QUASINORMAL SUBRELATIONS OF ERGODIC EQUIVALENCE RELATIONS

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Abstract. We introduce a notion of quasinormality for a nested pair $\mathcal{S} \subset \mathcal{R}$ of ergodic discrete hyperfinite equivalence relations of type II_1. (This is a natural extension of the normality concept due to Feldman-Sutherland-Zimmer [FSZ].) Such pairs are characterized by an irreducible pair $F \subset Q$ of countable amenable groups or rather (some special) their Polish closure $\overline{F} \subset \overline{Q}$. We show that "most" of ergodic subrelations of $\mathcal{R}$ are quasinormal and classify them. An example of non quasinormal subrelation is given. We prove as an auxiliary statement that two cocycles of $\mathcal{R}$ with dense ranges in a Polish group are weakly equivalent.

0. Introduction

It is well known that two ergodic finite measure preserving actions of countable amenable groups are orbit equivalent [Dy], [CFW]. This can be rephrased in equivalent terms of measured equivalence relations [FM]: there exists the unique (up to isomorphism) hyperfinite discrete ergodic equivalence relation, say $\mathcal{R}$, of type II_1. A natural subsequent problem that arises here is to study subrelations of $\mathcal{R}$ and this is the main concern of the present paper.

It was shown in [FSZ] how to associate to any pair $\mathcal{S} \subset \mathcal{R}$ of discrete ergodic type II_1 equivalence relations, a countable index set J and a cocycle $\sigma : \mathcal{R} \to \Sigma(J)$, where $\Sigma(J)$ is the full permutation group of J. The cardinality of J is called the *index* of $\mathcal{S} \subset \mathcal{R}$ and related closely to the Jones index in the study of sub-von-Neumann-algebras [Jo]. The cocycle σ is called *index cocycle* of $\mathcal{S} \subset \mathcal{R}$. The weak equivalence class of σ depends only on the isomorphism class of the pair $\mathcal{S} \subset \mathcal{R}$.

J. Feldman, C. E. Sutherland and R. J. Zimmer provided an elegant classification of ergodic hyperfinite pairs $\mathcal{S} \subset \mathcal{R}$ in the following two cases: (a) $\mathcal{S}$ is normal, (b) $\mathcal{S}$ is of finite index in $\mathcal{R}$ [FSZ]. Remark that the case (b) was considered earlier by M. Gerber in a different context—she classified the finite extensions of ergodic probability preserving transformations up to the "factor orbit equivalence" [Ge]. The purpose of this paper is to extend the above results to a wider class of subrelations, namely quasinormal ones.

We call $\mathcal{S}$ *quasinormal* if σ (or its restriction to $\mathcal{S}$) is regular, i.e. σ is cohomologous to a cocycle with dense range in a closed subgroup of $\Sigma(J)$. The concept of quasinormality was introduced in a different way in a previous paper of the author [Da, §4], where the problem of genericity for extensions of $\mathcal{S}$-cocycles to $\mathcal{R}$-cocycles with values in amenable locally compact groups was discussed (see also [GLS]). We show that the above definition is equivalent to [Da, Definition 4.1].

Before proceeding with the statements of our main results, we remind some standard notions of the orbit theory. Let $\mathcal{P}$ be a discrete measured equivalence relation on a standard probability space $(X, \mathfrak{B}, \mu)$. By the full group $[\mathcal{P}]$ we mean the group of automorphisms of X whose orbits are contained in $\mathcal{P}$-classes. The normalizer $N[\mathcal{R}]$ of $[\mathcal{P}]$ is the group of automorphisms of X which preserve $\mathcal{P}$ (see §2 for the rigorous definitions). Two $\mathcal{R}$-subrelations $\mathcal{S}_1$ and $\mathcal{S}_2$ are *$\mathcal{R}$-conjugate* if $\mathcal{S}_1 = (T \times T)\mathcal{S}_2$ for a transformation $T \in N[\mathcal{R}]$.

We say that a pair $F \subset Q$ of Polish groups is *irreducible* if F contains no nontrivial closed normal subgroups of Q.

Theorem 0.1 (Canonical Form for Quasinormal Subrelations). *Let $\mathcal{S}$ be an ergodic quasinormal subrelation of $\mathcal{R}$. There exist an ergodic subrelation $\mathcal{P} \subset \mathcal{S}$, a countable amenable group $Q \subset N[\mathcal{P}]$, and a subgroup F of Q such that $Q \cap [\mathcal{P}] = \{\mathrm{Id}\}$, $F \subset Q$ is irreducible, $\mathcal{R}$ is generated by $\mathcal{P}$ and Q and $\mathcal{S}$ is generated by $\mathcal{P}$ and F. Moreover, the index cocycle may be realized as $\sigma = \rho \circ \theta : \mathcal{R} \to \Sigma(F\backslash G)$, where $\theta : \mathcal{R} \to Q$ is given by $\theta(x, qy) = q$ for all $(x, y) \in \mathcal{P}$ and $q \in Q$, and ρ is the Cayley representation of Q in $\Sigma(F\backslash Q)$ as right translations.*

Notice that the pair $F \subset Q$ is not determined uniquely (up to isomorphism) by $\mathcal{S}$. That is why we need to introduce some special equivalence relation for these objects as follows. Denote by $\overline{Q}$ (resp. $\overline{F}$) the closure of $\rho(Q)$ (resp. the closure of $\rho(F)$) in $\Sigma(F\backslash Q)$ endowed with the usual Polish topology. It is easy to see that $\overline{F} = \{q \in \overline{Q} \mid q(F) = F\}$. Hence $\overline{F}$ is an open subgroup of $\overline{Q}$ and $\overline{F} \subset \overline{Q}$ is an irreducible pair of Polish groups.

Definition 0.2. We say that two irreducible pairs of countable groups $F_1 \subset Q_1$ and $F_2 \subset Q_2$ are weakly isomorphic if there exists a continuous isomorphism of $\overline{Q}_1$ onto $\overline{Q}_2$ which takes $\overline{F}_1$ onto $\overline{F}_2$.

Theorem 0.3 (Classification of Quasinormal Subrelations). *There is a bijective correspondence between the ergodic quasinormal subrelations $\mathcal{S}$ of $\mathcal{R}$ (up to $\mathcal{R}$-conjugacy) and the weak isomorphism classes of irreducible pairs of countable amenable groups $F \subset Q$. Furthermore, $F \subset Q$ is related to $\mathcal{S}$ as it is described in Theorem* 0.1.

Notice that the normal subrelations are quasinormal—they correspond exactly to the case where F is trivial. Clearly, the subrelations of finite index are also quasinormal, since the index cocycle as well as every cocycle with values in a finite group is regular. In both cases $\overline{Q} = Q$, $\overline{F} = F$ and Theorem 0.3 gives [FSZ, Theorems 3.1, 3.2].

The outline of the paper is as follows. §1 is of a preliminary nature. We study here cocycles of $\mathcal{R}$ with values in Polish groups and extend some results from [GS], where the groups were assumed to be locally compact. In particular, we prove that two cocycles with dense ranges in a Polish group are weakly equivalent. The second section introduces an idea of quasinormal pair $\mathcal{S} \subset \mathcal{R}$ (cf. with [Da, §4]). The proofs of Theorems 0.1 and 0.3 and related problems are contained here. In the final §3, we show that a "typical" (in the Baire category sense) ergodic subrelation of $\mathcal{R}$ is quasinormal but non normal. We also provide an example of non quasinormal subrelation.

Remark that throughout this paper equivalence relations are of type II_1. However, all the results are also valid for type II_∞ equivalence relations with minor modifications of the arguments. We hope to treat the type III case in a later paper.

1. Cocycles of measured equivalence relations with values in Polish groups

We begin this section with some background on orbit theory. Let $(X, \mathfrak{B}, \mu)$ be a standard probability space. Denote by $\mathrm{Aut}(X, \mathfrak{B}, \mu)$ the group of its automorphisms, i.e. Borel, one-to-one, onto, μ-preserving transformations; we do not distinguish between two of them which agree on a μ-conull subset. Let $\mathcal{R} \subset X \times X$ be a Borel discrete (i.e. each equivalence class is countable) equivalence relation. We shall assume that $\mathcal{R}$ is μ-preserving, i.e. there exists a countable subgroup $\Gamma \subset \mathrm{Aut}(X, \mu)$ such that $\mathcal{R}$ is the Γ-orbital equivalence relation. We endow $\mathcal{R}$ with the induced Borel structure and the σ-finite measure $\mu_{\mathcal{R}}$, $d\mu_{\mathcal{R}}(x, y) = d\mu(x)$, $(x, y) \in \mathcal{R}$. Write also

$$[\mathcal{R}] = \{q \in \mathrm{Aut}(X, \mu) \mid (qx, x) \in \mathcal{R} \text{ for } \mu\text{-a.a. } x \in X\},$$
$$N[\mathcal{R}] = \{q \in \mathrm{Aut}(X, \mu) \mid (qx, qy) \in \mathcal{R} \text{ for } \mu_{\mathcal{R}}\text{-a.a. } (x, y) \in \mathcal{R}\}$$

for the *full group* of $\mathcal{R}$ and the *normalizer* of $[\mathcal{R}]$ respectively. $\mathcal{R}$ is called *hyperfinite* if it can be generated by a single automorphism.

Let G be a Polish group and 1_G the identity of G. A Borel map $\alpha : \mathcal{R} \to G$ is a (1-)*cocycle* of $\mathcal{R}$ if

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z) \qquad \text{for a.a. } (x, y),\ (y, z) \in \mathcal{R}.$$

We do not distinguish between two cocycles if they agree $\mu_{\mathcal{R}}$-a.e. Two cocycles, $\alpha, \beta : \mathcal{R} \to G$, are *cohomologous* ($\alpha \approx \beta$), if

$$\alpha(x, y) = \phi(x)^{-1}\beta(x, y)\phi(y) \qquad \text{for } \mu_{\mathcal{R}}\text{-a.a. } (x, y),$$

where $\phi : X \to G$ is a Borel function (we call it a *transfer* function from α to β). A cocycle is a *coboundary* if it is cohomologous to the trivial one.

Two cocycles $\alpha, \beta : \mathcal{R} \to G$ are *weakly equivalent* if there is a transformation $T \in N[\mathcal{R}]$ such that $\alpha \approx \beta \circ T$, where the cocycle $\beta : \mathcal{R} \to G$ is defined by $\beta \circ T(x, z) = \beta(Tx, Tz)$.

We assume from now on that $\mathcal{R}$ is ergodic, i.e. every $\mathcal{R}$-saturated Borel subset is μ-null or μ-conull.

We say that α *has dense range* in G if for every $A \in \mathfrak{B}$, $\mu(A) > 0$, and an open subset $O \subset G$ there exists $B \in \mathfrak{B}$ and a transformation $q \in [\mathcal{R}]$ with $\mu(B) > 0$, $B \cup qB \subset A$, and $\alpha(x, qx) \in O$ for all $x \in B$.

Proposition 1.1. *Let F, H be closed subgroups of G and two cocycles $\alpha, \beta : \mathcal{R} \to G$ take values and have dense ranges in F and H respectively. If $\alpha \approx \beta$ then F and H are conjugate in G.*

Proof. Let $\alpha(x, y) = \phi(x)^{-1}\beta(x, y)\phi(y)$ at $\mu_{\mathcal{R}}$-a.e. $(x, y) \in \mathcal{R}$ for a Borel function $\phi : X \to G$. Take any *proper value* $g_0 \in G$ of ϕ, which means that $\mu(\phi^{-1}(O)) > 0$ for every neighborhood O of g_0. We shall prove that $F = g_0^{-1}Hg_0$. Given any $g \in H$ and a neighborhood V of $g_0^{-1}gg_0$, we choose neighborhoods U of g_0 and W of g with $U^{-1}WU \subset V$. Since β has dense range in H, there exists a Borel subset $A \subset X$ and a transformation $q \in [\mathcal{R}]$ such that $\mu(A) > 0$, $A \cup qA \subset \phi^{-1}(U)$ and $\beta(x, qx) \in W$ for all $x \in A$. Remind that $\alpha(\mathcal{R}) \subset F$ and hence $V \cap F \neq \emptyset$.

Since V is an arbitrary neighborhood of $g_0^{-1}gg_0$, we deduce that $g_0^{-1}gg_0 \in F$. Thus $g_0^{-1}Hg_0 \subset F$. The converse inclusion is established in a similar way. $\square$

Remark 1.2. It is easy to deduce from the above proof that the transfer function ϕ is of the form $\phi(x) = \psi(x)g'$ a.e. for some $g' \in G$ and a Borel function $\phi : X \to N_G(H)$, where $N_G(H) := \{g \in G \mid gHg^{-1} = H\}$ is the normalizer of H in G.

Definition 1.3. A cocycle $\alpha : \mathcal{R} \to G$ is called *regular* if it is cohomologous to a cocycle which takes values and has dense range in a closed subgroup H of G.

We denote by $\langle\alpha\rangle$ the conjugacy class of H , i.e. $\langle\alpha\rangle = \{gHg^{-1} \mid g \in G\}$. It is well defined by Proposition 1.1. It is obvious that given a cocycle α with dense range in G, then $\alpha \circ T$ also has dense range in G for every transformation $T \in N[\mathcal{R}]$. We deduce from this fact and Proposition 1.1

Corollary 1.4. *Let α and β be weakly equivalent cocycles. If α is regular then so is β and $\langle\alpha\rangle = \langle\beta\rangle$.*

Remind that an equivalence relation $\mathcal{P}$ is *of type I* if there is a Borel subset $A \subset X$, $\mu(A) > 0$, such that for a.e. $x \in X$ there is a unique $y \in A$ with $(x, y) \in \mathcal{R}$. We call such A a *$\mathcal{P}$-fundamental domain.* It is well known that every cocycle of an equivalence relation of type I is a coboundary [FM].

Lemma 1.5 (cf. with [GS, Proposition 1.1])**.** *Let $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$ for an increasing sequence of type I equivalence relations $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \dots$. Given two cocycles $\alpha, \beta : \mathcal{R} \to G$, consider two sequences of Borel maps $a_n, b_n : X \to G$ such that $\alpha(x, y) = a_n(x)a_n(y)^{-1}$, $\beta(x, y) = b_n(x)b_n(y)^{-1}$ for a.e. $(x, y) \in \mathcal{R}_n$. Define a sequence of maps $f_n : X \to G$ by setting $f_n(x) = a_n(x)b_n(x)^{-1}$. If f_n converges a.e. to a map $\phi : X \to G$ as $n \to \infty$ then $\alpha(x, y) = \phi(x)\beta(x, y)\phi(y)^{-1}$ for a.e. $(x, y) \in \mathcal{R}$.*

Proof. For a.e. $(x, y) \in \mathcal{R}_n$ and every $m > n$ we have

$$\begin{aligned} f_m(x)\beta(x, y)f_m(y)^{-1} &= a_m(x)b_m(x)^{-1}b_m(x)b_m(y)^{-1}b_m(y)a_m(y)^{-1} \\ &= a_m(x)a_m(y)^{-1} = \alpha(x, y), \end{aligned}$$

since $\mathcal{R}_n \subset \mathcal{R}_m$. Pass to the limit to obtain $\phi(x)\beta(x, y)\phi(y)^{-1} = \alpha(x, y)$ for a.e. $(x, y) \in \mathcal{R}_n$, $n \in \mathbb{N}$. $\square$

Proposition 1.6. *Let $\mathcal{R}$ be hyperfinite and G' a countable dense subgroup of G. Given a cocycle $\alpha : \mathcal{R} \to G$, there exists a cocycle $\beta \approx \alpha$ with $\beta(\mathcal{R}) \subset G'$.*

Proof. Since $\mathcal{R}$ is hyperfinite, there exists an increasing sequence of type I equivalence relations $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \dots$ with $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$. Let F_n stand for a $\mathcal{R}_n$-fundamental domain. We also put $F_0 = X$. Define a Borel map $T_n : X \to F_n$ by setting $T_n x = y$ if $(x, y) \in \mathcal{R}_n$. Notice that T_n is $\mathcal{R}_n$-invariant—i.e. $T_n x = T_n y$ for a.e. $(x, y) \in \mathcal{R}_n$—and

$$\alpha(x, y) = \alpha(x, T_n x)\alpha(T_n x, T_n y)\alpha(T_n y, y) = \alpha(x, T_n x)\alpha(y, T_n y)^{-1} \tag{2-1}$$

for a.e. $(x, y) \in \mathcal{R}_n$. Consider the family of Borel maps $a_n : F_{n-1} \to G$ given by $a_n(x) = \alpha(x, T_n x)$. Then

$$\begin{aligned} \alpha(x, T_n x) &= \alpha(x, T_1 x)\alpha(T_1 x, T_2 x) \dots \alpha(T_{n-1} x, T_n x) \\ &= a_1(x)a_2(T_1 x) \dots a_n(T_{n-1} x) \end{aligned} \tag{2-2}$$

for a.e. $x \in X$. Conversely, it is easy to see that an arbitrary family of Borel functions $a_n : F_{n-1} \to G$, $n \in \mathbb{N}$, determines a cocycle $\alpha : \mathcal{R} \to G$ by (2-1) and (2-2).

Let $\{W_n\}_{n=1}^\infty$ be a fundamental system of neighborhoods of 1_G with the following properties: $W_n^{-1} = W_n$ and $W_{n+1}W_{n+1}W_{n+1} \subset W_n$, $n \in \mathbb{N}$. Enumerate the elements of G': $G' = \{g_i\}_{i=1}^\infty$. For each $n \in \mathbb{N}$, we have $G = \bigcup_{i=1}^\infty W_n g_i$. Hence there is $m_n \in \mathbb{N}$ such that $\mu(A_n) > 1 - 2^{-n}$, where

$$A_n := \{x \in X \mid \alpha(x, T_n x) \in \bigcup_{i=1}^{m_n} W_{n+2} g_i\}.$$

Let V_n be a neighborhood of 1_G with $g_i V_n g_i^{-1} \subset W_n$ for all $i = 1, \dots, m_{n-2}$, $n > 2$. Take a family of Borel maps $b_n : F_{n-1} \to G'$ such that $a_n(x)b_n(x)^{-1} \in V_{n+1}$ for all $x \in F_{n-1}$. This family determines a cocycle $\beta : \mathcal{R} \to G$. We have for $k \in \mathbb{N}$ and $x \in \bigcap_{i=n}^{n+k-1} A_i$

$$\begin{aligned}
f_{n+k} &:= \alpha(x, T_{n+k}x)\beta(x, T_{n+k}x)^{-1}\\
&= \alpha(x, T_{n+k-1}x)a_{n+k}(T_{n+k-1}x)b_{n+k}(T_{n+k-1}x)^{-1}\beta(x, T_{n+k-1}x)^{-1}\\
&\in \alpha(x, T_{n+k-1}x)V_{n+k+1}\beta(x, T_{n+k-1}x)^{-1}\\
&= \alpha(x, T_{n+k-1}x)V_{n+k+1}\alpha(x, T_{n+k-1}x)^{-1}\alpha(x, T_{n+k-1}x)\beta(x, T_{n+k-1}x)^{-1}\\
&\subset W_{n+k+1}W_{n+k+1}W_{n+k+1}\alpha(x, T_{n+k-1}x)\beta(x, T_{n+k-1}x)^{-1}\\
&\subset W_{n+k}\alpha(x, T_{n+k-1}x)\beta(x, T_{n+k-1}x)^{-1} \subset \dots\\
&\subset W_{n+k}W_{n+k-1}\dots W_{n+1}\alpha(x, T_n x)\beta(x, T_n x)^{-1} \subset W_n f_n(x).
\end{aligned}$$

Since $\mu(\bigcap_{i=n}^{n+k-1} A_i) \geq 1 - 2^{-n} - 2^{-n-1} - \cdots - 2^{-n-k+1} \geq 1 - 2^{-n+1} \to 1$, the sequence f_n converges in measure as $n \to \infty$. Hence a subsequence of f_n converges a.e. and $\alpha \approx \beta$ by Lemma 1.4. $\square$

Remark 1.7. If G' is normal in G then the conclusion of Proposition 1.5 follows from the Connes-Krieger cohomology lemma (see [Su], [JT]). For G locally compact (and any G') the conclusion of the proposition was proved in [GS, Proposition 1.2]. We modified the argument of V. Ya. Golodets and S. D. Sinelshchikov in such a way to avoid the use of the local compactness.

Proposition 1.8. *Let $\mathcal{R}$ be hyperfinite. Given a cocycle $\alpha : \mathcal{R} \to G$ with dense range in G, there exists a cocycle $\beta \approx \alpha$ such that $\{(x, y) \in \mathcal{R} \mid \beta(x, y) = 1_G\}$ is an ergodic subrelation of $\mathcal{R}$.*

Proof. By virtue of Dye theorem [Dy] we may assume that $(X, \mathfrak{B}, \mu)$ and $\mathcal{R}$ are of the following special form:

(a) $(X, \mu) = (\{0, 1\}, \lambda)^{\mathbb{N}}$, where λ is the equidistribution on $\{0, 1\}$, i.e. $\lambda(0) = \lambda(1) = 0.5$,

(b) $\mathcal{R} = \bigcup_{n=1}^\infty \mathcal{R}_n$, where $\mathcal{R}_n = \{(x, y) \in X \times X \mid x_i = y_i \text{ for all } i \geq n\}$.

Let $\{W_n\}_{n=1}^\infty$ be a fundamental system of neighborhoods of 1_G with the properties as above. We construct inductively an increasing sequence $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \dots$ of type I subrelations of $\mathcal{R}$. Describe in general the n-th step.

Let $F_n := \{x \mid x_i = 0 \text{ for all } i < n\}$. Clearly, $\mu(F_{n-1} \setminus F_n) = \mu(F_n)$. Since α has dense range in G, we apply the standard exhaustion argument to construct a

Borel isomorphism $t_n : F_{n-1} \setminus F_n \to F_n$ such that $(x, t_n x) \in \mathcal{R}$ and $\alpha(x, t_n x) \in W_n$. Define a Borel map $T_n : X \to F_n$ by setting

$$T_n x = \begin{cases} x, & \text{for } x \in F_n \\ t_n T_{n-1} T_{n-2} \dots T_1, & \text{otherwise.} \end{cases}$$

Now we put $\mathcal{S}_n = \{(x,y) \mid T_n x = T_n y\}$. Since the S_n-class of a.e. $x \in X$ is finite, S_n is of type I. Moreover, F_n is a $\mathcal{S}_n$-fundamental domain. Clearly, $\mathcal{S}_1 \subset \cdots \subset \mathcal{S}_n \subset \mathcal{R}$.

Now we put $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$. Then $\mathcal{S}$ is an ergodic subrelation of $\mathcal{R}$. Actually, if a Borel function $f : X \to \mathbb{R}$ is $\mathcal{S}_n$-invariant then it does not depend on the first n-coordinates of x. Since n is arbitrary, f equals a.e. to a constant, as desired.

We claim that $\alpha \restriction \mathcal{S}$ is a coboundary. Notice that $\alpha(x,y) = \alpha(x, T_n x)\alpha(y, T_n y)^{-1}$ for a.e. $(x,y) \in \mathcal{S}_n$ and

$$\begin{aligned} f_{n+k} &:= \alpha(x, T_{n+k}) = \alpha(x, T_{n+k-1}x)\alpha(T_{n+k-1}x, t_{n+k}T_{n+k-1}x) \\ &\in \alpha(x, T_{n+k-1}x)W_{n+k} \subset \cdots \subset \alpha(x, T_n x)W_{n+1}W_{n+2}\dots W_{n+k} \subset f_n(x)W_n \end{aligned}$$

for a.e. $x \in X$. Hence f_n converges a.e. to a map $\phi : X \to G$. By Lemma 1.5 $\alpha(x,y) = \phi(x)\phi(y)^{-1}$ for a.e. $(x,y) \in \mathcal{S}$. This implies that the cocycle $\beta(x,y) := \phi(x)^{-1}\alpha(x,y)\phi(y)$, $(x,y) \in \mathcal{R}$, satisfies the conclusion of the proposition. $\square$

We conclude this section with an extension of the remarkable Uniqueness Theorem for Cocycles (due to V. Ya. Golodets and S. D. Sinelshchikov) to cocycles with dense ranges in Polish groups.

Theorem 1.9. *Let $\alpha, \beta : \mathcal{R} \to G$ be two cocycles with dense ranges in G. If $\mathcal{R}$ is hyperfinite then α and β are weakly conjugate.*

Proof. This is almost the same as that of [GS, Lemma 1.12], where G was assumed to be locally compact, but one should use Proposition 1.5 instead of [GS, Proposition 1.2]. $\square$

2. Quasinormal subrelations

We begin this section with a brief exposition of the basic notions of *measurable index theory* [FSZ].

Let $\mathcal{R}$ be an ergodic μ-preserving equivalence relation on $(X, \mathfrak{B}, \mu)$ and $\mathcal{S}$ an ergodic subrelation of $\mathcal{R}$. Then there exist $N \in \mathbb{N} \cup \{\infty\}$ and Borel functions $\phi_j : X \to X$ so that $\{\mathcal{S}[\phi_j(x)] \mid 0 \le j < N\}$ is a partition of $\mathcal{R}[x]$, where $\mathcal{R}[x]$ (resp. $\mathcal{S}[x]$) stands for the $\mathcal{R}$- (resp. $\mathcal{S}$-) class of x. N is called the *index* of $\mathcal{S}$ in $\mathcal{R}$ and $\{\phi_j\}_j$ are called *choice functions* for the pair $\mathcal{S} \subset \mathcal{R}$. We may assume without loss in generality that $\phi_j \in \mathrm{Aut}(X, \mu)$, $j \in J$, and $\phi_0(x) = x$ for all $x \in X$. Denote by $\Sigma(J)$ the full permutation group of the set $J \overset{\text{def}}{=} \{0, 1, \dots, N-1\}$ for $N < \infty$ or $J \overset{\text{def}}{=} \{0, 1, 2, \dots\}$ for $N = \infty$. We define the *index cocycle* $\sigma : \mathcal{R} \to \Sigma(J)$ by setting $\sigma(x,y)(i) = j$ if $\mathcal{S}[\phi_i(y)] = \mathcal{S}[\phi_j(x)]$. Notice that although choice functions are non-unique, the cohomological class of σ is independent of their particular choice and is an invariant of $\mathcal{S} \subset \mathcal{R}$. Moreover, any cocycle cohomologous to an index cocycle arises from a suitable selection of choice functions.

Two subrelations $\mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{R}$ are said to be $\mathcal{R}$-*conjugate* if there is a transformation $T \in N[\mathcal{R}]$ such that $(T \times T)\mathcal{S}_1 = \mathcal{S}_2$. In view of [FSZ, Theorem 1.6] $\mathcal{S}_1$ is isomorphic to $\mathcal{S}_2$ if and only if their indices are equal and their index cocycles are weakly equivalent.

Let σ stand for the index cocycle of $\mathcal{S} \subset \mathcal{R}$. Then $\mathcal{S}$ is said to be *normal* in $\mathcal{R}$ if the restriction $\sigma \restriction \mathcal{S}$ of σ to $\mathcal{S}$ is a coboundary. Equivalently, there are choice functions $\{\phi_j\}_{j\in J}$ with $\phi_j \in N[\mathcal{S}]$, $j \in J$. If, in addition, $\mathcal{R}$ is hyperfinite, then by [FSZ, §2] there is a countable amenable group $Q \subset N[S]$ with $Q \cap [\mathcal{S}] = 1_Q$ and such that $\mathcal{R}$ is generated by $\mathcal{S}$ and Q.

Definition 2.1. $\mathcal{S}$ is called *quasinormal* if σ is regular.

From now on $\mathcal{R}$ is an ergodic hyperfinite equivalence relation on $(X, \mathfrak{B}, \mu)$.

Proof of Theorem 0.1. By Proposition 1.7 there exists an index cocycle $\sigma : \mathcal{R} \to \Sigma(J)$ such that the subrelation $\mathcal{P} := \{(x, y) \in \mathcal{R} \mid \sigma(x, y) = \mathrm{Id}\}$ is ergodic. Replacing, if necessary, $\mathcal{S}$ by a $\mathcal{R}$-conjugated subrelation we may assume that σ is determined by a family of choice functions $\{\phi_j\}_{j\in J}$ with the properties: $\sigma(x, \phi_j(x))(0) = j$ for all $x \in X$, $j \in J$ and $\mathcal{S} = \{(x, y) \in \mathcal{R} \mid \sigma(x, y)(0) = 0\}$ (see [FSZ, Theorem 1.6]). Clearly, $\mathcal{P} \subset \mathcal{S}$ and $\phi_j \in N[\mathcal{P}]$, $j \in J$. Let $\{\psi_i\}_{i\in I}$ be choice functions for the pair $\mathcal{P} \subset \mathcal{S}$. We claim that $\psi_i \in N[\mathcal{P}]$. Actually, given $(x, y) \in \mathcal{P}$, we have

$$(\psi_i(x), \psi_i(y)) \in \mathcal{P} \iff \sigma(\psi_i(x), \psi_i(y))(j) = j \text{ for all } j \in J.$$

Since $\sigma(\psi_i(x), \psi_i(y)) = \sigma(\psi_i(x), \phi_j \circ \psi_i(x))\sigma(\phi_j \circ \psi_i(x), \phi_j \circ \psi_i(y))\sigma(\phi_j \circ \psi_i(y)), \psi_i(y))$ and $\sigma(\phi_j \circ \psi_i(x), \phi_j \circ \psi_i(y))(0) = 0$, we deduce that $\sigma(\psi_i(x), \psi_i(y)) = 0$ for all $j \in J$ and hence $\psi_i \in N[\mathcal{P}]$, as claimed. Notice that $\{\psi_i \circ \phi_j\}_{i\in I, j\in J}$ are choice functions for $\mathcal{P} \subset \mathcal{R}$. As in [FSZ], we define a multiplication law on $I \times J$ by setting

$$(i_1, j_1) * (i_2, j_2) = (i_3, j_3) \iff (\psi_{i_1} \circ \phi_{j_1} \circ \psi_{i_2} \circ \phi_{j_2}(x), \psi_{i_3} \circ \phi_{j_3}(x)) \in \mathcal{P} \text{ a.e.}$$

Then $(I \times J, *)$ is a countable amenable group, say Q, and $(I \times \{0\}, *)$ is a subgroup of Q, say F [FSZ]. Moreover, the map $v : Q \ni q = (i, j) \mapsto \psi_i \circ \phi_j \in N[\mathcal{P}]$ is an *outer near homomorphism*, i.e. (a) $v(q) \in [\mathcal{P}]$ if and only if $q = 1_Q$, (b) $v(q_1 * q_2) \in v(q_1)v(q_2)[\mathcal{P}]$. Since Q is amenable, there exists a map $w : Q \to [\mathcal{P}]$ such that the map $Q \ni q \mapsto v(q)w(q) \in N[\mathcal{P}]$ is an outer homomorphism [FSZ]. Thus Q can be viewed as a subgroup of $N[\mathcal{P}]$. Clearly, $\{\phi_j w((0, j))\}_{j\in J}$ are choice functions for $\mathcal{S} \subset \mathcal{R}$ (they determine the very same index cocycle σ) and $\{\psi_i w((i, 0))\}_{i\in I}$ are choice functions for $\mathcal{P} \subset \mathcal{S}$. Hence the following properties are satisfied: (a) $Q \cap [\mathcal{P}] = \mathrm{Id}$, (b) $\mathcal{R}$ is generated by $\mathcal{P}$ and Q, (c) $\mathcal{S}$ is generated by $\mathcal{P}$ and F. For $(i, j) \in Q$ and a.e. $(x, y) \in \mathcal{P}$ we have

$$\begin{aligned} &\sigma(x, \psi_i \circ \phi_j(y))(j_1) = j_2 \iff (\phi_{j_1} \circ \psi_i \circ \phi_j(x), \phi_{j_2}(y)) \in \mathcal{S} \\ &\iff \exists i_1 \in I \text{ with } (\psi_{i_1} \circ \phi_{j_1} \circ \psi_i \circ \phi_j(x), \phi_{j_2}(y)) \in \mathcal{P} \iff (i_1, j_1) * (i, j) = (0, j_2). \end{aligned}$$

It is clear that the map $\pi : Q \ni (i, j) \mapsto j \in J = F\backslash G$ is the F-quotient map taking F to $\{0\}$. Hence $\rho((i, j))(j_1) = \pi((i_1, j_1) * (i, j)) = \pi((0, j_2)) = j_2$. To put it in another way, $\sigma(x, qy)(j_1) = \rho(q)(j_1)$ for a.e. $(x, y) \in \mathcal{P}$, $q \in Q$, $j_1 \in F\backslash G$, i.e. $\sigma = \rho \circ \theta$, as desired. To complete the proof, we observe that the kernel of ρ is trivial, since $\Sigma(J)$ acts freely on J. This implies that $F \subset Q$ is irreducible. $\square$

Remark 2.2. We observe that σ takes values in $\rho(Q)$ and $\overline{\rho(Q)} \in \langle \sigma \rangle$. In a similar way, the restriction of σ to $\mathcal{S}$ takes values in $\rho(F)$ and $\overline{\rho(F)} \in \langle \sigma \restriction \mathcal{S} \rangle$.

The proof of Theorem 0.3 is divided into several lemmas.

Lemma 2.3. *Let $\mathcal{S}_1, \mathcal{S}_2$ be two ergodic quasinormal subrelations of $\mathcal{R}$. They are $\mathcal{R}$-conjugate if and only if their indices are equal and $\langle\sigma_1\rangle = \langle\sigma_2\rangle$, where σ_1 and σ_2 denote the index cocycles of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively.*

Proof follows from Theorem 1.9 and [FSZ, Theorem 1.6]. $\square$

Lemma 2.4. *Let $F_1 \subset Q_1$ and $F_2 \subset Q_2$ be two irreducible pairs of countable amenable groups corresponding to a quasinormal subrelation $\mathcal{S}$ of $\mathcal{R}$ as in Theorem* 0.1. *Then they are weakly isomorphic.*

Proof. Without loss of generality we may assume that Q_i is a transitive subgroup of $\Sigma(J)$, $F_i = \{q \in Q_i \mid q(0) = 0\}$, the index cocycle σ_i takes values in $Q_i \subset \Sigma(J)$ and has dense range in $\overline{Q_i}$, and the restriction $\sigma_i \restriction \mathcal{S}_i$ takes values in F_i and has dense range in $\overline{F_i}$, $i = 1, 2$. Since $\sigma_1 \approx \sigma_2$, there is a Borel function $\phi : X \to \Sigma(J)$ with $\sigma_1(x, y) = \phi(x)^{-1}\sigma_2(x, y)\phi(y)$ for a.e. $(x, y) \in \mathcal{R}$. Let $\tau \in \Sigma(J)$ be a proper value of ϕ. By Proposition 1.1 $\overline{Q_1} = \tau^{-1}\overline{Q_2}\tau$, $\overline{F_1} = \tau^{-1}\overline{F_2}\tau$ and hence the pairs $F_1 \subset Q_1$ and $F_2 \subset Q_2$ are weakly isomorphic. $\square$

Lemma 2.5. *Let $F \subset Q$ correspond to $\mathcal{S}$ as in Theorem* 0.1 *and T be any automorphism from $N[\mathcal{R}]$. Then $F \subset Q$ corresponds also to the $\mathcal{R}$-subrelation $(T \times T)\mathcal{S}$.*

Lemma 2.6. *For each irreducible pair of countable amenable groups $F \subset Q$ there exists a quasinormal ergodic subrelation $\mathcal{S} \subset \mathcal{R}$ such that $F \subset Q$ corresponds to $\mathcal{S}$.*

Proof. It is well known that Q can be embedded into $N[\mathcal{R}]$ in such a way that $Q \cap [\mathcal{R}] = \mathrm{Id}$. Denote by $\mathcal{R}'$ (resp. $\mathcal{S}'$) the equivalence relation generated by $\mathcal{R}$ and Q (resp. by $\mathcal{P}$ and F). Since $\mathcal{R}'$ is hyperfinite, there is a transformation $T \in \mathrm{Aut}(X, \mu)$ with $(T \times T)\mathcal{R}' = \mathcal{R}$. Clearly, the subrelation $\mathcal{S} := (T \times T)\mathcal{S}'$ is as desired. $\square$

Proof of Theorem 0.3. In view of Theorem 0.1 and Lemmas 2.4–2.6 the map $\{\mathcal{R}$-conjugacy class of $\mathcal{S}\} \mapsto \{$the weak isomorphism class of $F \subset Q$ as in Theorem 0.1$\}$ is well defined and onto. It remains to verify the injectivity. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two quasinormal subrelations of $\mathcal{R}$ such that the corresponding pairs $F_1 \subset Q_1$ and $F_2 \subset Q_2$ are weakly isomorphic. Since $\mathrm{Card}(F_1\backslash Q_1) = \mathrm{Card}(F_2\backslash Q_2)$, the $\mathcal{R}$-indices of $\mathcal{S}_1$ and $\mathcal{S}_2$ are equal. Let $\sigma_1, \sigma_2 : \mathcal{R} \to \Sigma(J)$ stand for the index cocycles of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. It is clear that $\overline{Q_1}$ and $\overline{Q_2}$ viewed as closed subgroups of $\Sigma(J)$ are conjugate. Since $\overline{Q_1} \in \langle\sigma_1\rangle$ and $\overline{Q_2} \in \langle\sigma_2\rangle$, it follows from Lemma 2.3 that $\mathcal{S}_1$ and $\mathcal{S}_2$ are $\mathcal{R}$-conjugate. $\square$

Remind that $\mathcal{S}$ is normal if $\sigma \restriction \mathcal{S}$ is a coboundary. Hence it is natural to state

Proposition 2.7. *$\mathcal{S}$ is quasinormal if and only if $\sigma \restriction \mathcal{S}$ is regular.*

Proof. ($\Longrightarrow$) Without loss of generality we may assume that σ takes values and has dense range in a closed transitive subgroup $G \subset \Sigma(J)$ and $\mathcal{S} = \{(x, y) \in \mathcal{R} \mid \sigma(x, y)(0) = 0\}$. Since $H := \{\tau \in G \mid \tau(0) = 0\}$ is an open subgroup of G and $\mathcal{S} = \sigma^{-1}(H)$, it follows that $\sigma \restriction \mathcal{S}$ has dense range in H.

($\Longleftarrow$) Let $\sigma \restriction \mathcal{S}$ take values and has dense range in a closed subgroup $H \subset \Sigma(J)$. By Proposition 1.8 we may assume that $\mathcal{P} := \{(x, y) \in \mathcal{S} \mid \sigma(x, y) = \mathrm{Id}\}$ is an ergodic subrelation. It remains to repeat the argument of Theorem 0.1 almost literally to deduce that $\mathcal{S} \subset \mathcal{R}$ has the structure described in Theorem 0.1. $\square$

3. Generic properties of subrelations

Denote by Z^1 the set of $\mathcal{R}$-cocycles with values in $\Sigma(J)$. Let λ be a $\mu_{\mathcal{R}}$-equivalent probability measure on $\mathcal{R}$. It is well known that Z^1 endowed with the topology of convergence in λ is a Polish space [FM]. This topology is unaffected if we replace λ with an equivalent probability measure. Let δ stand for Haar (σ-finite) measure on J. The group $\mathrm{Aut}(X \times J, \mu \times \delta)$ of $\mu \times \delta$-preserving automorphisms of $X \times J$ is Polish when endowed with the weak topology. Remind that $R_n \to R$ *weakly* in $\mathrm{Aut}(X \times J, \mu \times \delta)$ if $(\mu \times \delta)(R_n A \triangle RA) + (\mu \times \delta)(R_n^{-1} A \triangle R^{-1} A) \to 0$ as $n \to \infty$ for each Borel subset $A \subset X \times J$ with $(\mu \times \delta)(A) < \infty$. By [CK] the ergodics, say $\mathcal{E}$, form a dense G_δ in $\mathrm{Aut}(X \times J, \mu \times \delta)$. Since $\mathcal{R}$ is hyperfinite, there exists an ergodic transformation $T \in \mathrm{Aut}(X, \mu)$ such that $\mathcal{R}$ is the T-orbital equivalence relation. Consider the map $\Phi : Z^1 \ni \alpha \mapsto T_\alpha \in \mathrm{Aut}(X \times J, \mu \times \delta)$, where T_α is given by $T_\alpha(x, j) = (Tx, \alpha(x)(j))$. It is a routine to verify that Φ is continuous. Let Z^1_{ind} stand for the set of index cocycles, i.e.

$$Z^1_{\mathrm{ind}} = \{\alpha \in Z^1 \mid \alpha \text{ is the index cocycle of some ergodic subrelation } \mathcal{S} \subset \mathcal{R}\}$$

Clearly, $Z^1_{\mathrm{ind}} \neq \emptyset$. Since by [FSZ, Proposition 1.5 and Theorem 1.6(a)] $Z^1_{\mathrm{ind}} = \Phi^{-1}(\mathcal{E})$, it follows that Z^1_{ind} is a G_δ in Z^1 and hence a Polish space when endowed with the induced topology. We set

$$Z^1_{\max} := \{\alpha \in Z^1 \mid \alpha \text{ is quasinormal and } \langle \alpha \rangle = \{\Sigma(J)\}\}.$$

Let Q be the group of finite permutations of J, $F := \{\tau \in Q \mid \tau(0) = 0\}$, and $\mathcal{S}$ the quasinormal subrelation of $\mathcal{R}$ corresponding to $F \subset Q$ by Theorem 0.3. Since Q is dense in $\Sigma(J)$, the index cocycle of $\mathcal{S}$ belongs to $Z^1_{\max}$ and hence $Z^1_{\max} \neq \emptyset$. Only a slight modification of the routine argument from [PS] or [CHP, Theorem 3] is needed to prove that $Z^1_{\max}$ is a dense G_δ in Z^1. Since $Z^1_{\max} \subset Z^1_{\mathrm{ind}}$, we obtain

Proposition 3.1. *$Z^1_{\max}$ is a dense G_δ in Z^1_{ind}.*

In view of this statement it is of interest to give an example of non quasinormal ergodic subrelation.

Example 3.2. Let $(X, \mu) = (\{0, 1\}, \lambda)^{\mathbb{Q}}$, where λ is the equidistribution on $\{0, 1\}$. Let $H := \mathbb{Q} \rtimes \mathbb{Z}$ with multiplication as follows

$$(q, n)(p, m) = (q + 2^n p, n + m).$$

We define an action of H on X by setting $(hx)_p = x_{2^{-n}(p-q)}$ for all $p \in \mathbb{Q}$, where $h = (q, n)$ and $x = (x_p)_{p \in \mathbb{Q}}$. Clearly, (X, μ) is an ergodic H-space. Hence the Cartesian square $(Z, \nu) = (X, \mu) \times (X, \mu)$ is an ergodic H^2-space. Denote by $\mathcal{R}$ the H^2-orbit equivalence relation. Since H is amenable, $\mathcal{R}$ is hyperfinite. Consider the homomorphism $\pi : H^2 \to \Sigma(H)$ given by $\pi(h_1, h_2)(h) = h_1 h h_2^{-1}$. It is easy to verify that the kernel of π is isomorphic to the center of H. Since the center is trivial, π is one-to-one. We put

$$W := \{\tau \in \Sigma(H) \mid \tau(1_H) = 1_H,\ \tau(h') = h',\ \text{and } \tau(h'') = h''\},$$

where $h' = (1, 0)$ and $h'' = (0, 1)$. Clearly, W is an open neighborhood of the identity in $\Sigma(H)$. It is a routine to verify that $\pi(s)W \cap W = \emptyset$ for every nontrivial

$s \in H^2$. Now we define a cocycle $\sigma : \mathcal{R} \to \Sigma(H)$ by setting $\sigma(z, sz) = \pi(s)^{-1}$, $z \in Z$ and $s \in H^2$. Let $\mathcal{S} := \{(z, y) \in \mathcal{R} \mid \sigma(z, y)(1_H) = 1_H\}$. Notice that $\mathcal{S}$ is ergodic, since it contains an ergodic subrelation generated by the diagonal (Bernoulli) $\mathbb{Q}$-action on $Z = X \times X$. For each $h \in H$, we define a map $\phi_h : Z \to Z$ by setting $\phi_h(z) = sz$, where $s = (1_H, h) \in H^2$. Then $\sigma(z, \phi_h(z))(0) = h$, i.e. $\{\phi_h\}_{h\in H}$ are choice functions for $\mathcal{S} \subset \mathcal{R}$ and σ is the corresponding cocycle. We claim that $\mathcal{S}$ is not quasinormal in $\mathcal{R}$. Suppose the contrary: there exists a closed subgroup $G \subset \Sigma(H)$ and a Borel map $\phi : Z \to \Sigma(H)$ such that the cocycle $\beta : \mathcal{R} \ni (z, y) \mapsto \phi(z)^{-1}\sigma(z, y)\phi(y) \in \Sigma(H)$ takes values and has dense range in G. Choose an open set $U \subset \Sigma(H)$ and a neighborhood $O \subset \Sigma(H)$ of Id_H such that $UOU^{-1} \subset W$ and $\nu(\phi^{-1}(U)) > 0$. By assumption, there are a subset $A \subset Z$ and a nontrivial $s \in H^2$ with $\nu(A) > 0$, $A \cap sA \subset \phi^{-1}(U)$, and $\beta(sz, z) \in O$ for all $z \in A$. Then $W \not\ni \pi(s) = \sigma(sz, z) \in UOU^{-1} \subset W$ for all $z \in A$, a contradiction.

References

[CFW] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergod. Th. and Dynam. Sys. **1** (1981), 431–450.

[CK] J. Choksi and S. Kakutani, *Residuality of ergodic measurable transformations and of ergodic transformations which preserve an infinite measure*, Indiana Univ. Math. J. **28** (1979), 453–469.

[CHP] J. R. Choksi, J. M. Hawkins, and V. S. Prasad, *Abelian cocycles for nonsingular ergodic transformations and genericity of type III_1 transformations*, Mh. Math. **103** (1987), 187–205.

[Da] A. I. Danilenko, *Comparison of cocycles of measured equivalence relations and lifting problems*, Ergod. Th. and Dynam. Sys. **18** (1998), 125–151.

[Dy] H. A. Dye, *On groups of measure preserving transformations.* I, Amer. J. Math. **81** (1959), 119–159; II, Amer. J. Math. **85** (1963), 551–576.

[FM] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras.* I, Trans. Amer. Math. Soc. **234** (1977), 289–324.

[FSZ] J. Feldman, C. E. Sutherland, and R. J. Zimmer, *Subrelations of ergodic equivalence relations*, Ergod. Th. and Dynam. Syst. **9** (1989), 239–269.

[GLS] P. Gabriel, M. Lemańczyk, and K. Schmidt, *Extensions of cocycles for hyperfinite actions and applications*, Mh. Math. (to appear).

[Ge] M. Gerber, *Factor orbit equivalence and classification of finite extensions of ergodic transformations*, Isr. J. Math. **57** (1987), 28–48.

[GS] V. Ya. Golodets and S. D. Sinelshchikov, *Classification and structure of cocycles of amenable ergodic equivalence relation*, J. Funct. Anal. **121** (1994), 455–485.

[Jo] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.

[JT] V. F. R. Jones and M. Takesaki, *Actions of compact Abelian groups on semi-finite injective factors*, Acta Math. **153** (1984), 213–258.

[PS] K. R. Parthasarathy and K. Schmidt, *On the cohomology of a hyperfinite action*, Mh. Math. **84** (1977), 37–48.

[Su] C. Sutherland, *Notes on orbit equivalence; Krieger's theorem*, Lecture Notes Ser., vol. 23, Istitute of Mathematics, University of Oslo, Norway, 1976.

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