ON SUBRELATIONS OF ERGODIC MEASURED TYPE III EQUIVALENCE RELATIONS

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(Dedicated to the memory of Anzelm Iwanik)

Abstract. Given a nested pair of measured ergodic discrete hyperfinite equivalence relations $\mathcal{S} \subset \mathcal{R}$ of finite index, a classification of the inclusion up to orbit equivalence is discussed. In case of type III relations, the orbit equivalence classes of inclusions are completely classified in terms of a collection of a transitive permutation group G on a finite set (whose cardinality = the index of $\mathcal{S} \subset \mathcal{R}$), an compactification ergodic nonsingular $\mathbb{R}$-flow V and a homomorphism of G to the centralizer of V.

0. Introduction

We consider nonsingular discrete ergodic hyperfinite equivalence relations on a standard measure space. Our concern is to classify pairs of ergodic equivalence relation-subrelation $\mathcal{S} \subset \mathcal{R}$ of finite index (which means that the $\mathcal{R}$-equivaslence class of a.e. point consists of finitely many S-classes) up to orbit equivalence. This problem is closely related to the classification of subfactors in von Neumann algebras theory. For a single equivalence relation $\mathcal{R}$ the problem was solved by H. Dye [Dy] and W. Krieger [Kr] in terms of the associated flows. After that, in case where $\mathcal{R}$ is of type II_1 J. Feldman, C. Sutherland, and R. Zimmer [FSZ] provided a simple classification of ergodic $\mathcal{R}$-subrelations of finite index and normal $\mathcal{R}$-subrelations of an arbitrary index. (Remark that in an earlier paper [Ge] M. Gerber classified $\mathcal{R}$-subrelations of finite index in a different—but equivalent—context of finite extensions of ergodic probability preserving transformations.) These results were further extended in [Da1, §4] and [Da2], where quasinormal subrelations of type II_1 were introduced and studied.

Recently, T. Hamachi considered finite index subrelations of a type III_0 equivalence relation $\mathcal{R}$, introduced a system of invariants for orbit equivalence and claimed that it is complete [Ha]. However, in the present paper we construct orbitally non-equivalent subrelations of $\mathcal{R}$ which are non-distinguishable by these invariants. Moreover, for an arbitrary type III equivalence relation $\mathcal{R}$, we provide another system of invariants for orbit equivalence of $\mathcal{R}$-subrelations of finite index and show that it is complete. It consists of a transitive subgroup G of permutations on a finite set (whose cardinality equals to the index), an ergodic nonsingular $\mathbb{R}$-flow V and a homomorphism l of G to the centralizer of V such that the $l(G)$-quotient of V is conjugate to the associated flow of $\mathcal{R}$. Roughly speaking, Hamachi's invariants "remember" only the range and the kernel of l but not l itself and that is why

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they are not complete. It should be noted that the argument of [Ha] uses common discrete decomposition for $\mathcal{S}$ and $\mathcal{R}$, a lacunary measure, etc., i.e. modified techniques from [Kr] (see also [HO]). Our approach is different. We apply more recent advances in orbit theory ([FSZ], [GS1], [GS2]) which result to a short argument.

The outline of the paper is as follows. Section 1 contains background on orbit theory. Section 2 begins with the "measurable index theory" and contains our main classification result—Theorem 6. In Section 3 we provide a counterexample to [Ha, Theorem 6.1]. In the final Section 4, the case of type III_λ equivalence relations, $0 < \lambda \leq 1$, is considered in more detail. It turns out that our classification invariants have simpler (more explicit) form in this case.

1. Background on orbit theory

Let $(X, \mathfrak{B}, \mu)$ be a standard probability space. Denote by $\mathrm{Aut}\,(X, \mu)$ the group of its automorphisms, i.e. Borel one-to-one, onto, μ-nonsingular transformations. We do not distinguish between maps which agree on a μ-conull set. Given a Borel discrete μ-nonsingular equivalence relation $\mathcal{R} \subset X \times X$, we endow it with the induced Borel structure and σ-finite measure $\mu_{\mathcal{R}}$, $d\mu_{\mathcal{R}}(x, y) = d\mu(x)$, $(x, y) \in \mathcal{R}$. Write also

$$[\mathcal{R}] = \{\gamma \in \mathrm{Aut}(X, \mu) \mid (\gamma x, x) \in \mathcal{R} \text{ for } \mu\text{-a.e.} x \in X\},$$
$$N[\mathcal{R}] = \{\theta \in \mathrm{Aut}(X, \mu) \mid (\theta x, \theta y) \in \mathcal{R} \text{ iff } (x, y) \in \mathcal{R}\ \mu_{\mathcal{R}}\text{-a.e. }\}$$

for the *full group* of $\mathcal{R}$ and the *normalizer* of $[\mathcal{R}]$ respectively. For a countable subgroup Γ of $\mathrm{Aut}(X, \mu)$, we denote by $\mathcal{R}_\Gamma$ the Γ-orbital equivalence relation. It is known that each γ is of the form $\mathcal{R}_\Gamma$ [FM]. $\mathcal{R}$ is called *hyperfinite* if it can be generated by a single automorphism. We assume from now on that $\mathcal{R}$ is ergodic, i.e. every $\mathcal{R}$-saturated Borel subset is either μ-null or μ-conull.

Let G be a locally compact second countable (l.c.s.c.) group, 1_G the identity of G and λ_G the right Haar measure on G. A Borel map $\alpha : \mathcal{R} \to G$ is a (1-)cocycle of $\mathcal{R}$ if

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z) \qquad \text{for a.e. } (x, y), (y, z) \in \mathcal{R}.$$

Two cocycles, $\alpha, \beta : \mathcal{R} \to G$, are *cohomologous* ($\alpha \approx \beta$) if

$$\alpha(x, y) = \phi(x)^{-1}\beta(x, y)\phi(y) \qquad \text{for } \mu_{\mathcal{R}}\text{-a.e. } (x, y),$$

where $\phi : X \to G$ is a Borel function (we call it a *transfer* function). A cocycle is a *coboundary* if it is cohomologous to a trivial one. The set of all $\mathcal{R}$-cocycles with values in G will be denoted by $Z^1(\mathcal{R}, G)$. Let $\mathcal{R} = \mathcal{R}_\Gamma$. There is a cocycle $\rho \in Z^1(\mathcal{R}, G)$ such that $\rho(x, \gamma x) = \log \frac{d\mu\circ\gamma}{d\mu}(x)$ for all $\gamma \in \Gamma$ at a.e. $x \in X$. It is called the *Radon-Nikodym cocycle* of $\mathcal{R}$. Notice that it is independent on the particular choice of Γ. $\mathcal{R}$ is *of type II* if ρ is a coboundary. Otherwise $\mathcal{R}$ is *of type III*. Given $\alpha \in Z^1(\mathcal{R}, G)$, we denote by α_0 the "double" cocycle $\alpha \times \rho \in Z^1(\mathcal{R}, G \times \mathbb{R})$.

Remind that α and β are *weakly equivalent* if $\alpha \approx \beta \circ \theta$ for a transformation $\theta \in N[\mathcal{R}]$. Clearly, α and β are weakly equivalent if and only if the double cocycles α_0 and β_0 so are. Given $\alpha \in Z^1(\mathcal{R}, G)$, we define an equivalence relation $\mathcal{R}(\alpha)$ on $(X \times G, \mu \times \lambda_G)$ by setting $(x, g) \sim (y, h)$ if $(x, y) \in \mathcal{R}$ and $h = g\alpha(x, y)$. It is called the α-*skew product extension* of $\mathcal{R}$. If the $\mathcal{R}(\alpha)$-partition is measurable (i.e. admits a measurable cross-section) then α is called *transient*. Otherwise α is *recurrent*. By

[Sc] α is recurrent if and only if α_0 so is. We say that α *has dense range* in G if $\mathcal{R}(\alpha)$ is ergodic. It follows that α is recurrent.

Next, we define a Borel action V_α of G on $(X\times G, \mu\times\lambda_G)$ as follows $V_\alpha(h)(x,g) = (x, hg)$. Since $V_\alpha \in N[\mathcal{R}(\alpha)]$, it induces an automorphism, say $W_\alpha(h)$, on the measure space of $\mathcal{R}(\alpha)$-ergodic components. Moreover, $G \ni h \mapsto W_\alpha(h)$ is an ergodic G-action on this space. W_α is called the *Mackey action* of G associated to α. If two cocycles α and β are weakly equivalent, then they are both either transient or recurrent and the associated Mackey G-actions W_α and W_β are conjugate. We call $\mathbb{R}$-actions *flows*.

Theorem 1 (Golodets-Sinel'shchikov, [GS1], [GS2])**.**

(i) *Let $\mathcal{R}$ be an ergodic hyperfinite equivalence relation on (X,μ) and $\alpha, \beta \in Z^1(\mathcal{R}, G)$ recurrent cocycles. If the Mackey $G\times\mathbb{R}$-actions W_{α_0} and W_{β_0} are conjugate then α and β are weakly equivalent.*

(ii) *Given an ergodic $G\times\mathbb{R}$-action V, there exist a hyperfinite ergodic equivalence relation $\mathcal{R}$ on (X,μ) and a recurrent cocycle $\alpha \in Z^1(\mathcal{R}, G)$ such that V is conjugate to W_{α_0}.*

2. Subrelations of type III equivalence relations

Let $\mathcal{S}$ be an ergodic subrelation of $\mathcal{R}$. Then there exist $N \in \mathbb{N}\cup\{\infty\}$ and Borel functions $\{\phi_j : X \to X \mid 0 \le j < N\}$ such that $\{\mathcal{S}[\phi_j(x)] \mid 0 \le j < N\}$ is a partition of $\mathcal{R}[x]$, where $\mathcal{R}[x]$ (resp. $\mathcal{S}[x]$) stands for the $\mathcal{R}$- (resp. $\mathcal{S}$-) class of x [FSZ]. N is called the *index* of $\mathcal{S}$ in $\mathcal{R}$ and $\{\phi_j\}_j$ *choice functions* for $\mathcal{S}$. From now on we shall assume that $\operatorname{ind}\mathcal{S} := N$ is finite. Denote by $\Sigma(J)$ the full permutation group on the set $J := \{0, 1, \dots, N-1\}$ and define a cocycle $\sigma \in Z^1(\mathcal{R}, \Sigma(J))$ by setting $\sigma(x,y)(i) = j$ if $\mathcal{S}[\phi_i(y)] = \mathcal{S}[\phi_j(x)]$. Notice that although choice functions are nonunique, the cohomology class of σ is independent of their particular choice and is an invariant of $\mathcal{S}$. According to [FSZ] σ (or its cohomology class) is called the *index cocycle* of $\mathcal{S}$. Given a cocycle $\alpha \in Z^1(\mathcal{R}, \Sigma(J))$, we put

$$\mathcal{R}\times_\alpha J = \{(x,j,y,k) \in X\times J\times X\times J \mid (x,y)\in\mathcal{R} \text{ and } k = \sigma(x,y)[j]\}.$$

Clearly, $\mathcal{R}\times_\alpha J$ is a $(\mu\times\lambda_J)$-nonsingular discrete equivalence relation on $X\times J$, where λ_J is a "counting" measure on J. We set

$$Z^1_{\text{ind}} = \{\alpha \in Z^1(\mathcal{R}, \Sigma(J)) \mid \mathcal{R}\times_\alpha J \text{ is ergodic }\}.$$

Two subrelations $\mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{R}$ are said to be *$\mathcal{R}$-conjugate* if $\mathcal{S}_1 = (\theta\times\theta)\mathcal{S}_2$ for a transformation $\theta \in N[\mathcal{R}]$. We remind some fundamental facts on subrelations from [FSZ]:

Theorem 2. *Let $\mathcal{R}$ be a discrete ergodic hyperfinite equivalence relation and $\mathcal{S}\subset\mathcal{R}$ an ergodic subrelation with $\operatorname{ind}\mathcal{S} = N$. Then every index cocycle of $\mathcal{S}$ belongs to $Z^1_{\text{ind}}(\mathcal{R}, \Sigma(J))$. Conversely, for each $\sigma \in Z^1_{\text{ind}}(\mathcal{R}, \Sigma(J))$, there is an ergodic subrelation $\mathcal{S}\subset\mathcal{R}$ with $\operatorname{ind}\mathcal{S} = N$ and such that σ is an index cocycle of $\mathcal{S}$. Two ergodic subrelations $\mathcal{S}_1, \mathcal{S}_2$ of finite index in $\mathcal{R}$ are $\mathcal{R}$-conjugate if and only if $\operatorname{ind}\mathcal{S}_1 = \operatorname{ind}\mathcal{S}_2$ and their index cocycles are weakly equivalent.*

Thus the classification of ergodic $\mathcal{R}$-subrelations of index N up to the $\mathcal{R}$-conjugacy is equivalent to the classification of cocycles from $Z^1_{\text{ind}}(\mathcal{R}, \Sigma(J))$ up to the weak equivalence.

Theorem 3. *Let $\sigma \in Z^1_{\mathrm{ind}}(\mathcal{R}, \Sigma(J)$. Then there exists a transitive subgroup $G \subset \Sigma(J)$ and a cocycle $\sigma' : \mathcal{R} \to G$ with dense range in G such that $\sigma' \approx \sigma$. Two cocycles $\sigma_1 : \mathcal{R} \to G_1$ and $\sigma_2 : \mathcal{R} \to G_2$ with dense ranges in transitive subgroups G_1 and G_2 of $\Sigma(J)$ are weakly equivalent as elements of $Z^1(\mathcal{R}, \Sigma(J))$ if and only if there is $g \in \Sigma(J)$ such that $G_1 = gG_2g^{-1}$ and the cocycles σ_1 and $\mathrm{Ad}_g \circ \sigma_2$ are weakly equivalent as elements of $Z^1(\mathcal{R}, G_1)$, where Ad_g is the inner automorphism of $\Sigma(J)$ generated by g.*

Proof. The existence of G and σ' with the required properties follows from [Zi, Corollary 3.8]. Remark that G acts transitively on J because of $\mathcal{R} \times_\sigma J$ (and hence $\mathcal{R} \times_{\sigma'} J$) is ergodic. The last statement of the theorem can be easily deduced from [Zi, the argument of Theorem 6.1], where it was proved in a slightly weaker form: with "cohomologous" instead of "weakly equivalent". Observe also that although the theorems from [Zi] to which we refer were stated there only in type II, i.e. measure preserving, case they hold also in type III case with the same argument. □

Remark that every cocycle of $\mathcal{R}$ with values in a finite (or compact) group is recurrent. From Theorems 1 and 3 we deduce

Corollary 4. *Let $\sigma_1 : \mathcal{R} \to G_1$ and $\sigma_2 : \mathcal{R} \to G_2$ be two cocycles with dense ranges in transitive subgroups G_1 and G_2 of $\Sigma(J)$ respectively. Denote by $W_{(\sigma_1)_0}$ and $W_{(\sigma_2)_0}$ the Mackey $G_1 \times \mathbb{R}$- and $G_2 \times \mathbb{R}$-actions associated to the double cocycles $(\sigma_1)_0$ and $(\sigma_2)_0$ respectively. Then σ_1 and σ_2 are weakly equivalent as elements of $Z^1(\mathcal{R}, \Sigma(J))$ if and only if there is $g \in \Sigma(J)$ such that $G_1 = gG_2g^{-1}$ and the $G_2 \times \mathbb{R}$-actions $W_{(\sigma_2)_0}$ and $W_{(\sigma_1)_0} \circ (\mathrm{Ad}_g \times \mathrm{Id})$ are conjugate.*

Every measured $G \times \mathbb{R}$-action W on a space (Ω, ν) determines a measured flow V acting on the same measure space and a group homomorphism l from G to the centralizer $C(W)$ of W as follows: $V(t) = W(1_G, t)$, $l(g) = W(g, 0)$ for all $t \in \mathbb{R}$ and $g \in G$. Remind that

$$C(W) = \{R \in \mathrm{Aut}(\Omega, \nu) \mid RW(g, t) = W(g, t)R \text{ for all } t \in \mathbb{R} \text{ and } g \in G\}.$$

We call (V, l) the *constituents* of W.

Let an $\mathcal{R}$-cocycle σ take values and have dense range in a transitive subgroup $G \subset \Sigma(J)$. Denote by (V_σ, l_σ) the constituents of the Mackey $G \times \mathbb{R}$-action W_{σ_0} associated to the double cocycle σ_0. It is easy to verify (and well known) that the $l_\sigma(G)$-quotient of V_σ, i.e. the restriction of V_σ to the subalgebra of $l(G)$-invariant measured subsets, is conjugate to W_σ. On the other hand, the $V_\sigma(\mathbb{R})$-quotient of l_σ is a singleton, since σ has dense range in G and hence the associated Mackey action is trivial. It follows that V_σ is ergodic. We illustrate these with the commutative diagram

$$\begin{array}{ccccc} X \times G & \longleftarrow & X \times G \times \mathbb{R} & \longrightarrow & X \times \mathbb{R} \\ \downarrow & & \downarrow & & \downarrow \\ \{\bullet\} & \longleftarrow & \Omega_0 & \longrightarrow & \Omega, \end{array}$$

where $\{\bullet\}$, Ω_0, and Ω stand for the spaces of the Mackey actions associated to σ, σ_0, and the Radon-Nikodym cocycle of $\mathcal{R}$ respectively; the vertical arrows represent the corresponding ergodic decompositions (see §1); the upper horizontal arrows are natural projections, and the lower arrows are determined by the universality of the "middle" ergodic decomposition.

Definition 5. Let V_i be an ergodic nonsingular flow on a measure space (Ω_i, ν_i), G_i a transitive subgroup of $\Sigma(J)$, and $l_i : G_i \to C(V_i)$ a group homomorphism, $i = 1, 2$. We say that the triplets (V_1, G_1, l_1) and (V_2, G_2, l_2) are conjugate if there is a nonsingular isomorphism $\xi : \Omega_2 \to \Omega_1$ and $g \in \Sigma(J)$ such that $G_1 = gG_2g^{-1}$, $V_1(t) = \xi V_2(t)\xi^{-1}$ and $l_1(\mathrm{Ad}_g(g_2)) = \xi l_2(g_2)\xi^{-1}$ for all $t \in \mathbb{R}$ and $g_2 \in G_2$.

Now we are ready to record our main classification result.

Theorem 6. *Let $\mathcal{R}$ be an ergodic type III hyperfinite equivalence relation on $(X, \mathfrak{B}, \mu)$, and W_ρ its associated flow (ρ stands for the Radon-Nikodym cocycle).*

(i) *Given an ergodic subrelation of index N, we associate a triplet (V, G, l) consisting of an ergodic flow V, a transitive subgroup $G \subset \Sigma(J)$ and a homomorphism $l : G \to C(V)$ such that the $l(G)$-quotient flow of V is conjugate to W_ρ.*

(ii) *Conversely, given such a triplet, there exists an ergodic subrelation $\mathcal{S} \subset \mathcal{R}$, $\mathrm{ind}\,\mathcal{S} = N$, whose associated triplet is as given.*

(iii) *two ergodic $\mathcal{R}$-subrelations of index N are $\mathcal{R}$-conjugate if and only if their associated triplets are conjugate.*

Proof. (i) follows from Theorems 2, 3 and the remark before Definition 5.

(ii) Given a triplet (V, G, l), we consider a $G \times \mathbb{R}$-action W whose constituents are (V, l). By Theorem 1 there are an ergodic hyperfinite equivalence relation $\mathcal{R}'$ on $(X, \mathfrak{B}, \mu)$ and a cocycle $\sigma' : \mathcal{R}' \to G$ such that W is conjugate to the Mackey $G \times \mathbb{R}$-action associated to the double cocycle σ'_0. It is clear that the associated flow of $\mathcal{R}'$ is conjugate to the $l(G)$-quotient flow of V. By the assumptions on (V, G, l), this flow is conjugate to W_ρ. It follows from the Krieger theorem [Kr], [FM] that $\mathcal{R}$ and $\mathcal{R}'$ are orbit equivalent and hence we may identify them. Next, since V is ergodic, σ' has dense range in G. But G is a transitive subgroup of J-permutations and this implies $\sigma' \in Z^1_{\mathrm{ind}}(\mathcal{R}, \Sigma(J))$. It remains to apply Theorem 2.

(iii) follows from Theorem 2 and Corollary 4. □

3. On Hamachi's invariants

Let an $\mathcal{R}$-cocycle σ take values and have dense range in a transitive subgroup G of $\Sigma(J)$. Denote by H the G-stability group of 0, i.e. $H = \{g \in G \mid g[0] = 0\}$. Then $H \subset G$ is irreducible, i.e. H contains no nontrivial G-normal subgroups. If a subgroup $G_1 \subset \Sigma(J)$ is conjugate to G, then there exists $k \in \Sigma(J)$ such that $G_1 = kGk^{-1}$ and $k[0] = 0$ (remind that G is transitive). It follows that $H_1 = kHk^{-1}$, where H_1 is the G_1-stability group of 0. Thus the conjugacy classes of transitive subgroups of $\Sigma(J)$ are in one-to-one correspondence with the isomorphism classes of irreducible pairs of finite groups $H \subset G$ such that the cardinality of G/H is N. (We say that two pairs $H \subset G$ and $H' \subset G'$ are *isomorphic* if there is an isomorphism of G onto G' taking H onto H'.)

Let (V, G, l) be a triplet as in Theorem 6. Denote by G_0 the kernel of l and by (Ω, ν) the measure space of W_ρ. Then V is a G/G_0-extension of W_ρ, i.e. we may assume without loss in generality that V is defined on the space $(\Omega_0, \nu_0) := (\Omega \times G/G_0, \nu \times \lambda_{G/G_0})$ as follows

$$V(t)(\omega, h) = (W_\rho(t)\omega, h\alpha(\omega, t)), \tag{*}$$

where λ_{G/G_0} is Haar measure on G/G_0 and $\alpha : \Omega \times \mathbb{R}$ a measurable W-cocycle, i.e.

$$\alpha(\omega, t_1 + t_2) = \alpha(\omega, t_1)\alpha(W_\rho(t_1)\omega, t_2)$$

at a.e. $\omega \in \Omega$ for all $t_1, t_2 \in \mathbb{R}$. (Do not confuse cocycles of group actions with cocycles of equivalence relations.) Denote by $\pi : \Omega_0 \ni (\omega, h) \mapsto \omega \in \Omega$ the canonical projection. Then $\pi V(t) = W_\rho(t)\pi$ for all $t \in \mathbb{R}$. It is convenient to use the notation $\pi : V \xrightarrow{G/G_0} W_\rho$.

Remind that two group extensions $\pi : V \xrightarrow{G} W$ and $\pi' : V' \xrightarrow{G'} W'$ are *conjugate* if there are nonsingular isomorphisms $\psi : (\Omega_0, \nu_0) \to (\Omega_0', \nu_0')$ and $\phi : (\Omega, \nu) \to (\Omega', \nu')$ such that $\phi W(t)\phi^{-1} = W'(t)$, $\psi V(t)\psi^{-1} = V'(t)$, and $\psi\pi\psi^{-1} = \pi'$. This implies that G and G' are isomorphic.

Thus given a triplet (V, G, l), we associate a system $(G, H, G_0, \pi : V \xrightarrow{G/G_0} W_\rho)$ consisting of an irreducible pair of finite groups $H \subset G$, a normal subgroup $G_0 \subset G$ and a G/G_0-extension of W_ρ. We shall call it an *$\mathcal{H}$-system* (see [Ha]).

Definition 7 (see [Ha, Definition 6.1])**.** Two $\mathcal{H}$-systems $(G, H, G_0, \pi : V \xrightarrow{G/G_0} W_\rho)$ and $(G', H', G_0', \pi' : V' \xrightarrow{G'/G_0'} W_\rho)$ are *equivalent* if there is an isomorphism $\rho : G \to G'$ such that $\rho(H) = H'$, $\rho(G_0) = G_0'$ and the extensions π and π' are conjugate.

It is easy to see that if two triplets are conjugate then the associated $\mathcal{H}$-invariants are equivalent. It is claimed in [Ha] that the converse also holds which implies that $\mathcal{R}$-non-conjugate ergodic $\mathcal{R}$-subrelations of finite index have nonequivalent $\mathcal{H}$-invariants. Our purpose in this section is to construct a counterexample to this statement.

Example 8. Let Σ_3 be the permutation group of $\{0, 1, 2\}$ and A_5 the group of even permutations of $\{0, 1, \dots, 5\}$. We put $H := (\Sigma_3)^5 \rtimes A_5$ and $G := H^2$. It is easy to verify that $Z(H)$, the center of H, is trivial but $\operatorname{Out} H$, the group of outer automorphisms of H, is nontrivial. Denote by $\Sigma(H)$ the permutation group of H and consider a homomorphism $b : G \to \Sigma(H)$ as follows $b(h_1, h_2)[h] = h_1 h h_2^{-1}$, $h \in H$. Since the kernel of b is isomorphic to $Z(H)$, b is an embedding. It is obvious that G (or, more precisely, $b(G)$) acts transitively on H. Denote by G_0 the G-stability group of 1_H. Clearly, $G_0 = \{(h, h) \mid h \in H\}$. Define an automorphism κ of G by setting $\kappa(h_1, h_2) = (h_1, \tau(h_2))$, where τ is a non-innner automorphism of H. We claim that κ can non be extended to an automorphism of $\Sigma(H)$. Suppose the contrary: there exists $k \in \Sigma(H)$ such that $\kappa(g) = kgk^{-1}$ for all $g \in G$. (Remind that every automorphism of $\Sigma(H)$ is inner.) Put $h_0 := k[1_H] \in H$. Then $\kappa(g)[h_0] = h_0$ for all $g \in G_0$. Since G acts transitively on H, we deduce that $\kappa(G_0) = g_0 G_0 g_0^{-1}$ for an element $g_0 \in G$ with $g_0[0] = h_0$. Thus $\bigcup_{h \in H}(h, \tau(h)) = \bigcup_{h \in H}(h, h_1 h h_1^{-1})$ for some $h_1 \in H$. It follows that τ is an inner automorphism of H, a contradiction.

Let W be an ergodic properly non-transitive $\mathbb{R}$-flow on (Ω, ν) with trivial centralizer, i.e. $C(W) = W(\mathbb{R})$. Take a cocycle α of W with values in G such that the flow V determined by (*) with G_0 trivial is ergodic. Define a one-to-one homomorphism $l : G \to C(V)$ by setting

$$l(g')(\omega, g) = (\omega, g'g), \qquad \text{for all } (\omega, g) \in \Omega \times G,$$

and put $l_1 = l \circ \kappa$. We claim that the triplets (V, G, l) and (V, G, l_1) are non-conjugate. Suppose the contrary: there exist $\xi \in C(V)$ and $s \in \Sigma(H)$ such that

$$(**) \qquad l \circ \operatorname{Ad}_s(g) = \xi l(\kappa(g))\xi^{-1}, \qquad \text{for all } g \in G.$$

Since ξ passes through the natural projection $\Omega \times G \to \Omega$, it is well known (see, for example [Da1, Theorem 5.3 and §6]) that ξ is of the form $\xi(\omega, g) = (\zeta\omega, d(g)f(x))$ for a transformation $\zeta \in C(W)$, a G-automorphism d, and a measurable map $f : X \to G$. Hence $\zeta \in W(\mathbb{R})$. It follows from [Da1, Lemma 5.2 and §6] that d is inner. On the other hand, it is easy to verify that $\xi l(g)\xi^{-1} = l(d(g))$ for all $g \in G$. We deduce from (**) that $l \circ \mathrm{Ad}_s(g) = l \circ d \circ \kappa(g)$ and hence $\mathrm{Ad}_s = d \circ \kappa$. This contradicts to the fact that κ can not be extended to a $\Sigma(H)$-automorphism.

Since W is nontransitive, it is the associated flow of a type III_0 ergodic hyperfinite equivalence relation $\mathcal{R}$. By Theorem 6 there are ergodic $\mathcal{R}$-subrelations $\mathcal{S}$ and $\mathcal{S}_1$ of finite index whose associated triplets are (V, G, l) and (V, G, l_1) respectively. It follows that $\mathcal{S}$ and $\mathcal{S}_1$ are $\mathcal{R}$-nonconjugate. On the other hand, $\mathcal{H}$-invariants associated to (V, G, l) and (V, G, l_1) are obviously identical and we are done.

4. Case of III_λ equivalence relations, $0 < \lambda \le 1$

If $\mathcal{R}$ is of type III_λ, $0 < \lambda \le 1$, our invariants (see Theorem 6) can be described in a more apparent way.

We first consider the case where $\mathcal{R}$ is of type III_1. Then the associated flow W_ρ and any ergodic finite group extension V of W_ρ are trivial. Thus we deduce from Theorem 6

Corollary 9. *The set of $\mathcal{R}$-conjugacy classes of ergodic $\mathcal{R}$-subrelations of index N are in one-to-one correspondence with the (finite) family of conjugacy classes of transitive subgroup of $\Sigma(J)$, $J = \{0, 1, \dots, N-1\}$.*

Now let $\mathcal{R}$ be of type III_λ, $0 < \lambda < 1$. Then W_ρ is a transitive periodic flow with period $-\log\lambda$. If V is an ergodic finite group extension of W_ρ, then there is a non-negative integer n such that V is a periodic flow with the period $-n\log\lambda$ and V is a $\mathbb{Z}/n\mathbb{Z}$-extension of W.

Definition 10. A collection (n, G, l) consisting of a positive integer n, a transitive subgroup $G \subset \Sigma(J)$ and an onto homomorphism $l : G \to \mathbb{Z}/n\mathbb{Z}$ will be called a λ-*triplet*. Two λ-triplets (n, G, l) and (n', G', l') are *conjugate* if $n = n'$ and there is $s \in \Sigma(J)$ with $G = sG's^{-1}$ and $l \circ \mathrm{Ad}_s = l'$.

It is easy to deduce from Theorem 6

Corollary 11. *The set of $\mathcal{R}$-conjugacy classes of ergodic $\mathcal{R}$-subrelations of index N are in one-to-one correspondence with the (finite) family of conjugacy classes of λ-triplets.*

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