+2}, \ldots) \in X_m^{\bar{z}}$, we let

$$\text{(4-6)} \qquad S_{\bar{z}} x := (z_1 \cdots z_m f_m, z_{m+1} c_{m+1}, z_{m+2} c_{m+2}, \ldots).$$

Then it is easy to verify that (4-6) defines a measure preserving transformation of $(X, \mathfrak{B}, \mu)$. Moreover, $S_{\bar{z}} \in C(T)$.

Lemma 4.11. *Suppose that* (4-3) *holds. Let*

$$\text{(4-7)} \qquad C_m^\circ := \{c \in C_m \cap z_m^{-1} C_m \mid \alpha_m(c z_m) = v(\alpha_m(c))\}$$

If $\sum_{m=1}^\infty (1 - \#C_m^\circ/\#C_m) < \infty$ *then the cocycle* $\alpha \circ S_{\bar{z}}$ *is cohomologous to* $v \circ \alpha$.

Proof. By Borel-Cantelli lemma and (4-5) for μ-a.e. $x \in X$, there exists $m > 0$ such that $x = (f_m, c_{m+1}, \ldots) \in X_m$ with $f_m z_1 \cdots z_m \in F_m$ and $c_i \in C_i^\circ$ for all $i > m$. We now set

$$\phi(x) := \beta_m(f_m z_1 \cdots z_m) v(\beta_m(f_m))^{-1}.$$

The following calculation exploiting (4-2) and (4-7) shows that $\phi(x)$ is well defined:

$$\begin{aligned}
&\beta_{m+1}(f_m c_{m+1} z_1 \cdots z_{m+1}) v(\beta_{m+1}(f_{m+1} c_{m+1}))^{-1} \\
&\qquad = \beta_m(f_m z_1 \cdots z_m) \alpha_{m+1}(c_{m+1} z_{m+1}) v(\alpha_{m+1}(c_{m+1}))^{-1} v(\beta_m(f_m))^{-1} \\
&\qquad = \beta_m(f_m z_1 \cdots z_m) v(\beta_m(f_m))^{-1}.
\end{aligned}$$

Of course, ϕ is a Borel map from X to K. It remains to notice that if $x, x' \in X_m$ and (4-1) is satisfied for the pair (x, x') and moreover $f_m z_1 \cdots z_m, f'_m z_1 \cdots z_m \in F_m$ then

$$\begin{aligned}
\alpha \circ S_{\bar{z}}(x, x') &= \alpha(S_{\bar{z}} x, S_{\bar{z}} x') \\
&= \beta_m(f_m z_1 \cdots z_m) \beta_m(f'_m z_1 \cdots z_m)^{-1} \\
&= \phi(x) v(\beta_m(f_m)) v(\beta_m(f'_m))^{-1} \phi(x')^{-1} \\
&= \phi(x) v \circ \alpha(x, x') \phi(x')^{-1}.
\end{aligned}$$

□

Now (i) follows from Lemmata 4.11 and 4.5.

Our next step is to find a sufficient condition for (ii). We first state without proof a simple sufficient condition for a sequence of integers to be rigid for $T_{e_{n+1}}$ (use the fact that $e_{n+1} \in C(G)$).

Lemma 4.12. *Let $m_1 < m_2 < \cdots$ and*

$$\frac{\#(e_{n+1}^{m_i} C_i \triangle C_i)}{\# C_i} \to 0 \quad as \quad i \to \infty.$$

Then the sequence $(m_i)_{i \geq 1}$ is rigid for $T_{e_{n+1}}$.

For each $\eta \in \widehat{K}$, we denote by $p(\eta)$ the smallest positive integer p such that $\eta \circ v^p = \eta$. We also define a function $\tau_\eta \in L^2(K, \lambda_K)$ by setting

$$\tau_\eta := \frac{1}{p(\eta)} \sum_{j=0}^{p(\eta)-1} \eta \circ v^j.$$

Of course, $\|\tau_\eta\|_2 = 1$. Denote by $\widehat{K}_2$ the set of pairs $(\eta, \eta') \in \widehat{K} \times \widehat{K}$ such that $\eta' \neq \eta \circ v^j$ for any $j \in \mathbb{Z}$. If $(\eta, \eta') \in \widehat{K}_2$ then $\tau_\eta \perp \tau_{\eta'}$ in $L^2(K, \lambda_K)$ and hence there exists $z_{\eta,\eta'} \in K$ with $\tau_\eta(z_{\eta,\eta'}) \neq \tau_{\eta'}(z_{\eta,\eta'})$. In particular, for each non-trivial $\eta \in \widehat{K}$, there exists $z_\eta \in K$ such that $\tau_\eta(z_\eta) \neq 1$. This condition is enough to show the ergodicity of $T^\alpha_{e_{n+1}}$. We however want $T^\alpha_{e_{n+1}}$ to be weakly mixing. For this, we claim that for each non-trivial $\eta \in \widehat{K}$, there are two points $z_\eta, z'_\eta \in K$ such that

$$\frac{1}{2} |\tau_\eta(z_\eta) + \tau_\eta(z'_\eta)| < 1.$$

Indeed this claim is obvious if $p(\eta) = 1$. If $p(\eta) > 1$ then there exists $z_\eta \in K$ with $|\tau_\eta(z_\eta)| < 1$ and we put $z'_\eta := z_\eta$. Partition $\mathbb{N}$ into infinite subsets indexed as follows

$$\mathbb{N} = \mathcal{N} \sqcup \bigsqcup_{1_{\widehat{K}} \neq \eta \in \widehat{K}} \mathcal{N}_\eta \sqcup \bigsqcup_{(\eta,\eta') \in \widehat{K}_2} \mathcal{N}_{\eta,\eta'}.$$

Now we refer the reader to Step 3 from Section 3, where the sets L_k were defined. Recall that we tiled there a 'large' parallelepiped $J(l_1^{(k)},\dots,l_{n+1}^{(k)})$ with mutually disjoint translations of a cube $I(r_kb_k)$ up to $\frac{1}{a_k}$. Then $\phi_{5k-2}(L_k)$ was defined as the corresponding set of 'tiling centers'. Enlarging, if necessary, $l_1^{(k)},\dots,l_{n+1}^{(k)}$ we may assume without loss of generality that L_k is rather regular, i.e. it has a shape of 'almost parallelepiped'. Therefore for each $k\in\mathbb{N}\setminus\mathcal{N}$, there exist a finite subset $D_k\subset H$ and positive integers p_k,q_k such that

$$\#D_k \text{ is even;}$$

$$e_{n+1}^jD_k\cap e_{n+1}^{j'}D_k=\emptyset \text{ whenever } 0\le j\ne j'<p_kq_k;$$

$$\text{the set } L_k':=\bigsqcup_{j=0}^{q_kp_k-1}e_{n+1}^jD_k \text{ is contained in } L_k; \tag{4-8}$$

$$\frac{\#(L_k\setminus L_k')}{\#L_k}\to 0,\ p_k\to\infty \text{ as } k\to\infty \text{ along } \mathbb{N}\setminus\mathcal{N}; \tag{4-9}$$

$$\sum_{k\in\mathbb{N}\setminus\mathcal{N}}\frac{1}{q_k}<\infty; \tag{4-10}$$

$$\text{if } k\in\mathcal{N}_\eta \text{ for a non-trivial } \eta\in\widehat{K} \text{ then } p(\eta)\mid q_k; \tag{4-11}$$

$$\text{if } k\in\mathcal{N}_{\eta,\eta'} \text{ for a } (\eta,\eta')\in\widehat{K}_2 \text{ then } \operatorname{lcm}(p(\eta),p(\eta'))\mid q_k.$$

Recall that $C_{5k+3}:=\phi_{5k+2}(L_k)$.

Lemma 4.13. *For each $k\in\mathbb{N}\setminus\mathcal{N}$, we partition D_k into two subsets D_k' and D_k'' of equal cardinality. If for each $0\le j<q_k$ and $0\le i<p_k$, we have*

$$\alpha_{5k+3}(\phi_{5k+2}(e_{n+1}^{jp_k+i}d))=\begin{cases} v^j(z_\eta^{-i}), & \text{if } k\in\mathcal{N}_\eta \text{ and } d\in D_k'\\ v^j(z_\eta'^{\,-i}), & \text{if } k\in\mathcal{N}_\eta \text{ and } d\in D_k''\\ v^j(z_{\eta,\eta'}^{-i}), & \text{if } k\in\mathcal{N}_{\eta,\eta'}\end{cases}$$

then

(a) *for each non-trivial $\eta\in\widehat{K}$, the operator $U_\eta(e_{n+1})$ is $\frac12(\tau_\eta(z_\eta)+\tau_\eta(z_\eta'))$-weakly mixing.*
(b) *for each pair $(\eta,\eta')\in\widehat{K}_2$, the operators $U_\eta(e_{n_1})$ and $U_{\eta'}(e_{n_1})$ are $\tau_\eta(z_{\eta,\eta'})$- and $\tau_{\eta'}(z_{\eta,\eta'})$-weakly mixing respectively along the same sequence.*

Proof. Since $T_{\phi_{5k+2}(e_{n+1})}=T_{e_{n+1}}^{b_kr_k}$, it follows from (4-8), (4-9) and Lemma 4.12 that the sequence $(b_kr_k)_{k\in\mathbb{N}\setminus\mathcal{N}}$ is rigid for $T_{e_{n+1}}$. Now we set

$$C_{5k+3}':=\{\phi_{5k+2}(e_{n+1}^{jp_k+i}d)\mid 0\le j<q_k, 0<i<p_k \text{ and } d\in D_k\} \text{ and}$$
$$X_{5k+2}':=F_{5k+2}\times C_{5k+3}'\times C_{5k+4}\times C_{5k+5}\times\cdots.$$

Then we have

$$\frac{\mu(X_{5k+2}')}{\mu(X_{5k+2})}\ge\frac{\#L_k'}{\#L_k}-\frac{1}{p_k}.$$

Moreover, if $x \in X'_{5k+2}$ and $x = (f_{5k+2}, \phi_{5k+2}(e_{n+1}^{jp_k+i}d), \dots)$ then $T^{-1}_{\phi_{5k+2}(e_{n+1})}x \in X_{5k+2}$. If $k \in \mathcal{N}_\eta$ then

$$\alpha(T^{-1}_{\phi_{5k+2}(e_{n+1})}x, x) = v^j(z^{-i+1})(v^j(z^{-i}))^{-1} = v^j(z), \tag{4-12}$$

where $z = z_\eta$ if $d \in D'_k$ or $z = z'_\eta$ if $d \in D''_k$. Using (4-9), (4-12) and the fact $\#D'_k = \#D''_k = 0.5\#D_k$ we obtain

$$\begin{aligned}\int_X \eta(\alpha(T^{-1}_{\phi_{5k+2}(e_{n+1})}x, x))\, d\mu(x) &= \int_{X'_{5k+2}} \eta(\alpha(T^{-1}_{\phi_{5k+2}(e_{n+1})}x, x))\, d\mu(x) + \overline{o}(1)\\ &= \frac{1}{q_k}\sum_{j=0}^{q_k-1}\frac{1}{p_k}\sum_{i=1}^{p_k-1}\frac{1}{\#D_k}\Bigg(\sum_{d\in D'_k}\eta(v^j(z_\eta)) + \sum_{d\in D''_k}\eta(v^j(z'_\eta))\Bigg) + \overline{o}(1)\\ &= \frac{1}{2q_k}\Bigg(\sum_{j=0}^{q_k-1}\eta(v^j(z_\eta)) + \sum_{j=0}^{q_k-1}\eta(v^j(z'_\eta))\Bigg) + \overline{o}(1),\end{aligned}$$

where $\overline{o}(1)$ denotes a sequence that tends to 0 as $k \to \infty$. Using (4-11) and then passing to the limit along $\mathcal{N}_\eta$ we obtain

$$\int_X \eta(\alpha(T^{-1}_{\phi_{5k+2}(e_{n+1})}x, x))\, d\mu(x) \to \frac{1}{2}(\tau_\eta(z_\eta) + \tau_\eta(z'_\eta)).$$

Since any subsequence of a rigid sequence is rigid itself, (a) follows now from Lemma 4.8.

The claim (b) is demonstrated in a similar way. $\square$

Now we show how to satisfy (iv) of the statement of Theorem 4.4. For this, we are going to adapt the ideas used in the proof of Lemma 4.13. First, we need a sufficient condition for a sequence of integers to be rigid for $T_{e_{l+1}e_1^{-1}}$. Notice that Lemma 4.12 stated for $T_{e_{n+1}}$ does not work for $T_{e_{l+1}e_1^{-1}}$ since $e_{l+1}e_1^{-1} \notin C(G)$. However it is not difficult to modify the lemma as follows.

Lemma 4.14. *Let $g \in H$, $g \neq 1_H$. Let $m_1 < m_2 < \cdots$ and*

$$\max_{0\le\sigma<n}\frac{\#(e_0^{-\sigma}g^{m_i}e_0^{\sigma}C_i \triangle C_i)}{\#C_i} \to 0 \quad \text{as } i \to \infty.$$

Then the sequence $(m_i)_{i\ge1}$ is rigid for T_g.

Proof. For each $f \in F_{i-1}$, there are $h \in H$ and $0 \le \sigma < n$ such that $f = he_0^{\sigma}$. It remains to notice that $g^{m_i}fc = fe_0^{-\sigma}g^{m_i}e_0^{\sigma}c$ for each $c \in C_i$. $\square$

It is easy to see that the elements $A^\kappa(e_{l+1}e_1^{-1})$, $\kappa = 0, \dots, n-2$, are rationally independent in H and $A^{n-1}(e_{l+1}e_1^{-1}) = \prod_{\kappa=0}^{n-2} A^\kappa(e_{l+1}e_1^{-1})^{-1}$. Recall that the group automorphism $A : H \to H$ was defined in Section 1. Partition $\mathcal{N}$ into infinite subsets indexed as follows

$$\mathcal{N} = \bigsqcup_{l=1}^{n-1}\ \bigsqcup_{1_{\widehat{K}}\neq\eta\in\widehat{K}} \mathcal{N}_\eta^{(l)}.$$

Without loss of generality we may assume that for each $k \in \mathcal{N}_\eta^{(l)}$, there exist a finite subset $D_k \subset H$ and positive integers p_k, q_k such that the following are satisfied:

— the sets $e_{n+1}^j D_k \prod_{\kappa=0}^{n-2} A^\kappa (e_{l+1}e_1^{-1})^{i_\kappa}$, $0 \le j < q_k$, $0 \le i_0, \dots, i_{n-2} < p_k$, are pairwise disjoint;

— their union $L'_k := \bigsqcup_{j=0}^{q_k-1} \bigsqcup_{i_0,\dots,i_{n-2}=0}^{p_k-1} e_{n+1}^j D_k \prod_{\kappa=0}^{n-2} A^\kappa (e_{l+1}^i e_1^{-1})^{i_\kappa}$ is contained in L_k;

— $\#(L_k \setminus L'_k)/\#L_k \to 0$ and $p_k \to \infty$ as $k \to \infty$ along $\mathcal{N}$;

— $p(\eta) \mid q_k$ and

$$\sum_{k \in \mathcal{N}} \frac{1}{q_k} < \infty. \tag{4-13}$$

Lemma 4.15. *If for each $k \in \mathcal{N}_\eta^{(l)}$, we have*

$$\alpha_{5k+3}(\phi_{5k+2}(e_{n+1}^j d \prod_{\kappa=0}^{n-2} A^\kappa (e_{l+1}e_1^{-1})^{i_\kappa})) = v^j (z_\eta^{-i_0 - \dots - i_{n-2}})$$

for all $0 \le j < q_k$, $0 \le i_0, \dots, i_{n-2} < p_k$ and $d \in D_k$ then the operator $U_\eta(e_{l+1}e_1^{-1})$ is $(\frac{n-1}{n}\tau_\eta(z_\eta) + \frac{1}{n}\tau_\eta(z_\eta^{-n+1}))$-weakly mixing.

Proof. We consider only the case $l = 1$ (the other cases are similar).

Since $T_{\phi_{5k+2}(e_2 e_1^{-1})} = T_{e_2 e_1^{-1}}^{b_k r_k}$, it is easy to deduce from the definition of L'_k and Lemma 4.14 that the sequence $(b_k r_k)_{k \in \mathbb{N} \setminus \mathcal{N}}$ is rigid for $T_{e_2 e_1^{-1}}$. Recall that $F_{5k+2} = \bigsqcup_{0 \le \sigma < n} e_0^\sigma I(r_k b_k)$. If $x \in [e_0^\sigma I(r_k b_k)]_{5k+2}$ and

$$x = (f_{5k+2}, \phi_{5k+2}(e_{n+1}^j d \prod_{\kappa=2}^{n} (e_\kappa e_{\kappa-1}^{-1})^{i_k}), c_{5k+4}, \dots)$$

with $0 < i_2, \dots, i_n < p_k - 1$ then $T_{\phi_{5k+2}(e_2 e_1^{-1})}^{-1} x = (f_{5k+2}, c'_{5k+3}, c_{5k+4}, \dots) \in X_{5k+2}$, where

$$c'_{5k+3} = \begin{cases} \phi_{5k+2}(e_{n+1}^j d e_2^{-1} e_1 \prod_{\kappa=2}^n (e_\kappa e_{\kappa-1}^{-1})^{i_k}), & \text{if } \sigma = 0 \\ \phi_{5k+2}(e_{n+1}^j d e_{2-\sigma+n}^{-1} e_{1-\sigma+n} \prod_{\kappa=2}^n (e_\kappa e_{\kappa-1}^{-1})^{i_k}), & \text{if } 2 \le \sigma < n \\ \phi_{5k+2}(e_{n+1}^j d \prod_{\kappa=2}^n (e_\kappa e_{\kappa-1}^{-1})^{i_k+1}), & \text{if } \sigma = 1. \end{cases}$$

Hence

$$\alpha(T_{\phi_{5k+2}(e_2 e_1^{-1})}^{-1} x, x) = \begin{cases} v^j(z_\eta), & \text{if } \sigma \ne 1 \\ v^j(z_\eta^{-n+1}), & \text{if } \sigma = 1. \end{cases}$$

This yields

$$\begin{aligned} \int_X \eta(\alpha(T_{\phi_{5k+2}(e_2 e_1^{-1})}^{-1} x, x))\, d\mu(x) &= \sum_{\sigma \ne 1} \int_{[e_0^\sigma I(r_k b_k)]_{5k+2}} + \int_{[e_0 I(r_k b_k)]_{5k+2}} + \overline{o}(1) \\ &= \frac{n-1}{n q_k} \sum_{j=0}^{q_k-1} \eta(v^j(z_\eta)) + \frac{1}{n q_k} \sum_{j=0}^{q_k-1} \eta(v^j(z_\eta^{-n+1})) + \overline{o}(1) \\ &= \frac{n-1}{n} \tau_\eta(z_\eta) + \frac{1}{n} \tau_\eta(z_\eta^{-n+1}) + \overline{o}(1). \end{aligned}$$

It remains to pass to the limit when $k \to \infty$ along $\mathcal{N}_\eta^{(2)}$ and apply Lemma 4.8. $\square$

Proof of Theorem 4.1. By Lemma 4.3, there exists a compact Polish Abelian group $K_0 \subset K$ and a continuous group automorphism $v : K \to K$ such that $\mathcal{O}(v, K_0) = M$. Let $(X, \mathfrak{B}, \mu, T)$ be the dynamical system constructed in Section 3. To define a sequence of maps $\alpha_m : C_m \to K$ we consider two cases. If $m \not\equiv 3 \pmod 5$ then we set $\alpha_m \equiv 1_K$. In the case $m = 5k + 3$ for some $k \in \mathbb{N}$, we first define α_m on $\phi_{5k+2}(L'_k) \subset C_m$ via the formulae from the statements of Lemmata 4.13 and 4.15 and then extend α_m in an arbitrary way to the rest of C_m. Let α denote the (C, F)-cocycle associated with $(\alpha_m)_{m=1}^\infty$. Now let

$$z_m := \begin{cases} 1_G, & \text{if } m \not\equiv 3 \pmod 5 \\ \phi_{5k+2}(e_{n+1}^{p_k}), & \text{if } m = 5k+3 \text{ for some } k \in \mathbb{N} \setminus \mathcal{N} \\ \phi_{5k+2}(e_{n+1}), & \text{if } m = 5k+3 \text{ for some } k \in \mathcal{N} \end{cases}$$

and let $\bar{z} := (z_m)_{m=1}^\infty$. Since $\sum_{k=1}^\infty \frac{1}{q_k} < \infty$ by (4-10) and (4-13), it follows from the definition of α and Lemma 4.11 that the cocycle $\alpha \circ S_{\bar{z}}$ is cohomologous to $v \circ \alpha$. Hence Lemma 4.5 implies Theorem 4.4(i). Since $\tau_\eta(z_{\eta,\eta'}) \neq \tau_{\eta'}(z_{\eta,\eta'})$ whenever $(\eta, \eta') \in \widehat{K}_2$, Lemmata 4.13(b) and 4.7 imply Theorem 4.4(ii). We deduce Theorem 4.4(iii) from Lemma 4.10. Notice that the fact $\tau_\eta(z_\eta) \neq 1$ entails $\frac{n-1}{n}\tau_\eta(z_\eta) + \frac{1}{n}\tau_\eta(z_\eta^{-n+1}) \neq 1$. Therefore Theorem 4.4(iv) follows from Lemmata 4.15 and 3.6. Thus all the conditions of Theorem 4.4 hold and we deduce from it that $\mathcal{M}(T_{e_{n+1}}^{\alpha,K_0}) = n \cdot \mathcal{O}(v, K_0) = n \cdot M$. Since $|\frac{1}{2}(\tau_\eta(z_\eta) + \tau_\eta(z'_\eta))| \neq 1$ for any non-trivial $\eta \in \widehat{K}$, it follows from Lemma 4.13(a) that the transformation $T_{e_{n+1}}^\alpha$ is weakly mixing. Hence $T_{e_{n+1}}^{\alpha,K_0}$ is weakly mixing too. $\square$

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Institute for Low Temperature Physics & Engineering of Ukrainian National Academy of Sciences, 47 Lenin Ave., Kharkov, 61164, UKRAINE

E-mail address: danilenko@ilt.kharkov.ua