ON COCYCLES WITH VALUES IN GROUP EXTENSIONS. GENERIC RESULTS

Alexandre I. Danilenko

Abstract. Let $1 \to A \to E \to G \to 1$ be a short exact sequence of locally compact groups, A amenable. Given a recurrent G-valued cocycle ω of an ergodic nonsingular transformation, we consider the subset of those E-valued cocycles whose G-projection is ω. It is proved that for a generic cocycle from this subset the restriction of the associated (Mackey) E-action to A is trivial. This improves the results of K. Dajani [D1, D3] and answers a question from [D1].

0. Introduction

Consider a short exact sequence of locally compact second countable (l.c.s.c.) groups

$$1 \to A \to E \xrightarrow{\pi} G \to 1. \tag{0-1}$$

Let T be an ergodic transformation of a Lebesgue space. We endow the subset of all E-valued cocycles of T with the (Polish) topology of convergence in measure. It is well known that for a generic E-valued cocycle of T the associated Mackey action of E is trivial [PS, CHP]. We study in this paper a more subtle problem. Given a G-valued cocycle ω of T, we denote by Z^1_ω the closed subset of E-valued cocycles whose G-quotient is ω. Our purpose is to show that for a generic cocycle from Z^1_ω the associated Mackey E-action restricted to A is trivial, i.e. A is contained in the stability group of a.e. point, or, equivalently, A is a subgroup of the kernel of the unitary representation of E generated by this action (provided that ω is recurrent and A amenable). This strengthens the main result of K. Dajani from [D3], where it was proved that a generic cocycle from Z^1_ω is recurrent and the intersection $\overline{r}_A$ of its essential range (i.e. an analogue of the Krieger-Araki-Woods asymptotic ratio set) with A contains infinity. (She additionally assumed that A is Abelian and noncompact and (0-1) splits.) She also asked in [D1] whether the subset of cocycles from Z^1_ω with $\overline{r}_A = \{1, \infty\}$ is residual? As it follows from our result stating that generically $\overline{r}_A = G \cup \infty$ the answer is quite opposite: this subset is of first category.

Notice that the main result of this paper turned out to be useful in studying the centralizer of ergodic skew products [DL]. We also study the A-cohomology relation on the space of E-valued cocycles: two cocycles are A-cohomologous if they are cohomologous in an ordinary sense and, besides, admit a transfer function with values in A. It suffices to restrict ourselves to studying Z^1_ω, since it is saturated with

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respect to the A-cohomology relation—see Sections 2, 3 for details. Remark that the subject of this work is related closely to the theory of the so-called 'H-cocycles' elaborated by several authors [D1–D3, Ul, Be]. In this paper we demonstrate a different approach to the problems considered there: we first translate them into the language of usual cocycles and then apply a wealth of the low dimensional cohomology theory for hyperfinite equivalence relations [FM, S1, S3, Z2, GS1, GS2, Da]. This permits us to extend and refine many of the results from [D1–D3, Be] providing them with new short proofs.

The outline of the paper is as follows. Section 1 contains a background on measured equivalence relations and their cohomology. In Section 2 we find an "ergodic" criterion for the sequence (0-1) to split (Theorem 2.4). We also find conditions under which an ergodic A-action can be extended up to an E-action on the same measure space (Proposition 2.5, cf. [Be]). Furthermore, the A-cohomology relation for cocycles is studied. For example, A-analogues of the essential range and the normalized essential range of a cocycle from Z^1_ω are investigated in Proposition 2.7. (The second one is an invariant for the A-cohomology). We also consider a natural map from Z^1_ω to the set of cocycles of the ω-skew product extension of T and study its functoral (cohomological) properties in Remark 2.2 and Theorem 2.3. The interplay between these results and the theory of 'H-cocycles' is explained in details in Section 3. The last Section 4 contains the main result of the paper—Theorem 4.4—(about the generic property of Z^1_ω) and explains its relation to K. Dajani's work.

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1. Preliminaries

Let $(X, \mathfrak{B}, \mu)$ be a standard probability space. Denote by $\mathrm{Aut}(X, \mu)$ the group of its automorphisms, i.e. Borel, one-to-one, onto, μ-nonsingular transformations. Throughout this paper we do not distinguish between two measurable maps which agree on a conull subset. Given a Borel discrete μ-nonsingular equivalence relation $\mathcal{R} \subset X \times X$, we endow it with the induced Borel structure and the σ-finite measure $\mu_\mathcal{R}$, $d\mu_\mathcal{R}(x, y) = d\mu(x)$, $(x, y) \in \mathcal{R}$, and write

$$[\mathcal{R}] = \{\gamma \in \mathrm{Aut}(X, \mu) \mid (\gamma x, x) \in \mathcal{R} \text{ for } \mu\text{-a.a. } x \in X\},$$
$$N[\mathcal{R}] = \{\theta \in \mathrm{Aut}(X, \mu) \mid (\theta x, \theta y) \in \mathcal{R} \text{ for } \mu_\mathcal{R}\text{-a.a. } (x, y) \in \mathcal{R}\}$$

for the *full group* of $\mathcal{R}$ and the *normalizer* of $[\mathcal{R}]$ respectively. For a countable subgroup Γ of $\mathrm{Aut}(X, \mu)$ we denote by $\mathcal{R}_\Gamma$ the Γ-orbital equivalence relation (and it is known that each $\mathcal{R}$ is of the form $\mathcal{R}_\Gamma$ [FM]). $\mathcal{R}$ is called *hyperfinite* if it is generated by a single transformation. $\mathcal{R}$ is *of type I* if the $\mathcal{R}$-partition of X is measurable. $\mathcal{R}$ is *conservative* if μ-a.e. ergodic component of this partition is properly ergodic.

Let G be a l.c.s.c. group, 1_G the identity in G, and λ_G the right Haar measure on G. A Borel map $\alpha : \mathcal{R} \to G$ is a (1-)*cocycle* of $\mathcal{R}$ if

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z)$$

for a.e. $(x, y), (y, z) \in \mathcal{R}$. Two cocycles, $\alpha, \beta : \mathcal{R} \to G$, are *cohomologous* ($\alpha \approx \beta$), if

$$\alpha(x, y) = \phi(x)^{-1}\beta(x, y)\phi(y),$$

for $\mu_{\mathcal{R}}$-a.e. $(x,y)\in\mathcal{R}$, where $\phi: X\to G$ is a Borel function (we call it a *transfer* function). A cocycle is a *coboundary* if it is cohomologous to the trivial one. The set of cocycles of $\mathcal{R}$ with values in G will be denoted by $Z^1(\mathcal{R},G)$. It is well known that $Z^1(\mathcal{R},G)$ endowed with the topology of convergence in $\mu_{\mathcal{R}}$ is a Polish space. By $H^1(\mathcal{R},G)$ we denote the quotient space $Z^1(\mathcal{R},G)/\approx$.

Let $\mathcal{R}=\mathcal{R}_\Gamma$. There is a cocycle $\rho_\mu\in Z^1(\mathcal{R},\mathbb{R})$ such that $\rho_\mu(x,\gamma x)=\log\frac{d\mu\circ\gamma}{d\mu}(x)$ for all $\gamma\in\Gamma$ at μ-a.e. $x\in X$. It is called the *Radon-Nikodym* cocycle of $\mathcal{R}$. If $\rho\equiv 1$ then μ is $[\mathcal{R}]$-invariant. $\mathcal{R}$ is said to be of type II_1 (II_∞) if there exists a finite (infinite, σ-finite) $[\mathcal{R}]$-invariant measure on X equivalent to μ. Otherwise $\mathcal{R}$ is *of type* III.

Let $\mathcal{R}$ and $\mathcal{S}$ be two equivalence relations on measure spaces $(X,\mathfrak{B},\mu)$ and $(Y,\mathfrak{G},\nu)$ respectively and some cocycles $\alpha\in Z^1(\mathcal{R},G)$ and $\beta\in Z^1(\mathcal{S},G)$ be given. The pairs $(\mathcal{R},\alpha)$ and $(\mathcal{S},\beta)$ (or, simply, α and β) are *weakly equivalent* if there is a Borel isomorphism $\theta: X\to Y$ so that $\mu\sim\nu\circ\theta$, $(\theta\times\theta)(\mathcal{R})=\mathcal{S}$, and $\alpha\approx\beta\circ\theta$, where the cocycle $\beta\circ\theta\in Z^1(\mathcal{R},G)$ is defined by $\beta\circ\theta(x,z)=\beta(\theta x,\theta z)$.

Given a cocycle $\alpha\in Z^1(\mathcal{R},G)$, we define an equivalence relation $\mathcal{R}\times_\alpha G$ on $(X\times G,\mu\times\lambda_G)$ by setting $(x,g)\sim(y,h)$ if $(x,y)\in\mathcal{R}$ and $h=g\alpha(x,y)$. It is called the *α-skew product extension* of $\mathcal{R}$. We define a Borel nonsingular action V_α of G on $(X\times G,\mu\times\lambda_G)$ as follows $V_\alpha(h)(x,g)=(x,hg)$. It is clear that $V_\alpha(h)\in N[\mathcal{R}\times_\alpha G]$ for all $h\in G$. Hence V_α induces a new G-action on the measure space of $\mathcal{R}\times_\alpha G$-ergodic components. It is called the *Mackey action* of G associated to α. We denote it by W_α. Remark that W_α is ergodic if and only if $\mathcal{R}$ is. If two cocycles, α,β, are weakly equivalent, then W_α and W_β are conjugate.

α is *recurrent* (resp. *transient*) if $\mathcal{R}\times_\alpha G$ is conservative (resp. of type I). α has *dense range* in G if W_α is the trivial action on a one-point set, i.e. the α-skew product extension of $\mathcal{R}$ is ergodic. It is well known that the properties of cocycles to be recurrent, transient, or to have dense range are invariant under the weak equivalence.

We need the following results. Associate to every cocycle $\alpha\in Z^1(\mathcal{R},G)$ a *double cocycle* $\alpha_0=\alpha\times\rho_\mu\in Z^1(\mathcal{R},G\times\mathbb{R})$, where ρ_μ the Radon-Nikodym cocycle of $\mathcal{R}$.

Uniqueness Theorem for Cocycles [GS2, Theorem 3.1]**.** *Let $\mathcal{R}$ and $\mathcal{S}$ be two ergodic hyperfinite equivalence relations on standard probability spaces $(X,\mathfrak{B},\mu)$ and $(Y,\mathfrak{C},\nu)$ respectively, and $\alpha\in Z^1(\mathcal{R},G)$, $\beta\in Z^1(\mathcal{S},G)$ recurrent cocycles.*

(i) *If $\mathcal{R}$ and $\mathcal{S}$ are both of type II_1 or II_∞ and the Mackey actions W_α and W_β of G are conjugate then α and β are weakly equivalent.*
(ii) *If $\mathcal{R}$ and $\mathcal{S}$ are both of type III and the Mackey actions W_{α_0} and W_{β_0} of the group $G\times\mathbb{R}$ associated to the double cocycles α_0 and β_0 respectively are conjugate then α and β are weakly equivalent.*

Existence Theorem for Cocycles [GS1, Corollary]**.** *Let G be a l.c.s.c. group and let an amenable nonsingular ergodic action V of $G\times\mathbb{R}$ be given. Then there are a discrete ergodic hyperfinite equivalence relation $\mathcal{R}$ and a recurrent cocycle $\alpha\in Z^1(\mathcal{R},G)$ such that V is conjugate to the Mackey action associated to α_0.*

Theorem on Amenability of Group Actions [GS1, p.523]**.** *Let $V=\{V(g)\}_{g\in G}$ be a nonsingular ergodic action of a l.c.s.c. group G on a standard measure space (X,μ). Then V is amenable if and only if the $V(G)$-orbital equivalence relation on X is amenable and for μ-a.e. $x\in X$ the stability group $G_x\overset{\mathrm{def}}{=}\{g\in G\mid V(g)x=x\}$ at x is amenable.*

The definitions of an amenable equivalence relation and an amenable group action can be found in [Z1, Z2]. We remark that a discrete equivalence relation $\mathcal{R}$ is amenable if and only if it is hyperfinite [CFW]. For such a $\mathcal{R}$ and an arbitrary l.c.s.c. group G the Mackey action associated to every cocycle $\alpha \in Z^1(\mathcal{R}, G)$ is amenable [Z1, Theorem 3.3].

Denote by $\overline{G}$ the one point compactification of G. An element $g \in \overline{G}$ is an *essential value* of α if for every neighborhood U of g in $\overline{G}$ and every subset $B \subset X$ of positive measure there is a subset $C \subset \mathcal{R} \cap (B \times B)$ with $\mu_{\mathcal{R}}(C) > 0$ and $\alpha(C) \subset U$. By $\overline{r}(\alpha)$ we denote the *essential range* of α, i.e. the set of all its essential values. Notice that $r(\alpha) \stackrel{\text{def}}{=} \overline{r}(\alpha) \cap G$ is a closed subgroup of G and α has dense range in G if and only if $r(\alpha) = G$. We put also $r^{(\text{nor})}(\alpha) \stackrel{\text{def}}{=} \bigcap_{g \in G} g^{-1} r(\alpha) g$ for the *normalized essential range* of α. Notice that $r^{(\text{nor})}(\alpha)$ is the kernel of the unitary representation of G generated by W_α. If α and β are weakly equivalent cocycles then $r^{(\text{nor})}(\alpha) = r^{(\text{nor})}(\beta)$.

For a more detailed exposition of these concepts we refer to [FM, S1, S3, Z2].

2. Cocycles with values in group extensions

Turn back to the sequence (0-1). We have a continuous map

$$\pi_* : Z^1(\mathcal{R}, E) \ni \alpha \mapsto \pi_*(\alpha) \in Z^1(\mathcal{R}, G),$$

where $\pi_*(\alpha)(x, y) = \pi(\alpha(x, y))$ for all $(x, y) \in \mathcal{R}$ (throughout this paper $Z^1(\mathcal{R}, G)$ is endowed with the topology of convergence in measure). It is clear that π_* factors through the cohomology relations. Moreover, $\pi_*(\alpha \circ \theta) = (\pi_*(\alpha)) \circ \theta$ for each automorphism $\theta \in N[\mathcal{R}]$. Hence if two cocycles $\alpha, \beta \in Z^1(\mathcal{R}, E)$ are weakly equivalent, the same is also valid for $\pi_*(\alpha)$ and $\pi_*(\beta)$.

Now for a cocycle $\omega \in Z^1(\mathcal{R}, G)$ we put $Z^1_\omega(\mathcal{R}, E) \stackrel{\text{def}}{=} \pi_*^{-1}(\omega)$ and endow this set with the induced (Polish) topology. It is easy to see that $Z^1_\omega(\mathcal{R}, E) \neq \varnothing$ whenever $\mathcal{R}$ is hyperfinite.

Definition 2.1. We say that two cocycles $\alpha, \beta \in Z^1(\mathcal{R}, E)$ are *A-cohomologous* ($\alpha \approx_A \beta$) if $\alpha \approx \beta$ and there is a transfer function with values in A.

Notice that $\approx_A$ is an equivalence relation on $Z^1(\mathcal{R}, E)$ and $Z^1_\omega(\mathcal{R}, E)$ is $\approx_A$-saturated, i.e. if $\alpha \in Z^1_\omega(\mathcal{R}, E)$ and $\beta \in Z^1(\mathcal{R}, E)$ then $\alpha \approx_A \beta$ implies $\beta \in Z^1_\omega(\mathcal{R}, E)$. We put $H^1_\omega(\mathcal{R}, E) = Z^1_\omega(\mathcal{R}, E)/{\approx_A}$. Let $s : G \to E$ be a Borel normalized cross-section of π (which means that $\pi \circ s = \text{Id}$ and $s(1_G) = 1_E$). Suppose that ω is weakly equivalent to a cocycle ω_1, i.e. $\omega_1 \approx \omega \circ \theta$ with a transfer function $\phi : X \to G$ and an automorphism $\theta \in N[\mathcal{R}]$ (we assume for simplicity's sake that ω and ω_1 are defined on the same equivalence relation). Then the map $Z^1_\omega(\mathcal{R}, E) \ni \alpha \mapsto \alpha_1 \in Z^1_{\omega_1}(\mathcal{R}, E)$ is a homeomorphism, where $\alpha_1(x, y) = s(\phi(x))^{-1} \alpha \circ \theta(x, y) s(\phi(y))$. This map passes through the $\approx_A$-cohomology relation and generates the canonical map $H^1_\omega(\mathcal{R}, E) \to H^1_{\omega_1}(\mathcal{R}, E)$.

Let $\lambda_A, \lambda_E, \lambda_G$ be right Haar measures on A, E, G (resp.) and $\widehat{\lambda}_A, \widehat{\lambda}_G$ probability measures equivalent to λ_A, λ_G (resp.). Then the map

$$q_s : X \times G \times A \ni (x, g, a) \mapsto q_s(x, g, a) = (x, as(g)) \in X \times E \tag{2-1}$$

is a Borel isomorphism sending the measure $\mu \times \widehat{\lambda}_G \times \widehat{\lambda}_A$ to $\mu \times \widehat{\lambda}_E$, where $\widehat{\lambda}_E$ is a λ_E-equivalent probability measure on E. We call q_s the *s-map*. If two arbitrary points $(x, as(g)), (y, bs(h)) \in X \times E$ are $\mathcal{R} \times_\alpha E$-equivalent, we obtain easily that $b = as(g)\alpha(x,y)s(h)^{-1}$ and hence $h = g\omega(x,y)$ (we assume that $a, b \in A$ and $g, h \in G$). Notice that the map

$$\alpha^{(s)} : \mathcal{R} \times_\omega G \ni ((x,g),(y,h)) \mapsto \alpha^{(s)}((x,g),(y,h)) \stackrel{\text{def}}{=} s(g)\alpha(x,y)s(h)^{-1} \in A \tag{2-2}$$

is a cocycle from $Z^1(\mathcal{R} \times_\omega G, A)$. Hence we deduce from (2-1) and (2-2) that

$$\mathcal{R} \times_\alpha E = (q_s \times q_s)((\mathcal{R} \times_\omega G) \times_{\alpha^{(s)}} A). \tag{2-3}$$

In view of this the cocycles α and $\alpha^{(s)}$ are transient, recurrent, or have dense ranges simultaneously.

Next, if we choose another normalized cross-section $s_1 : G \to E$ then $\alpha^{(s_1)} \approx \alpha^{(s)}$. Thus the cohomology class of $\alpha^{(s)}$ is well defined by (0-1) and ω only. Notice also that $\alpha^{(s)} \approx \beta^{(s)}$ for every pair of A-cohomologous cocycles from $Z^1_\omega(\mathcal{R}, E)$. Hence (0-1) and ω well define the canonical map

$$H^1_\omega(\mathcal{R}, E) \ni [\alpha] \mapsto [\alpha^{(s)}] \in H^1(\mathcal{R} \times_\omega G, A).$$

Its image in $H^1(\mathcal{R} \times_\omega G, A)$ will be denoted by $\mathrm{Inv}(\omega)$.

Recall that a transformation $\theta \in N[\mathcal{R}]$ is *compatible with* ω if $\omega \circ \theta \approx \omega$ [DG, Da]. Denote by $\omega(\theta)$ the set of all transfer functions for the pair $\omega, \omega \circ \theta$. The group of all ω-compatible transformations is denoted by $D(\mathcal{R}, \omega)$. Let $L(\mathcal{R}, \omega)$ be the set of all transformations of $(X \times G, \mu \times \lambda_G)$ of the form $\widehat{\theta}(x,g) = (\theta x, g\phi(x))$ for some transformation $\theta \in D(\mathcal{R}, G)$ and function $\phi \in \omega(\theta)$. One can easily see that $L(\mathcal{R}, \omega)$ is a subgroup of $N[\mathcal{R} \times_\omega G]$.

Remark 2.2. Suppose that two cocycles, $\alpha, \beta \in Z^1_\omega(\mathcal{R}, E)$ are weakly equivalent, i.e. there exists a transformation $\theta \in N[\mathcal{R}]$ such that $\beta \circ \theta \approx \alpha$ with a transfer function $\psi : X \to E$. This implies that $\theta \in D(\mathcal{R}, \omega)$ and the function $\phi : X \ni x \mapsto \pi(\psi(x)) \in G$ lies in $\omega(\theta)$. We define two measure space transformations $\widetilde{\theta} : (X \times E, \mu \times \lambda_E) \to (X \times E, \mu \times \lambda_E)$ and $\widehat{\theta} : (X \times G, \mu \times \lambda_G) \to (X \times G, \mu \times \lambda_G)$ by setting $\widetilde{\theta}(x,e) = (\theta x, e\psi(x))$ and $\widehat{\theta}(x,g) = (\theta x, g\pi(\phi(x)))$. It is easy to verify (and well known) that $(\widetilde{\theta} \times \widetilde{\theta})(\mathcal{R} \times_\alpha E) = \mathcal{R} \times_\beta E$. Since $\widehat{\theta} \in L(\mathcal{R}, \omega)$ and $q_s^{-1}\widetilde{\theta}q_s(x,g,a) = (\widehat{\theta}(x,g), af(x,g))$ for a measured function $f : X \times G \to A$, it follows from (2-3) that $\beta^{(s)} \circ \widehat{\theta} \approx \alpha^{(s)}$ with f being a transfer function. Thus we obtain that the cocycles $\alpha^{(s)}$ and $\beta^{(s)}$ are also weakly equivalent and, besides, admit an intertwining transformation—we mean $\widehat{\theta}$—from $L(\mathcal{R}, \omega)$.

In the case of Abelian A this observation may be refined. Notice first of all that A is a G-module ($g \cdot a \stackrel{\text{def}}{=} s(g)as(g)^{-1}$ for all $g \in G$ and $a \in A$). Denote by $\mathrm{Aut}\, E$ the group of continuous group automorphisms of E and set up

$$\mathrm{Aut}\,(E; A, G) = \{l \in \mathrm{Aut}\, E \mid l(a) = a \text{ for all } a \in A \text{ and } l(eA) = eA \text{ for all } e \in E\}.$$

It is easy to verify (and well known) that an automorphism l of E belongs to this subgroup if and only if $l(e) = ev(\pi(e))$ for all $e \in E$, where the map $v : G \to A$ is a continuous *skew homomorphism*, i.e. $v(gh) = h^{-1} \cdot v(g) + v(h)$ for all $g, h \in G$.

Given a cocycle $\alpha : \mathcal{R} \to E$ and an automorphism l of E, we denote by $l_*(\alpha)$ the l-image of α, i.e. $l_*(\alpha) : \mathcal{R} \ni (x,y) \mapsto l(\alpha(x,y)) \in E$.

Theorem 2.3. *Let $\mathcal{R}$ be ergodic and hyperfinite, A Abelian, G countable, and ω have dense range in G. Given two cocycles, $\alpha, \beta \in Z^1_\omega(\mathcal{R}, E)$, we have $\alpha^{(s)} \approx \beta^{(s)}$ if and only if $l(\alpha) \approx_A \beta$ for an automorphism $l \in$Aut$(E; A, G)$. In a similar way, $\alpha^{(s)} \approx \beta^{(s)} \circ \widehat{\theta}$ for a transformation $\widehat{\theta} \in L(\mathcal{R}, \omega)$ if and only if $l_*(\alpha) \approx_A \beta \circ \theta$ for an automorphism $l \in$Aut$(E; A, G)$, where the transformation $\theta \in D(\mathcal{R}, \omega)$ is the "first coordinate" of $\widehat{\theta}$, i.e. $\widehat{\theta}(x, .) = (\theta x, .)$ a.e.*

Proof. We prove only the first conclusion of the theorem, since the second one is similar.

($\Longrightarrow$) Let $\alpha^{(s)} \approx \beta^{(s)}$. In view of (2-2) and (2-3) there is a Borel function $f : X \times G \to A$ with

$$\alpha(x, y) = f(x, g)^{-1}\beta(x, y)f(y, g\omega(x, y)) \tag{2-4}$$

for $\mu_{\mathcal{R}} \times \lambda_G$-a.e. $(x, y, g) \in \mathcal{R} \times G$. Since we are free to replace ω by a cohomologous cocycle, we may assume that the subrelation $\mathcal{S} = \{(x, y) \in \mathcal{R} \mid \omega(x, y) = 1_G\}$ is ergodic [GS2, Lemma 1.6]. It follows from the exactness of (0-1) that the restrictions of α and β to $\mathcal{S}$ take values in A. We deduce from (2-4) that

$$\alpha(x, y) = -f(x, g)^{-1} + \beta(x, y) + f(y, g)$$

for a.e. $(x, y, g) \in \mathcal{S} \times G$. Thus $\alpha \restriction \mathcal{S} \approx_A \beta \restriction \mathcal{S}$. Moreover, since $\mathcal{S}$ is ergodic, the transfer function is determined up to an additive constant (see [DG, §1.3]). Hence $f(x, g) = \phi(x) + v(g)$ for some Borel maps $\phi : X \to A$ and $v : G \to A$. We put $\widetilde{\beta}(x, y) = \phi(x)^{-1}\beta(x, y)\phi(y)$ for $(x, y) \in \mathcal{R}$. Perturb ϕ (and hence v), if necessary, by a constant function to deduce from (2-4) and the Fubini theorem that

$$\alpha(x, y) = \widetilde{\beta}(x, y)v(\omega(x, y)) \tag{2-5}$$

for a.e. $(x, y) \in \mathcal{R}$. This implies

$$\begin{aligned} v(\omega(x, y)) &= \widetilde{\beta}(y, x)\alpha(x, y) = \widetilde{\beta}(y, z)\widetilde{\beta}(z, x)\alpha(x, z)\alpha(z, y) = \\ &\widetilde{\beta}(y, z)v(\omega(x, z))\widetilde{\beta}(z, y)v(\omega(x, y)) = \widetilde{\beta}(y, z)v(\omega(x, z))\widetilde{\beta}(y, z)^{-1} + v(\omega(z, y)) = \\ &\qquad \omega(y, z) \cdot v(\omega(x, z)) + v(\omega(z, y)) \end{aligned}$$

for a.e. (x, y), $(y, z) \in \mathcal{R}$ (we use the fact that $\pi_*(\widetilde{\beta}) = \omega$). Since ω has dense range in G, it is a routine to deduce from this equality (apply the standard exhaustion argument) that $v(gh) = h^{-1} \cdot v(g) + v(h)$ for all $(g, h) \in G \times G$. It follows from (2-5) that $l^{-1}(\alpha) \approx_A \beta$, where l is the automorphism from Aut$(E; A, G)$ associated to v.

($\Longleftarrow$) Let $l(\alpha) \approx_A \beta$ with a transfer function $\phi : X \to A$. Denote by $v : G \to A$ the skew homomorphism associated to l^{-1} and put $f(x, g) = v(g) + \phi(x)$, $x \in X$, $g \in G$. It is easy to check that (2-4) is satisfied. But this is equivalent to the relation $\alpha^{(s)} \approx \beta^{(s)}$ and we are done. □

I was informed by S. Bezuglyi that he considered problems which are close to Remark 2.2 and Theorem 2.3 in the case of Abelian A and topologically trivial extension (0-1). But his approach is different. Notice that Theorem 2.3 holds also for continuous G, but the proof is a little complicated and will be omitted.

The next statement provides an "ergodic description" of split short exact sequences of l.c.s.c. groups (from now on we do not assume that A is Abelian).

Theorem 2.4. *Let $\mathcal{R}$ be ergodic and ω have dense range in G. Then* (0-1) *splits if and only if* $\operatorname{Inv}(\omega)$ *contains the coboundary class.*

Proof. Assume that (0-1) splits and s is a (continuous) group homomorphism. Then $s_*(\omega) \in Z^1_\omega(\mathcal{R}, E)$ and it follows from (2-2) that $(s_*(\omega))^{(s)}$ is trivial, as desired. Conversely, suppose that there are a measurable function $\eta : X \times G \to E$ and a cocycle $\alpha \in Z^1_\omega(\mathcal{R}, E)$ with $\alpha(x,y) = \eta(x,g)^{-1}\eta(y, g\omega(x,y))$. We let $F(x,g,h) = \eta(x,g)\eta(x,hg)^{-1}$ for all $(x,h,g) \in X \times G \times G$. Then $F(x,g,h) = F(y, g\omega(x,y), h)$ a.e. Since ω has dense range in G, it follows that $F(x,g,h) = l(h)$ a.e. for some measurable function $l : G \to E$. Reproduce the argument of [Da, Theorem 5.3] to obtain that $\eta(x,g) = l(g)a(x)$, where $l : G \to E$ is a continuous homomorphism and $a : X \to E$. Hence $\alpha(x,y) = a(x)^{-1}l(\omega(x,y))a(y)$, $(x,y) \in \mathcal{R}$. We obtain that $\omega \approx (\pi \circ l)_*(\omega)$. It follows that $(\omega \times \omega) \approx (\omega \times (\pi \circ l)_*(\omega))$ as cocycles of $\mathcal{R}$ with values in the group $G \times G$. The two cocycles take values and have dense ranges in the subgroups $r(\omega \times \omega) = \{(g,g) \mid g \in G\}$ and $r(\omega \times (\pi \circ l)_*(\omega)) = \{(g, \pi(l(g))) \mid g \in G\}$ respectively. Hence the Mackey actions associated to these cocycles are the transitive actions of $G \times G$ on the homogeneous spaces generated by these subgroups respectively. Since the cocycles are cohomologous, the Mackey actions are conjugate and hence these subgroups are conjugate in $G \times G$. It is easy to deduce that $\pi \circ l \circ \tau = \mathrm{Id}$ for some inner automorphism τ of G, i.e. $l \circ \tau$ is a cross-section of π. $\square$

Let $p_1 : (X \times E, \mu \times \widetilde{\lambda}_E) \to (\Omega_1, \nu_1)$ and $p_2 : (X \times G \times A, \mu \times \widetilde{\lambda}_G \times \widetilde{\lambda}_A) \to (\Omega_2, \nu_2)$ be the $(\mathcal{R} \times_\alpha E)$- and the $((\mathcal{R} \times_\omega G) \times_{\alpha^{(s)}} A)$-ergodic decompositions and $W_\alpha(E)$ and $W_{\alpha^{(s)}}(A)$ the associated Mackey actions on Ω_1 and Ω_2 respectively. It follows from (2-1) and (2-3) that there is a measure space isomorphism $j : (\Omega_2, \nu_2) \to (\Omega_1, \nu_1)$ such that the following diagram is commutative:

$$\begin{array}{ccc}
(X \times E, \mu \times \widetilde{\lambda}_E) & \xrightarrow{p_1} & (\Omega_1, \nu_1) \\
{\scriptstyle q_s}\big\uparrow & & \big\uparrow{\scriptstyle j} \\
(X \times G \times A, \mu \times \widetilde{\lambda}_G \times \widetilde{\lambda}_A) & \xrightarrow[p_2]{} & (\Omega_2, \nu_2)
\end{array}$$

Moreover, since for each $a \in A$ the map q_s intertwines the left a-shift along A with the left a-shift along E, we have $jW_{\alpha^{(s)}}(a) = W_\alpha(a)j$. Identifying Ω_1 with Ω_2 via j we can view $W_\alpha(E)$ as an extension of the A-action $W_{\alpha^{(s)}}$ to an E-action on the same measure space. It is also useful to notice that $W_\omega(G)$ can be viewed naturally as the quotient action of $W_\alpha(E)$ on the space of $W_\alpha(A)$-ergodic components. Thus, $W_\alpha(A)$ is ergodic if and only if $W_\omega(G)$ is the trivial G-action on a one-point set, i.e. ω $(= \pi_*(\alpha))$ has dense range in G.

Proposition 2.5. *Let E be amenable, $\mathcal{R}$ ergodic hyperfinite type II, and ω a cocycle of $\mathcal{R}$ with dense range in G. Given an ergodic nonsingular action V of A, the following statements are equivalent:*

- (i) *V can be extended up to a nonsingular E-action on the same measure space,*
- (ii) *V is conjugate to $W_\alpha \restriction A$ for some recurrent cocycle $\alpha \in Z^1_\omega(\mathcal{R}, E)$,*
- (iii) *V is conjugate to $W_\beta(A)$ for some recurrent cocycle $\beta \in Z^1(\mathcal{R} \times_\omega G, A)$ with the cohomology class $[\beta] \in \operatorname{Inv}(\omega)$.*

Proof (cf. [Be]). (ii)⇒(i) is trivial, (iii)⇒(ii) follows from the remark before the Proposition. It remains to prove (i)⇒(iii). Suppose that $V(A)$ can be extended up to some E-action which we denote by the same symbol V. It is amenable because every action of an amenable group is amenable [Z2]. By the Existence Theorem for Cocycles there is a recurrent $\alpha \in Z^1(\mathcal{R}, E)$ such that $W_\alpha(E)$ is conjugate to $V(E)$. Since $W_\alpha(A)$ is ergodic, $\pi_*(\alpha)$ has dense range in G. By the Uniqueness Theorem for Cocycles $\pi_*(\alpha)$ and ω are weakly equivalent and hence without loss in generality we may assume that $\pi_*(\alpha) = \omega$. Now we set up $\beta = \alpha^{(s)}$ for a Borel normalized cross-section $s : G \to E$. It is easy to see that β is as desired. □

Corollary 2.6. (i) *Let p be a positive integer, T be an ergodic nonsingular transformation and* (0-1) *as follows*

$$0 \to \mathbb{Z} \stackrel{\times p}{\to} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

Let $\mathcal{R}$ satisfy the hypotheses of Proposition 2.5 *and and $\omega : \mathcal{R} \to \mathbb{Z}/p\mathbb{Z}$ a be cocycle with dense range. Take an arbitrary cocycle $\beta \in Z^1(\mathcal{R} \times_\omega (\mathbb{Z}/p\mathbb{Z}), \mathbb{Z})$ such that the transformation $W_\beta(1)$ is conjugate to T. Then $[\beta] \in \mathrm{Inv}(\omega)$ if and only if T admits a p-root.*

(ii) *For $p = 2$, let $\mathcal{R}$, ω, T, β be as above but* (0-1) *as follows*

$$1 \to \mathbb{Z} \to \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

Then $[\beta] \in \mathrm{Inv}(\omega)$ if and only if there is an involution in $\mathrm{Aut}(X, \mu)$ *conjugating T and T^{-1}. For example, if T has simple spectrum, then $[\beta] \in \mathrm{Inv}(\omega)$ by* [GJLR].

Let $\alpha \in Z^1(\mathcal{R}, E)$. One can extend the topological embedding $A \subset E$ up to a topological embedding $\overline{A} \subset \overline{E}$ and set up $\overline{r}_A(\alpha) = \overline{r}(\alpha) \cap \overline{A}$, $r_A(\alpha) = \overline{r}_A(\alpha) \cap A$, and $r_A^{(\mathrm{nor})}(\alpha) = \bigcap_{a \in A} a^{-1} r_A(\alpha) a$. We summarize some properties of these objects in

Proposition 2.7. (i) *$r_A(\alpha)$ and $r_A^{(\mathrm{nor})}(\alpha)$ are closed subgroups of A, the second one is normal,*

(ii) *$r^{(\mathrm{nor})}(\alpha) \cap A \subset r_A^{(\mathrm{nor})}(\alpha) \subset r_A(\alpha)$,*
(iii) *if $r_A(\alpha) = A$ then $r_A^{(\mathrm{nor})}(\alpha) = r^{(\mathrm{nor})}(\alpha) \cap A = A$,*
(iv) *if $r_A(\alpha)$ is noncompact then $\infty \in \overline{r}_A(\alpha)$,*
(v) *if $\alpha \approx_A \beta$ then $r_A^{(\mathrm{nor})}(\alpha) = r_A^{(\mathrm{nor})}(\beta)$, and $\infty \in \overline{r}_A(\alpha)$ whenever $\infty \in \overline{r}_A(\beta)$,*
(vi) *if α is transient and A noncompact, then $\overline{r}_A(\alpha) = \{0, \infty\}$.*

Proof. (i), (iv) follow from [FM, Proposition 8.5], (vi) from [S3, §2], (ii) and (iii) are obvious.

(v) Let $\beta(x, y) = \phi(x)^{-1}\alpha(x, y)\phi(y)$ for some measurable function $\phi : X \to A$ and suppose that $a \in r_A^{(\mathrm{nor})}(\alpha)$. We want to show that $a \in r_A^{(\mathrm{nor})}(\beta)$. Given a neighborhood U of a in E and a subset $B \in \mathfrak{B}$, $\mu(B) > 0$, one can find an element $b \in A$, a neighborhood O of b and a neighborhood V of bab^{-1} such that $O^{-1}VO \subset U$ and $\mu(B \cap \phi^{-1}(O)) > 0$. Since $bab^{-1} \in r_A(\alpha)$, there are a subset $C \in \mathfrak{B}$, $\mu(C) > 0$, and a transformation $\gamma \in [\Gamma]$ with $C \cup \gamma C \subset B \cap \phi^{-1}(O)$ and $\phi^{-1}(x)\alpha(x, \gamma x)\phi(\gamma x) \in O^{-1}VO \subset U$ and we are done. The second statement of (v) is not difficult and we leave its proof to the reader. □

Remark 2.7. It is easy to see that the short sequence of groups (in the algebraic sense, without topologies)

$$1 \to r_A(\alpha) \to r(\alpha) \xrightarrow{\pi} \pi(r(\alpha)) \to 1 \tag{2-6}$$

is exact. Let us assume that (0-1) splits and the cross-section $s : G \to E$ is a group homomorphism. Then, clearly, (2-6) splits whenever $s(\pi(r(\alpha))) \subset r(\alpha)$—for example, if $r_A(\alpha) = A$.

3. On H-cocycles

Let $\mathcal{R}^{(2)} = \{(x, y, z) \in X \times X \times X \mid (x, y), (y, z) \in \mathcal{R}\}$. We fix a normalized cross-section $s : G \to E$ and define two Borel functions $f_2 : G \times G \to A$ and $f_2^{(s)} : \mathcal{R}^{(2)} \to A$ by setting

$$\begin{aligned} f_2(g_1, g_2) &= s(g_1)s(g_2)s(g_1g_2)^{-1}, \\ f_2^{(s)}(x, y, z) &= f_2(\omega(x, y), \omega(y, z)). \end{aligned}$$

Then we have

$$\begin{aligned} f_2(g_1, g_2)f_2(g_1g_2, g_3) =& \mathrm{Ad}_{s(g_1)}[f_2(g_2, g_3)]f_2(g_1, g_2g_3), \\ f_2^{(s)}(x, y, z)f_2^{(s)}(x, z, w) =& \mathrm{Ad}_{s(\omega(x,y))}[f_2^{(s)}(y, z, w)]f_2^{(s)}(x, y, w), \end{aligned}$$

i.e. f_2 and $f_2^{(s)}$ are "noncommutative 2-cocycles" of G and $\mathcal{R}$ respectively with value in A. (As usual, $\mathrm{Ad}_b(a) = bab^{-1}$.) Moreover, they are normalized, i.e. $f_2(g, g) = 1_A$ and $f_2^{(s)}(., ., .) = 1_A$ whenever two of three variables are the same.

Given a cocycle $\alpha \in Z^1_\omega(\mathcal{R}, E)$, we set up

$$\widetilde{\alpha}(x, y) = \alpha(x, y)s(\omega(x, y))^{-1} \tag{3-1}$$

for a.e. $(x, y) \in \mathcal{R}$. Then $\widetilde{\alpha}$ is a Borel map $\mathcal{R} \to A$ with

$$\widetilde{\alpha}(x, z) = \widetilde{\alpha}(x, y)\mathrm{Ad}_{s(\omega(x,y))}[\widetilde{\alpha}(y, z)]f_2^{(s)}(x, y, z) \tag{3-2}$$

for $(x, y, z) \in \mathcal{R}^{(2)}$. Conversely, as one can easily see, every map $\widetilde{\alpha} : \mathcal{R} \to A$ satisfying (3-2) determines a cocycle $\alpha \in Z^1_\omega(\mathcal{R}, E)$ by (3-1). Such maps are called '*H-cocycles*'. The set of all 'H-cocycles' is denoted by $\mathcal{Z}^1_{\omega,s}(\mathcal{R}, A)$. If $\mathcal{Z}^1_{\omega,s}(\mathcal{R}, A)$ is furnished with the topology of convergence in measure then, clearly, the map

$$Z^1_\omega(\mathcal{R}, E) \ni \alpha \mapsto \widetilde{\alpha} \in \mathcal{Z}^1_{\omega,s}(\mathcal{R}, A)$$

is a homeomorphism. Two 'H-cocycles', $\widetilde{\alpha}$, $\widetilde{\beta}$, are said to be '*H-cohomologous*' if $\alpha \approx_A \beta$ or, equivalently,

$$\widetilde{\alpha}(x, y) = \phi(x)^{-1}\widetilde{\beta}(x, y)\mathrm{Ad}_{s(\omega(x,y))}[\phi(y)]$$

for a.e. $(x, y) \in \mathcal{R}$ and some measurable function $\phi : X \to A$.

Suppose from now on that A is Abelian. Then A is a G-module (see Section 2) and f_2 and $f_2^{(s)}$ are Borel 2-cocycles of G and $\mathcal{R}$ respectively in an ordinary sense [Mo, FM]. Moreover, $\mathcal{Z}^1_{\omega,s}(\mathcal{R}, A)$ with the pointwise addition is an Abelian Polish group.

Remark 3.1. We observe that if $\mathcal{R}$ is hyperfinite and ergodic, then $f_2^{(s)}$ can be represented in the form $f_2^{(s)}(x,y,z) = \omega(x,y)\cdot\phi(y,z) - \phi(x,z) + \phi(x,y)$ a.e. for some Borel map $\phi : \mathcal{R} \to A$ with $\phi(x,x) = 1_A$ [FM, Theorem 6]. Let us perturb an 'H-cocycle' $\widetilde{\alpha}$ by adding ϕ. Then the resulting map $\widehat{\alpha} : \mathcal{R} \to A$ satisfies

$$\widehat{\alpha}(x,z) = \widehat{\alpha}(x,y) + \omega(x,y)\cdot\widehat{\alpha}(y,z)$$

for a.e. $(x,y,z) \in \mathcal{R}^{(2)}$. If (0-1) is central, i.e. A is the trivial G-module, we obtain $\widehat{\alpha} \in Z^1(\mathcal{R}, A)$.

Remark 3.2. In the particular case when (0-1) is topologically trivial—i.e. s can be chosen to be continuous—our definitions of 'H-cocycles' and 'H-coboundaries' are equivalent to those from [Be] (and to those from [D1, D2] if, in addition, (0-1) splits and s is a group homomorphism; this implies that f_2 and $f_2^{(s)}$ are trivial). We also observe that the concepts of the '*H-superrecurrence*' and the '*H-supertransitivity*' of 'H-cocycles' from $\mathcal{Z}^1_{\omega,s}(\mathcal{R}, A)$ considered in [D1–D3, Be] correspond exactly to the recurrence and the transitivity of the related cocycles from $Z^1_\omega(\mathcal{R}, E)$. Hence most of the results from [Be], [D2], [D1, §§2,3], [D3, §§1–3] follow from our Section 2.

4. Generic results

In this section we prove the main result of the paper—Theorem 4.4. For this we need several auxiliary statements. We first recall the definition of the weak topology on $\mathrm{Aut}(X, \mu)$. Every transformation θ induces a linear bounded operator U_θ on $L^1(X, \mu)$ as follows

$$(U_\theta f)(x) = f(\theta x)\frac{d\mu\circ\theta}{d\mu}(x).$$

By a classical result of Banach the map $\theta \mapsto U_\theta$ is a bijection of $\mathrm{Aut}(X, \mu)$ onto the group of positive invertible isometries in $L^1(X, \mu)$. The strong operator topology when restricted to $\mathrm{Aut}(X, \mu)$ is called the *weak* topology.

Let $(X, \mu) = (Z, \kappa) \times (Y, \lambda)$. Denote by $\mathrm{Aut}_Z(X, \mu)$ the set of μ-nonsingular transformations which factor trivially through the first coordinate mapping $Z \times Y \to Z$. More exactly, a transformation θ belongs to $\mathrm{Aut}_Z(X, \mu)$ if

$$\theta(z, y) = (z, \theta_z y) \qquad \text{a.e.} \tag{4-1}$$

for a measurable field $Z \ni z \mapsto \theta_z \in \mathrm{Aut}(Y, \lambda)$. Clearly, $\mathrm{Aut}_Z(X, \mu)$ is a closed subgroup of $\mathrm{Aut}(X, \mu)$ endowed with the weak topology. It is well known that the ergodic transformations form a (dense) G_δ in $\mathrm{Aut}(X, \mu)$ [CK, Theorem 3]. We need a generalization of this fact. Let $\mathcal{E}$ stand for the set of all measurable fields of ergodic transformations on Y. More exactly, a transformation $\theta \in \mathcal{E}$ if (4-1) is satisfied with θ_z being ergodic for κ-a.e. z.

Lemma 4.1. *$\mathcal{E}$ is a G_δ in* $\mathrm{Aut}(X,\mu)$ *with the weak topology.*

Proof. The forthcoming argument is similar to that of [CK, Theorem 1] and [Iw, Lemma 3]. Denote by $\mathrm{Cons}(X,\mu)$ the set of nonsingular conservative transformations on X. It is clear that $\mathcal{E} \subset \mathrm{Cons}(X,\mu)$. For a transformation $\theta \in \mathrm{Aut}(X,\mu)$, we denote by $\mathfrak{B}(\theta)$ the σ-algebra of all θ-invariant measurable sets. By virtue of the Halmos-Hurevich-Oxtoby Ratio Ergodic Theorem [Ha, Ch] for every $\theta \in \mathrm{Cons}(X,\mu)$ and $f \in L^1(X,\mu)$ we have

$$\left(\sum_{i=1}^{n} U_\theta^i f\right) \Big/ \sum_{i=1}^{n} U_\theta^i 1 \to E(f|\mathfrak{B}(\theta)) \qquad \text{a.e. as } n \to \infty,$$

where E stands for the conditional expectation. Clearly, $\theta \in \mathcal{E}$ if and only if the limit equals the function $\int f d\lambda$ given by $(\int f\, d\lambda)(z,y) = \int_Y f(z,y')\, d\lambda(y')$. Since the map $f \to E(f|\mathfrak{B}(\theta))$ is continuous on $L^1(X,\mu)$ and since convergence a.e. implies convergence in measure, we have $\theta \in \mathcal{E}$ if and only if

$$\left(\sum_{i=1}^{n} U_\theta^i f_k\right) \Big/ \sum_{i=1}^{n} U_\theta^i 1 \to \int f_k\, d\lambda \qquad \text{in measure as } n \to \infty$$

for every k, where $\{f_k\}_{k=1}^{\infty}$ is a fixed norm dense sequence in $L^1(X,\mu)$. By a standard argument, the set

$$\mathcal{A}(k,n,r,p) = \left\{\theta \in \mathrm{Aut}(X,\mu) : \mu\left(\left\{\left|\left(\sum_{i=1}^{n} U_\theta^i f_k\right) \Big/ \sum_{i=1}^{n} U_\theta^i 1 - \int f_k\, d\lambda\right| \geq \frac{1}{r}\right\}\right) < \frac{1}{p}\right\}$$

is open in $\mathrm{Aut}(X,\mu)$. Since $\mathrm{Cons}(X,\mu)$ is a G_δ in $\mathrm{Aut}(X,\mu)$ (see [Iw, Lemma 2]), we deduce that

$$\mathcal{E} = \mathrm{Cons}(X,\mu) \cap \mathcal{E} = \mathrm{Cons}(X,\mu) \cap \bigcap_{p=1}^{\infty} \bigcap_{r=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \mathcal{A}(k,n,r,p)$$

is a G_δ in $\mathrm{Aut}(X,\mu)$ and hence in $\mathrm{Aut}_Z(X,\mu)$. $\square$

Let $\mathcal{M}(X,\mu)$ stand for the set of all measurable functions on X with values in a l.c.s.c. group G.

Lemma 4.2 [CHP, Theorem 1]**.** *The map* $\mathrm{Aut}(X,\mu) \times \mathcal{M}(X,G) \ni (\theta,f) \mapsto \theta_f \in \mathrm{Aut}(X \times G, \mu \times \lambda_G)$*, where $\theta_f(x,g) = (\theta x, gf(x))$ is continuous when* $\mathrm{Aut}(X,\mu)$ *and* $\mathrm{Aut}(X \times G, \mu \times \lambda_G)$ *are assumed to have the weak topology and $\mathcal{M}(X,G)$ the topology of convergence in measure.*

Let μ' be a probability measure on $\mathcal{R}$ equivalent to $\mu_{\mathcal{R}}$.

Corollary 4.3. *Let T be an ergodic transformation on $(X,\mathfrak{B},\mu)$ and $\mathcal{R}$ the T-orbital equivalence relation. Then the map $Z^1(\mathcal{R},G) \ni \alpha \mapsto T_\alpha \in$*$\mathrm{Aut}(X \times G, \mu \times \lambda_G)$*, where $T_\alpha(x,g) = (Tx, g\alpha(x,Tx))$, is continuous.* (*Recall that the topology of convergence in μ' on $Z^1(\mathcal{R},G)$ is implicit*).

Proof. It is enough to notice that the map $Z^1(\mathcal{R},G) \ni \alpha \mapsto f_\alpha \in \mathcal{M}(X,G)$ is a homeomorphism, where $f_\alpha(x) = \alpha(x,Tx)$, and then to apply Lemma 4.2. □

Let $\mathcal{R}$ be a type III equivalence relation on $(X,\mathfrak{B},\mu)$ and $\rho_\mu : \mathcal{R} \to \mathbb{R}$ stand for the corresponding Radon-Nikodym cocycle. Consider an "extension" of (0-1) as follows

$$1 \to A \to E \times \mathbb{R} \overset{\pi\times \mathrm{Id}}{\longrightarrow} G \times \mathbb{R} \to 1,$$

where A is embedded into $E \times \mathbb{R}$ via the map $a \mapsto a \times \{0\}$. Notice that $\widehat{s} \overset{\mathrm{def}}{=} s\times \mathrm{Id}$ is a cross-section of $\pi\times \mathrm{Id}$.

Theorem 4.4. *Let $\mathcal{R}$ be ergodic and hyperfinite, G arbitrary l.c.s.c., A amenable and ω recurrent. Then the subset $Z^*_\omega = \{\alpha \in Z^1_\omega(\mathcal{R},E) \mid r_A(\alpha_0) = A\}$ is a dense G_δ in $Z^1_\omega(\mathcal{R},E)$, where $\alpha_0 = \alpha \times \rho_\mu$.*

Proof. We proceed in several steps.

Step 1. Consider the ω_0-skew product extension $\mathcal{R}\times_{\omega_0}(G\times\mathbb{R})$ of $\mathcal{R}$ on the space $(X\times G\times\mathbb{R}, \mu\times\lambda_G\times\lambda_\mathbb{R})$. Since ω is recurrent, so is ω_0. (This fact was proved in [S2] in a particular case where $G = \mathbb{R}^n$. However only a slight and obvious modification of this proof is needed to adopt it in the general situation.) Therefore $\mathcal{R}\times_{\omega_0}(G\times\mathbb{R})$ is a conservative type II equivalence relation. Denote by (Z,κ) the measure space of its ergodic components. As it was shown in [GS2, the proof of Proposition 3.2] only two possibilities may be realized: either κ-a.e. ergodic component is of type II_∞ or κ-a.e. ergodic component is of type II_1. Let $\mathcal{T}$ be an ergodic hyperfinite type II equivalence relation of the corresponding type (we mean II_1 or II_∞) on a standard measure space (Y,ν). Since $\mathcal{R}$ is hyperfinite, there is a Borel nonsingular isomorphism

$$i : (Z \times Y, \kappa \times \nu) \to (X \times G \times \mathbb{R}, \mu \times \lambda_G \times \lambda_\mathbb{R})$$

such that $(i \times i)(\mathcal{D} \times \mathcal{T}) = \mathcal{R} \times_{\omega_0} (G \times \mathbb{R})$, where $\mathcal{D}$ is the diagonal equivalence relation on Z [HO, S1]. Then every cocycle $\beta \in Z^1(\mathcal{R} \times_{\omega_0} (G \times \mathbb{R}), A)$ splits into a measurable field of cocycles $Z \ni z \mapsto {}_z\beta \in Z^1(\mathcal{T}, A)$ as follows

$${}_z\beta(y_1, y_2) = \beta((z, y_1), (z, y_2))$$

for all $(y_1, y_2) \in \mathcal{T}$. Conversely, every measurable field $Z \ni z \mapsto {}_z\beta \in Z^1(\mathcal{T}, A)$ determines a cocycle $\beta \in Z^1(\mathcal{R} \times_{\omega_0} (G \times \mathbb{R}), A)$. Let $q_{\widehat{s}} : X \times (G \times \mathbb{R}) \times A \to X \times (E \times \mathbb{R})$ be the $\widehat{s}$-map (see (2-1)). Recall that $\mathcal{R} \times_\delta (E \times \mathbb{R}) = (q_{\widehat{s}} \times q_{\widehat{s}})((\mathcal{R} \times_{\omega_0} (G \times \mathbb{R})) \times_{\delta(\widehat{s})} A)$ for every cocycle $\delta \in Z^1_{\omega_0}(\mathcal{R}, E \times \mathbb{R})$ (see (2-3)). We define a weakly continuous group isomorphism

$$\Psi : \mathrm{Aut}(X \times E \times \mathbb{R}, \mu \times \lambda_E \times \lambda_\mathbb{R}) \to \mathrm{Aut}(Z \times Y \times A, \kappa \times \nu \times \lambda_A)$$

by setting $\Psi(\theta) = (i \times \mathrm{Id})^{-1} q_{\widehat{s}}^{-1} \theta q_{\widehat{s}} (i \times \mathrm{Id})$.

Let T be an ergodic transformation on (X,μ) generating $\mathcal{R}$, $\alpha : \mathcal{R} \to E$ a cocycle, and T_{α_0} the α_0-skew product extension of T, i.e. $T_{\alpha_0}(x, e, t) = (Tx, e\alpha(x, Tx), t + \rho_\mu(x, Tx))$. We consider the following chain of maps

$$\begin{aligned} &Z^1_\omega(\mathcal{R}, E) \ni \alpha \mapsto \alpha_0 \in Z^1_{\omega_0}(\mathcal{R}, E \times \mathbb{R}) \ni \alpha_0 \to T_{\alpha_0} \in \\ &\quad \mathrm{Aut}(X \times E \times \mathbb{R}, \mu \times \lambda_E \times \lambda_\mathbb{R}) \ni \theta \mapsto \Psi(\theta) \in \mathrm{Aut}(Z \times Y \times A, \kappa \times \nu \times \lambda_A). \end{aligned}$$

The first map is, clearly, a homeomorphism, the second one is continuous by Corollary 4.3. We observe also that $\Psi(T_{\alpha_0})$ belongs to the subgroup $\mathrm{Aut}_Z(Z\times Y\times A,\kappa\times\nu\times\lambda_A)$ consisting of $\kappa\times\nu\times\lambda_A$-nonsingular transformations which factor trivially through the first coordinate mapping $Z\times Y\times A\to Z$. One can check easily that $r_A(\alpha_0)=A$ if and only if the cocycle $(\alpha_0)^{(\widehat{s})}\in Z^1(\mathcal{R}\times_{\omega_0}(G\times\mathbb{R}),A)$ splits into a measurable field of cocycles on $\mathcal{T}$ with dense ranges in A. This, in turn, is equivalent to the fact that $\Psi(T_{\alpha_0})\in\mathcal{E}$, where $\mathcal{E}\subset\mathrm{Aut}_Z(Z\times Y\times A,\kappa\times\nu\times\lambda_A)$ is the set of measurable fields of ergodics on $Y\times A$. We obtain that Z^*_ω is a preimage of $\mathcal{E}$ with respect to a continuous map. It follows from Lemma 4.1 that Z^*_ω is a G_δ in $Z^1_\omega(\mathcal{R},E)$.

Step 2. Let α and β be two arbitrary cocycles from $Z^1_\omega(\mathcal{R},E)$. Given $\epsilon>0$ and $N\in\mathbb{N}$ we find a Borel subset $B\subset X$ such that $T^nB\cap T^mB=\varnothing$ for all $0\le n<m<N$ and $\mu(\bigcup_{n=0}^{N-1}T^nB)>1-\epsilon$ (Rokhlin's lemma for nonsingular ergodic transformations [S3, §3]). Then there is a Borel function $b:X\to A$ with $b(x)^{-1}\beta(x,Tx)b(Tx)=\alpha(x,Tx)$ for all $x\in\bigcup_{n=0}^{N-2}T^nB$. Actually, put $b(x)=1_A$ for all $x\in B$, $b(x)=\beta(T^{-1}x,x)^{-1}\alpha(T^{-1}x,x)$ for $x\in TB$, $b(x)=\beta(T^{-1}x,x)^{-1}b(T^{-1}x)\alpha(T^{-1}x,x)$ for $x\in T^2B$, etc. Notice that b takes values in A because $\pi_*(\alpha)=\pi_*(\beta)$. As $N\to\infty$ and $\epsilon\to0$ we obtain a sequence of Borel maps $b_N:X\to A$ such that $b_N(x)^{-1}\beta(x,Tx)b_N(Tx)\to\alpha(x,Tx)$ in measure. Since $\mathcal{R}$ is generated by T, it follows that the A-cohomology class of β is dense in $Z^1_\omega(\mathcal{R},E)$.

Thus to complete the proof we need to establish that $Z^*_\omega\neq\emptyset$.

Step 3. Let $W_{\omega_0}=\{W_{\omega_0}(g,r)\}_{(g,r)\in G\times\mathbb{R}}$ be the Mackey action of the group $G\times\mathbb{R}$ associated to the double cocycle ω_0 and (Z,ν) the space of this action. We define a nonsingular action $V=\{V(e,r)\}_{(e,r)\in E\times\mathbb{R}}$ of the group $E\times\mathbb{R}$ on (Z,ν) by setting $V(e,r)=W_{\omega_0}(\pi(e),r)$. Denote by $(G\times\mathbb{R})_z$ and $(E\times\mathbb{R})_z$ the stability subgroups at $z\in Z$ for W_{ω_0} and V respectively. Since $\mathcal{R}$ is hyperfinite, W_{ω_0} is amenable. By the Theorem on Amenability of Group Actions $(G\times\mathbb{R})_z$ is amenable for ν-a.e. $z\in Z$ and the $W_{\omega_0}(G\times\mathbb{R})$-orbital equivalence relation and hence $V(E\times\mathbb{R})$-orbital equivalence relation on Z is amenable. Since $(E\times\mathbb{R})_z$ is an extension of $(G\times\mathbb{R})_z$ via A and A is amenable, $(E\times\mathbb{R})_z$ is amenable for ν-a.e. $z\in Z$. Hence again by the Theorem on Amenability of Group Actions V is amenable. It follows from the Existence Theorem for Cocycles that there exists an ergodic discrete hyperfinite equivalence relation $\mathcal{S}$ on (X,μ) and a recurrent cocycle $\beta\in Z^1(\mathcal{S},E)$ such that V is conjugate to the Mackey action W_{β_0} of $E\times\mathbb{R}$ associated to the double cocycle $\beta_0=\beta\times\rho'_\mu$, where ρ'_μ is the Radon-Nikodym cocycle of $\mathcal{S}$. Since $W_{\beta_0}\restriction A$ is trivial, $r^{(\mathrm{nor})}(\beta_0)\supset A$ and the Mackey action of $G\times\mathbb{R}$ associated to the projection cocycle $(\pi\times\mathrm{Id})_*(\beta_0)=\pi_*(\beta)\times\rho'_\mu$ is conjugate to W_{ω_0} (see the remark before Proposition 2.5). $\pi_*(\beta)$ is recurrent because β is. We deduce from the Uniqueness Theorem for Cocycles that ω and $\pi_*(\beta)$ are weakly equivalent. Then every cocycle from $Z^1_{\pi_*(\beta)}(\mathcal{S},E)$ is weakly equivalent to a cocycle from $Z^1_\omega(\mathcal{R},E)$. Thus there exists a cocycle $\alpha\in Z^1_\omega(\mathcal{R},E)$ which is weakly equivalent to β. Since the double cocycles α_0 and β_0 are weakly equivalent as well, $r^{(\mathrm{nor})}(\alpha_0)=r^{(\mathrm{nor})}(\beta_0)$. By Proposition 2.7 $A\supset r_A^{(\mathrm{nor})}(\alpha_0)\supset r^{(\mathrm{nor})}(\alpha_0)\cap A=A$ and hence $\alpha\in Z^*_\omega$. $\square$

Remark 4.5. Notice that if $\alpha\in Z^*_\omega$ then $r_A(\alpha)=A$. However, the converse is not true: there exist cocycles α with $r_A(\alpha_0)=\{1_A\}$ but $r_A(\alpha)=A$ (see, for example [BG, Example 7.4]).

Remark 4.6. The main result of [D3]—Theorem 4.10—states that for an arbitrary l.c.s.c. group G and a noncompact Abelian group A the subset

$$Z_\omega^{**} = \{\alpha \in Z_\omega^1(\mathcal{R}, E) \mid \alpha \text{ is recurrent and } \infty \in \overline{r}_A(\alpha)\}$$

is residual in $Z_\omega^1(\mathcal{R}, E)$ provided that the double cocycle ω_0 is recurrent. We see that our Theorem 4.4 improves this assertion, since $Z_\omega^{**} \supset Z_\omega^*$.

Remark 4.7. Let $A = G = \mathbb{R}$, $E = A \rtimes G$, $\mathcal{R}$ be type III_λ, $0 < \lambda < 1$, and ω the Radon-Nikodym cocycle of $\mathcal{R}$. K. Dajani asks in [D1, p.131] whether the subset $Z_\omega^0 \overset{\text{def}}{=} \{\alpha \in Z_\omega^1(\mathcal{R}, E) \mid \overline{r}_A(\alpha) = \{0, \infty\}$ is residual in $Z_\omega^1(\mathcal{R}, A)$? As it follows from the same theorem, the answer is negative—Z_ω^0 is of first category. Recall that the Radon-Nikodym cocycle of every ergodic equivalence relation is recurrent [S2].

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Department of Mechanics and Mathematics, Kharkov State University, Freedom square 4, Kharkov, 310077, UKRAINE

E-mail address: `danilenko@ilt.kharkov.ua`