ENDOMORPHISMS OF MEASURED EQUIVALENCE RELATIONS, COCYCLES WITH VALUES IN NON LOCALLY COMPACT GROUPS AND APPLICATIONS

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Abstract. Consider a class of Polish groups arising from the subclass of amenable locally compact ones via operations of countable projective limit and group extensions. We show that for each group from this class there exists a cocycle of an ergodic transformation with dense range in it. This is applied to extend and provide a short (orbital) proof for one of the main result from [ALV] on noncoalescence of some ergodic skew product extensions.

0. Introduction

Let T be an ergodic measure preserving automorphism of a standard σ-finite measure space $(X, \mathfrak{B}, \mu)$, G an amenable locally compact second countable (l.c.s.c.) group, and $\phi : X \to G$ a Borel function (T-cocycle) such that the ϕ-skew product extension $T_\phi : X \times G \to X \times G$ given by

$$T_\phi(x, g) = (Tx, g\phi(x))$$

is ergodic. (We endow $X \times G$ with the product Borel structure and the T_ϕ-invariant product measure $\mu \times \lambda_G$, where λ_G is right Haar measure on G). Recall that the centralizer of an automorphism $R : X \to X$ is the collection of endomorphisms of X which commute with R. The collection of invertible commutors is denoted by $C(R)$. R is said to be coalescent if every commutor is invertible. We study the commutors of T_ϕ of the form

$$Q(x, g) = (Sx, l(g)f(x)), \tag{*}$$

where S is a commutor of T, $l : G \to G$ a continuous group endomorphism and $f : X \to G$ a Borel function. It is shown in [Ne], [ALMN], [Da3] that if T has a pure point spectrum then every commutor of T_ϕ is of the form (*). Remark that Q commutes with T_ϕ if and only if $\phi(Sx) = f(x)^{-1}l(\phi(x))f(Tx)$ for a.e. x, i.e. the T-cocycles $\phi \circ S$ and $l \circ \phi$ are cohomologous. It was shown in [LLT] that for T a probability preserving transformation and an invertible commutor S such that the action $\mathbb{Z}^2 \ni (m, n) \mapsto S^mT^n$ is free and $G = \mathbb{T}$, there is $\phi : X \to \mathbb{T}$ such that $\phi \circ S$ is cohomologous to $l \circ \phi$, where $l(t) = t^2$. This includes the first example of a noncoalescent Anzai skew product (i.e. a $\mathbb{T}$-skew product extension of an irrational rotation on $\mathbb{T}$). In [ALV] this was generalized to all Abelian l.c.s.c.

groups G and pairwise commuting countable families of T-commutors as follows: if $S_1, \dots, S_d \in C(T)$, $d \le \infty$, are such that $(T, S_1, \dots, S_d)$ generate a free $\mathbb{Z}^{d+1}$-action and if $w_1, \dots, w_d \in \operatorname{End} G$ pairwise commute, then there is a cocycle $\phi : X \to G$ such that T_ϕ is ergodic and $\phi \circ S_i$ is cohomologous to $w_i \circ \phi$.

The main purpose of the present paper is to extend this result to non-Abelian G and non-Abelian families of T-commutors. We also permit T to be infinite measure preserving. Remind first the definition of an ordered group which will work as $S_1, \dots, S_d$ above.

Let I be a countable group with a left invariant partial order $\succ$ such that $(I, \succ)$ is a directed set. Set $I_+ = \{i \in I \mid i \succ 1_I\}$. Then we have

(1) $I_+ I_+ \subset I_+$,
(2) $I_+ \cap I_+^{-1} = \{1_I\}$,
(3) $I_+ I_+^{-1} = I$,
(4) $i \succ j$ if and only if $j^{-1} i \in I_+$.

Conversely, given a subset $I_+ \subset I$ such that (1)–(3) are satisfied, then (4) determines a partial order on I and $(I, \succ)$ is a directed set. A pair (I, I_+) is called a *left ordered* group (see [Ef] and [Go], where it is assumed additionally that I is Abelian). We give several examples of left ordered groups: $(\mathbb{Q}, \mathbb{Q}_+)$, $(\mathbb{Z}^d, \mathbb{Z}^d_+)$, $d \le \infty$, where $\mathbb{Z}^\infty$ is the group of $\mathbb{Z}$-valued finite sequences. Consider also $I = \mathbb{Q} \rtimes_\lambda \mathbb{Z}$ with the multiplication law as follows

$$(q, n)(p, m) = (q + \lambda^n p, n + m), \qquad p, q \in \mathbb{Q},\ n, m \in \mathbb{Z},$$

where λ is some positive rational. Set $I_+ = \{(q, n) \in I \mid q \ge 0,\ n \ge 0\}$. Then (I, I_+) is a non-Abelian left ordered group.

Main Theorem. *Let T be an ergodic measure preserving automorphism of a σ-finite standard measure space, (I, I_+) a left ordered amenable group, and G an amenable l.c.s.c. group. Suppose that two monoid homomorphisms are given: $S : I_+ \to C(T)$ and $\omega : I_+ \to \operatorname{End} G$ such that $S(i)S(j)^{-1} \in \{T^n \mid n \in \mathbb{Z}\}$ implies $i = j$ (i.e. the joint action of I and T is free). Then there is a cocycle $\phi : X \to G$ such that T_ϕ is ergodic and $\omega(i) \circ \phi$ is cohomologous to $\phi \circ S(i)$ for all $i \in I_+$.*

Even in the particular case studied by Aaronson–Lemańczyk–Volny our proof is completely different. The main tool of [ALV] is Rokhlin lemma for $\mathbb{Z}^d$-actions [KW]. Those authors also claim that their approach can be adopted to actions of more general amenable groups of commutors if one applies the Ornstein–Weiss variant of Rokhlin lemma [OW]. Our argument does not use any version of Rokhlin lemma. Instead, we develop further an orbital approach to lifting problems suggested in our previous paper [Da3], where only invertible transformations were studied. Our main tools—here and in [Da3]—are the Existence and Uniqueness theorems for cocycles of an ergodic transformation (Golodets–Sinel'shchikov, [GS1]—[GS3]). Notice that the particular case of Main Theorem—all $\omega(i)$ are invertible—is proved in [Da3, Theorem 5.8 and Corollary 6.9]. To preserve the structure of that proof in the general situation we need first to determine a sort of "cross product" for G and I via ω. To this end a construction is suggested which can be viewed as a "topological group analogue" of the Rokhlin natural extension of measure space endomorphisms. However the Polish group appearing this way is not locally compact in general. Therefore a problem arises to extend the theorem on existence of cocycles with dense ranges in l.c.s.c. amenable groups to a wider class of Polish groups which is of independent interest. We say that a cocycle has dense range in a Polish group A if A is just the group of all essential values of the cocycle. If A is l.c.s.c. this is equivalent to the fact that the associated skew product extension is ergodic ([FM1], [Sc]). We show in this paper that the class of Polish groups admitting a cocycle with dense range is closed under the operations of passing to countable projective limits and group extension. This is fairly enough for our main result here. It is also worthwhile to observe that in the situation considered in [ALV]—G and I are Abelian—one can avoid the use of the extended Existence theorem for cocycles—which is the most technical ingredient of our paper—and thus obtain a short proof of [ALV, Theorem 1].

The outline of the paper is as follows. Section 1 begins with the study of measure multiplying endomorphisms of a σ-finite measure space. The background on orbit theory is also contained here. We pay special attention to class-bijective endomorphisms of measured equivalence relations. They enjoy many of the algebraic and topological properties of their invertible counterparts—elements from the normalizer of the full group. It appears that these endomorphisms admit a reasonable interpretation in the context of the von Neumann algebras associated to the equivalence relation. From our point of view the monoid of class-bijective endomorphisms is an orbit analogue of the centralizer of a measure space automorphism. This explains our interest to them. Section 2 is devoted entirely to cocycles of measured equivalence relations. Here we provide explicit constructions of product cocycles with dense ranges in

(i) countable projective limits of amenable l.c.s.c. groups,
(ii) extensions of amenable l.c.s.c. groups by (i)-type groups.

Generic properties of cocycles with values in these groups are discussed. The last Section 3 is organized like §§5,6 from [Da3]. We start with the lifting theory in orbital setting. Many of the results here are only slight modifications of their "invertible" analogues from [Da3]. Then we do not reproduce the arguments but replace them with exact references to [Da3]. Only new phenomena are discussed. In the second part of Section 3 we deduce some results of the lifting theory in "classical" setting—i.e. in the context of the centralizers of ergodic transformations and their skew product extensions—from their orbital counterparts. Main Theorem appears as one of these corollaries.

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1. Endomorphisms of measured equivalence relations

Measure multiplying endomorphisms. Let $(X, \mathfrak{B}, \mu)$ be a standard probability space. A Borel map $T : X \to X$ is an *endomorphism* if $\mu \circ T^{-1} \sim \mu$. Throughout this paper we do not distinguish between maps which coincide a.e. and sets which differ by null sets. Denote by $\mathrm{End}(X, \mu)$ the monoid of all endomorphisms of $(X, \mathfrak{B}, \mu)$ and by $\mathrm{Aut}(X, \mu)$ the subgroup of invertible ones. For a σ-finite μ-equivalent measure λ, let $\mathrm{End}_\times(X, \lambda)$ stand for the submonoid of λ-multiplying endomorphisms, i.e. $T \in \mathrm{End}_\times(X, \lambda)$ if and only if $\lambda \circ T^{-1} = c\lambda$ for some constant $c \in \mathbb{R}_+$. The *dilation* function

$$\Delta_\lambda : \mathrm{End}_\times(X, \lambda) \ni T \mapsto \Delta_\lambda(T) := \frac{d\lambda}{d\lambda \circ T^{-1}} \in \mathbb{R}_+$$

is a monoid homomorphism. Every $T \in \mathrm{End}_\times(X,\lambda)$ generates a positive isometry U_T in $L^1(X,\lambda)$ as follows

$$(U_T f)(x) = \Delta_\lambda(T) f(Tx), \qquad f \in L^1(X,\lambda),\ x \in X.$$

We let $C = \{a \cdot I_B \mid a \in \mathbb{R} \text{ and } B \in \mathfrak{B} \text{ with } \lambda(B) < \infty\}$, where I_B stands for the indicator of B. Clearly, C is a closed subset of $L^1(X,\lambda)$. It is a routine to verify that the map $T \mapsto U_T$ is an antiisomorphism of $\mathrm{End}_\times(X,\lambda)$ onto the monoid

$$\{P \in \mathcal{L}(L^1(X,\lambda)) \mid P \text{ is a positive isometry and } Pf \in C \text{ for all } f \in C\}.$$

The last one is closed in $\mathcal{L}(L^1(X,\mu))$ with respect to the strong operator topology (s.o.t.). Hence $\mathrm{End}_\times(X,\lambda)$ is a Polish monoid when equipped with the *weak* topology inherited from s.o.t. The following statement is easy and we omit the proof.

Lemma 1.1. *A sequence of λ-multiplying endomorphisms T_n weakly converges to a λ-multiplying endomorphism T if and only if $\lambda(T_n^{-1}B \triangle T^{-1}B) \to 0$ for every $B \in \mathfrak{B}$ with $\lambda(B) < \infty$ as $n \to \infty$.*

It is easy to deduce from Lemma 1.1 that the dilation function is continuous.

Remind that the *uniform* topology on $\mathrm{Aut}(X,\mu)$ is generated by the family of pseudometrics d_B:

$$d_B(T,S) := \lambda(\{x \in B \mid Tx \neq Sx\}) + \lambda(\{x \in B \mid T^{-1}x \neq S^{-1}x\}),$$

for all $B \in \mathfrak{B}$ with $\lambda(B) < \infty$. It is unaffected if one replaces λ with an equivalent σ-finite measure. It is well known that $\mathrm{Aut}(X,\mu)$ with the uniform topology is a complete nonseparable group. Remark that the uniform topology is stronger than the weak one. Clearly, $\mathrm{Aut}_\times(X,\lambda) := \mathrm{Aut}(X,\mu) \cap \mathrm{End}_\times(X,\lambda)$ is a closed subgroup of $\mathrm{Aut}(X,\mu)$.

Measured equivalence relations. Let $\mathcal{R}$ be a Borel discrete equivalence relation on X. We assume that each $\mathcal{R}$-class is countable. $\mathcal{R}$ is called μ-nonsingular if $\mu(B) = 0$ implies $\mu(\mathcal{R}(B)) = 0$, where

$$\mathcal{R}(B) = \{x \in X \mid \text{ there is } y \in B \text{ with } (x,y) \in \mathcal{R}\}$$

stands for the $\mathcal{R}$-saturation of $B \in \mathfrak{B}$. Given a countable subgroup $\Gamma \subset \mathrm{Aut}(X,\mu)$, we denote by $\mathcal{R}_\Gamma$ the Γ-orbital equivalence relation. Clearly, $\mathcal{R}_\Gamma$ is μ-nonsingular. According to [FM1] every nonsingular $\mathcal{R}$ arises this way. Remark that the corresponding Γ is highly non unique. Given a σ-finite μ-equivalent measure λ, we say that $\mathcal{R}$ is λ-preserving if so is Γ. Clearly, this property does not depend on the choice of Γ.

$\mathcal{R}$ is called *hyperfinite* if it is generated by a single transformation. $\mathcal{R}$ is said to be *ergodic* if every $\mathcal{R}$-saturated Borel subset B, i.e. $B = \mathcal{R}(B)$, is either null or conull. Notice that $\mathcal{R}_\Gamma$ is ergodic if and only if so is Γ. Let $\mathcal{S}$ be another equivalence relation on a probability space $(Y,\mathfrak{C},\nu)$. We say that $\mathcal{R}$ and $\mathcal{S}$ are *isomorphic* if there exists a Borel isomorphism $T : X \to Y$ such that $\nu \circ T \sim \mu$ and $T(\mathcal{R}(x)) = \mathcal{S}(Tx)$ at a.e. x. It is well known that every two ergodic hyperfinite probability (or infinite σ-finite measure) preserving equivalence relations are isomorphic (see [Dy], [HO],

[FM1]). Moreover, without loss of generality we may assume that the corresponding isomorphism is measure (not only measure class) preserving. Let

$$[\mathcal{R}] = \{T \in \mathrm{Aut}(X,\mu) \mid (x,Tx) \in \mathcal{R} \text{ for a.e. } x\},$$
$$N[\mathcal{R}] = \{T \in \mathrm{Aut}(X,\mu) \mid T[\mathcal{R}]T^{-1} = [\mathcal{R}]\}$$

stand for the full group of $\mathcal{R}$ and its normalizer in $\mathrm{Aut}(X,\mu)$. It is well known that $[\mathcal{R}]$ is a Polish group when endowed with the uniform topology [HO]. We need a "noninvertible" analogue of $N[\mathcal{R}]$. Set

$$\mathcal{N}[\mathcal{R}] = \{T \in \mathrm{End}(X,\mu) \mid (T\times T)\mathcal{R} = \mathcal{R}\}.$$

To put this in another way, an endomorphism T of (X,μ) lies in $\mathcal{N}[\mathcal{R}]$ if and only if $T(\mathcal{R}(x)) = \mathcal{R}(Tx)$ for a.e. $x \in X$. Clearly, $\mathcal{N}[\mathcal{R}]$ is a submonoid of $\mathrm{End}(X,\mu)$ and $N[\mathcal{R}] = \mathcal{N}[\mathcal{R}] \cap \mathrm{Aut}(X,\mu)$.

Let λ be a σ-finite μ-equivalent measure on X. We define a σ-finite measure $\lambda_{\mathcal{R}}$ on $\mathcal{R}$ with the induced Borel structure by setting

$$\lambda_{\mathcal{R}} = \int_X \sum_{y\in\mathcal{R}(x)} \delta_{(y,x)}\, d\lambda(x),$$

where $\delta_{(y,x)}$ is the Dirac measure concentrated at (y,x). Let ν be a finite measure equivalent to $\lambda_{\mathcal{R}}$. For each $T \in \mathcal{N}[\mathcal{R}]$, it is easy to verify that $T\times T \in \mathrm{End}(\mathcal{R},\nu)$. Furthermore, the map $\mathcal{N}[\mathcal{R}] \ni T \mapsto T\times T \in \mathrm{End}(\mathcal{R},\nu)$ is a one-to-one monoid homomorphism. We set $\mathrm{Ker}\, T := \{(x,y) \in \mathcal{R} \mid Tx = Ty\}$, i.e., $\mathrm{Ker}\, T = \mathcal{R} \cap (T\times T)^{-1}\mathcal{D}$, where $\mathcal{D}$ is the diagonal (the least) equivalence relation on X.

Class-bijective endomorphisms. Now we isolate a distinguished family of endomorphisms from $\mathcal{N}[\mathcal{R}]$.

Definition 1.2. An endomorphism $T \in \mathcal{N}[\mathcal{R}]$ is *class-bijective* if for a.e. x the restriction of T to $\mathcal{R}(x)$ is a bijection onto $\mathcal{R}(Tx)$.

Denote by $\mathcal{N}^*[\mathcal{R}]$ the monoid of class-bijective endomorphisms. Clearly, it contains $N[\mathcal{R}]$.

Proposition 1.3. *Let $T \in \mathcal{N}[\mathcal{R}]$. The following properties are equivalent*

(i) *T is class-bijective,*
(ii) $\mathrm{Ker}\, T = \mathcal{D}$,
(iii) *there is a map $[\mathcal{R}] \ni S \mapsto S_T \in [\mathcal{R}]$ such that $ST = TS_T$.*

Proof. is obvious. □

It is easy to see that the map defined in (iii) is a one-to-one continuous group homomorphism when $[\mathcal{R}]$ is endowed with the uniform topology. Moreover, this map is onto if and only if T is invertible. Notice that if $\mathcal{R} = \mathcal{R}_\Gamma$, Γ is a countable transformation group, then $\{S_T \mid S \in \Gamma\}$ generates the whole $\mathcal{R}$.

Lemma 1.4. *Let $T \in \mathcal{N}^*[\mathcal{R}]$ and λ a μ-equivalent σ-finite measure such that $\lambda \restriction T^{-1}\mathfrak{B}$ is σ-finite. Then the measure $\lambda_{\mathcal{R}} \circ (T\times T)^{-1}$ is σ-finite and equivalent to $\lambda_{\mathcal{R}}$. Moreover,*

$$\frac{d\lambda_{\mathcal{R}}}{d\lambda_{\mathcal{R}} \circ (T\times T)^{-1}}(x,y) = \frac{d\lambda}{d\lambda\circ T^{-1}}(x) \qquad \text{for a.e. } (x,y) \in \mathcal{R}.$$

Proof. follows directly from the definition of $\lambda_{\mathcal{R}}$. □

It follows that $T\times T$ is $\lambda_{\mathcal{R}}$-multiplying whenever T is λ-multiplying and $\Delta_{\lambda_{\mathcal{R}}}(T\times T) = \Delta_\lambda(T)$.

Proposition 1.5. *Let $\mathcal{R}$ be ergodic and preserve a σ-finite μ-equivalent measure λ. Then every endomorphism $T \in \mathcal{N}^*[\mathcal{R}]$ with $\lambda \restriction T^{-1}\mathfrak{B}$ being σ-finite is λ-multiplying.*

Proof. We define a function $f : X \to \mathbb{R}_+$ by setting $f(x) = (d\lambda/(d\lambda \circ T^{-1}))(Tx)$. Take any $S \in \Gamma$, where Γ is as above. Then

$$f(S_T x) = \frac{d\lambda}{d\lambda \circ T^{-1}}(STx) = \frac{d\lambda \circ S}{d\lambda}(Tx)\frac{d\lambda}{d(\lambda \circ T^{-1}) \circ S}(Tx) = \frac{d\lambda}{d(\lambda \circ S_T) \circ T^{-1}}(Tx) = \frac{d\lambda}{d\lambda \circ T^{-1}}(Tx) = f(x).$$

Since $\{S_T \mid S \in \Gamma\}$ is an ergodic group, it follows that $f(x) = c$ a.e. for some $c > 0$. Remind that $T : X \to X$ is onto (mod 0). Hence $(d\lambda/(d\lambda \circ T^{-1}))(x) = c$ a.e. □

We set $\mathcal{N}^*_\times[\mathcal{R}] := \mathcal{N}^*[\mathcal{R}] \cap \mathrm{End}_\times(X, \lambda)$.

Corollary 1.6. *Under the hypothesis of Proposition* 1.5*, the one-to-one monoid homomorphism $T \mapsto T \times T$ maps $\mathcal{N}^*_\times[\mathcal{R}]$ onto a closed submonoid of* $\mathrm{End}_\times(\mathcal{R}, \lambda_\mathcal{R})$ *furnished with the weak topology.*

Proof. Denote by $\sigma : \mathcal{R} \ni (x, y) \mapsto (y, x) \in \mathcal{R}$ the Sakai flip. It is well known that $\sigma \in \mathrm{Aut}(\mathcal{R}, \lambda_\mathcal{R})$ [FM1]. We deduce from Propositions 1.3 and 1.5 that the image of $\mathcal{N}^*_\times[\mathcal{R}]$ is

$$\{S \in \mathrm{End}_\times(\mathcal{R}, \lambda_\mathcal{R}) \mid \sigma S = S\sigma, S^{-1}(\mathcal{D}) = \mathcal{D}, \text{ and } S \text{ passes through each of the coordinate maps } (x, y) \mapsto x,\ (x, y) \mapsto y\}.$$

Clearly, this subset is weakly closed. □

It follows that $\mathcal{N}^*_\times[\mathcal{R}]$ is a Polish monoid when endowed with the topology inherited from the weak one on $\mathrm{End}_\times(\mathcal{R}, \lambda_\mathcal{R})$. We call this topology *normal*.

Lemma 1.7. *Let $\mathcal{R}$, λ, and Γ be as above. A sequence of endomorphisms $T_n \in \mathcal{N}^*_\times[\mathcal{R}]$ converges to an endomorphism $T \in \mathcal{N}^*_\times[\mathcal{R}]$ in the normal topology if and only if*

(i) *$T_n \to T$ weakly,*
(ii) *$S_{T_n} \to S_T$ uniformly for each $S \in \Gamma$.*

Moreover, if (ii) *is satisfied then $S_{T_n} \to S_T$ uniformly for each $S \in [\mathcal{R}]$.*

Proof. Let $B \in \mathfrak{B}$ with $\lambda(B) < \infty$ and $S \in \Gamma$. We set $\mathfrak{g}_{B,S} = \{(x, Sx) \mid x \in B\}$. Clearly $\mathfrak{g}_{B,S} \subset \mathcal{R}$ and $\lambda_\mathcal{R}(\mathfrak{g}_{B,S}) < \infty$. For a.e. $(y, z) \in \mathcal{R} \cap (T \times T)^{-1}\mathfrak{g}_{B,S}$, we have $Ty \in B$ and $Tz = STy = TS_Ty$. Since T is class-bijective, it follows that $z = S_T y$. Thus $(T \times T)^{-1}\mathfrak{g}_{B,S} = \mathfrak{g}_{T^{-1}B, S_T}$ (mod 0). In a similar way, $(T_n \times T_n)^{-1}\mathfrak{g}_{B,S} = \mathfrak{g}_{T_n^{-1}B, S_{T_n}}$. This implies

$$\lambda_\mathcal{R}((T \times T)^{-1}\mathfrak{g}_{B,S} \triangle (T_n \times T_n)^{-1}\mathfrak{g}_{B,S_n}) = \lambda(T^{-1}B \triangle T_n^{-1}B) + \lambda(\{x \in T^{-1}B \cap T_n^{-1}B \mid S_T x \neq S_{T_n} x\}).$$

Since $\{\mathfrak{g}_{B,S} \mid S \in \Gamma, \lambda(B) < \infty\}$ generates the whole σ-algebra $(\mathfrak{B} \times \mathfrak{B}) \restriction \mathcal{R}$, the conclusion of the proposition follows from Lemma 1.1 and the definition of the uniform topology. □

Corollary 1.8 (cf. [Da2, §3.1])**.** *The normal topology when restricted to* $N[\mathcal{R}]$ *coincides with the topology introduced by T. Hamachi and M. Osikawa in* [HO]*.*

Von Neumann algebras interpretation. Let $\mathcal{R}$ be as above. We set $\mathcal{H} = L^2(\mathcal{R}, \lambda_{\mathcal{R}})$ and consider representations

$$U : [\mathcal{R}] \ni S \mapsto U_S \in \mathcal{U}(\mathcal{H}),$$
$$\pi : L^\infty(X, \lambda) \ni a \mapsto \pi(a) \in \mathcal{L}(\mathcal{H}),$$

where $(U_S f)(x, y) = f(S^{-1}x, y)$ and $(\pi(a)f)(x, y) = a(x)f(x, y)$. Then

$$\mathcal{M}(\mathcal{R}) := (\{\pi(a) \mid a \in L^\infty(X, \lambda)\} \cup \{U_S \mid S \in [\mathcal{R}]\})''$$

is called the von Neumann algebra of $\mathcal{R}$ [FM2]. It follows from our assumptions on $\mathcal{R}$ that $\mathcal{M}(\mathcal{R})$ is a factor of type II. Notice that $\mathcal{A}(\mathcal{R}) := \{\pi(a) \mid a \in L^\infty(X, \lambda)\}$ is a Cartan subalgebra of $\mathcal{M}(\mathcal{R})$, i.e. a maximal Abelian subalgebra such that $(\{V \in \mathcal{U}(\mathcal{H}) \mid V\mathcal{A}(\mathcal{R})V^* = \mathcal{A}(\mathcal{R})\})'' = \mathcal{M}(\mathcal{R})$ and there is a faithful normal conditional expectation of $\mathcal{M}(\mathcal{R})$ onto $\mathcal{A}(\mathcal{R})$. Given an endomorphism $T \in \mathcal{N}^*_\times[\mathcal{R}]$, we define a map $\Phi : \mathcal{M}(\mathcal{R}) \to \mathcal{M}(\mathcal{R})$ by setting

$$\Phi(\pi(a)) = \pi(a \circ T), \qquad a \in L^\infty(X, \lambda)$$
$$\Phi(U_S) = U_{S_T}, \qquad S \in [\mathcal{R}].$$

It is easy to verify that Φ is a well defined one-to-one homomorphism with $\mathcal{A}(\mathcal{R}) \supset \Phi(\mathcal{A}(\mathcal{R}))$. Given $A \in \mathcal{M}(\mathcal{R})_+$, put $\omega(A) = \langle A\mathcal{D}, \mathcal{D}\rangle$, where $\mathcal{D}$ is the indicator of the diagonal in $X \times X$ and $\langle ., . \rangle$ the inner product in $L^2(\mathcal{R}, \lambda_{\mathcal{R}})$. Then $\omega : \mathcal{M}(\mathcal{R})_+ \to [0; +\infty]$ is a faithful normal semifinite trace and $\omega \circ \Phi = \Delta_\lambda(T)\omega$.

2. Cocycles of measured equivalence relations

Background on cocycles. Let $\mathcal{R}$ be an ergodic λ-preserving equivalence relation on $(X, \mathfrak{B})$, $\lambda \sim \mu$, and G a Polish group. A measurable map $\alpha : \mathcal{R} \to G$ is a *cocycle* of $\mathcal{R}$ if

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z)$$

for a.e. $(x, y), (y, z) \in \mathcal{R}$. Remind that we do not distinguish between two cocycles if they agree $\lambda_{\mathcal{R}}$-a.e. Two cocycles, $\alpha, \beta : \mathcal{R} \to G$, are *cohomologous* ($\alpha \approx \beta$) if

$$\alpha(x, y) = \phi(x)^{-1}\beta(x, y)\phi(y)$$

for $\lambda_{\mathcal{R}}$-a.e. (x, y), where $\phi : X \to G$ is a Borel map (we call it a transfer function from α to β). The set of all cocycles of $\mathcal{R}$ with values in G will be denoted by $Z^1(\mathcal{R}, G)$. By [FM1] $Z^1(\mathcal{R}, G)$ endowed with the topology of convergence in measure (any probability equivalent to $\lambda_{\mathcal{R}}$) is a Polish space.

Let $\mathcal{S}$ be an equivalence relation on a space $(Y, \mathfrak{C}, \nu)$ and a cocycle $\beta : \mathcal{S} \to G$ be given. $(\mathcal{R}, \alpha)$ and $(\mathcal{S}, \beta)$ or simply α and β are said to be *weakly equivalent* if there is a nonsingular isomorphism $T : X \to Y$ which takes $\mathcal{R}$ to $\mathcal{S}$ and such that $\alpha \approx \beta \circ T$, where the cocycle $\beta \circ T$ is defined by $\beta \circ T(x, x') = \beta(Tx, Tx')$. Remark that $\beta \circ T$ is well defined also for each endomorphism $T \in \mathcal{N}[\mathcal{R}]$.

Definition 2.1. A cocycle $\alpha : \mathcal{R} \to G$ *has dense range* in G if for every Borel subset $B \subset X$, $\mu(B) > 0$, and open $O \subset G$ there are a subset A, $\mu(A) > 0$, and a transformation $S \in [\mathcal{R}]$ with $A \cup SA \subset B$ and $\alpha(x, Sx) \in O$ for all $x \in A$.

It is easy to deduce from this definition

Lemma 2.2. *Let $l : G \to H$ be a continuous group homomorphism with $l(G)$ dense in H. If $\alpha \in Z^1(\mathcal{R}, G)$ has dense range in G, then $l \circ \alpha \in Z^1(\mathcal{R}, H)$ has dense range in H.*

The following well known statement can be proved easily via the standard exhaustion argument.

Lemma 2.3. *Let $\alpha \in Z^1(\mathcal{R}, G)$ has dense range in G. Then for every open subset $O \subset G$ there is a transformation $S \in [\mathcal{R}]$ with $\alpha(x, Sx) \in O$.*

Let G be l.c.s.c. and λ_G right Haar measure on G. Given a cocycle $\alpha : \mathcal{R} \to G$, we define the *skew product* equivalence relation $\mathcal{R}(\alpha)$ on $(X \times G, \mu \times \lambda_G)$ by setting $(x, g) \sim (y, h)$ if $(x, y) \in \mathcal{R}$ and $h = g\alpha(x, y)$. It is easy to see that $\mathcal{R}(\alpha)$ is $\lambda \times \lambda_G$-preserving. The following criterion is well known.

Lemma 2.4 ([Sc], [FM1]). *Let G be l.c.s.c. Then α has dense range in G if and only if $\mathcal{R}(\alpha)$ is ergodic.*

The next two statements are of fundamental importance in cocycle studying.

Lemma 2.5 ([Z1], [GS1], [Is]). *Let $\mathcal{R}$ be an ergodic hyperfinite measure preserving* (*either finite or σ-finite*) *equivalence relation. Then for each amenable l.c.s.c. G there exists a cocycle $\alpha : \mathcal{R} \to G$ with dense range in G.*

Lemma 2.6 ([GS1], [GS3]). *Let α and β be cocycles of $\mathcal{R}$ and $\mathcal{S}$ respectively with dense ranges in G. If $\mathcal{R}$ and $\mathcal{S}$ are both ergodic hyperfinite finite* (*or infinite*) *measure preserving, then α and β are weakly equivalent.*

Cocycles with values in countable projective limits of l.c.s.c. groups. Let I be a countable partially ordered right filtering set. Remind that a family of Polish groups $\{G_i\}_{i \in I}$ and continuous epimorphisms $\{p_{ij} : G_i \to G_j \mid i > j\}$ is an *inverse spectrum* if $p_{ii} = \mathrm{id}$ for all $i \in I$ and $p_{jk}p_{ij} = p_{ik}$ for all $i > j > k$. We form $G = \operatorname{proj\,lim}_{i \in I} G_i$ and denote by $p^{(i)} : G \to G_i$, $i \in I$, the canonical projections. Then G endowed with the projective limit topology is a Polish group.

Proposition 2.7. *Let (G_i, p_{ij}) be an inverse spectrum of amenable l.c.s.c. groups and $\mathcal{R}$ an ergodic hyperfinite measure preserving equivalence relation. Then there is a cocycle of $\mathcal{R}$ with dense range in $G := \operatorname{proj\,lim}_{i \in I} G_i$.*

Proof. Without loss of generality we may assume that $I = \mathbb{N}$ with the natural ordering. Denote by $s_i : G_i \to G_{i+1}$ a Borel cross-section of $p_{i+1,i}$. By Lemma 2.5 there are cocycles $\alpha : \mathcal{R} \to G_1$ and $\beta : \mathcal{R} \to G_2$ with dense ranges in G_1 and G_2 respectively. We deduce from Lemmas 2.2 and 2.6 that there is a transformation $T \in N[\mathcal{R}]$ and function $\phi : X \to G_1$ such that

$$\alpha_1(x, y) = \phi(x)^{-1} p_{21} \circ \beta \circ T(x, y)\phi(y)$$

for all $(x, y) \in \mathcal{R}$. We set

$$\alpha_2(x, y) = s_1(\phi_1(x))^{-1} \beta \circ T(x, y) s_1(\phi_1(y)).$$

Clearly, α_2 is a cocycle of $\mathcal{R}$ with dense range in G_2 and $p_{21} \circ \alpha_2 = \alpha_1$. Continue inductively to obtain a sequence $\{\alpha_i\}_{i \in I}$ of cocycles of $\mathcal{R}$ such that α_i takes values and has dense range in G_i and $p_{i+1,i} \circ \alpha_{i+1} = \alpha_i$ for all $i \in \mathbb{N}$. It follows that a cocycle $\alpha \in Z^1(\mathcal{R}, G)$ is well defined by $p^{(i)} \circ \alpha = \alpha_i$ for all $i \in \mathbb{N}$. By the definition of projective limit topology α has dense range in G. □

It is worthwhile to observe that if there is a cocycle of an ergodic hyperfinite $\mathcal{R}$ with dense range in the projective limit of an inverse spectra of l.c.s.c. groups (G_i, p_{ij}) then each G_i is amenable—it follows from [Z2] and Lemma 2.2.

Theorem 2.8. *Let $\mathcal{R}$ and G be as above. Then the subset*

$$Z_\infty = \{\alpha \in Z^1(\mathcal{R}, G) \mid \alpha \text{ has dense range in } G\}$$

is a dense G_δ in $Z^1(\mathcal{R}, G)$ endowed with the topology of convergence in measure.

Proof. According to Proposition 2.7 Z_∞ is not empty. A standard application of Rokhlin lemma (to a transformation generating the whole $\mathcal{R}$) implies that the cohomology class of every cocycle is dense in $Z^1(\mathcal{R}, G)$. Thus Z_∞ is dense in $Z^1(\mathcal{R}, G)$. We let

$$Z_n = \{\beta \in Z^1(\mathcal{R}, G_n) \mid \beta \text{ has dense range in } G_n\}.$$

It is well known that Z_n is a dense G_δ in $Z^1(\mathcal{R}, G_n)$ (see [PS], [CHP]). It is easy to see that the map

$$\pi_n : Z^1(\mathcal{R}, G) \ni \alpha \mapsto p^{(n)} \circ \alpha \in Z^1(\mathcal{R}, G_n)$$

is continuous. Since $Z_\infty = \bigcap_{n \in \mathbb{N}} \pi_n^{-1}(Z_n)$, we deduce that Z_∞ is a G_δ. □

Product cocycles. Let X_k be a finite set and μ_k the equidistribution on X_k, $k \in \mathbb{N}$. We endow the product space $X = \prod_{k=1}^\infty X_k$ with the product Borel structure and set $\mu = \bigotimes_{k=1}^\infty \mu_k$. Given $x_1' \in X_1, \dots, x_n' \in X_n$, we call the subset $I = \{x = (x_k) \in X \mid x_1 = x_1', \dots, x_n = x_n'\}$ a *n-cylinder*. A union of n-cylinders is an *n-cylindric* subset. Let $\mathcal{R}^{(n)} = \{(x, x') \in X \times X \mid x_k = x_k' \text{ for all } k > n\}$. Clearly, $\mathcal{R}^{(0)} \subset \mathcal{R}^{(1)} \subset \dots$ and $\mathcal{R} \overset{\text{def}}{=} \bigcup_{n=1}^\infty \mathcal{R}^{(n)}$ is a Borel discrete μ-preserving ergodic equivalence relation on X. By $[[\mathcal{R}^{(n)}]]$ we denote the (finite) subgroup of $[\mathcal{R}^{(n)}]$ consisting of all transformations which map each n-cylinder onto an n-cylinder. Given a sequence of functions $a_k : X_k \to G$, we define a cocycle $\alpha \in Z^1(\mathcal{R}, G)$ by setting

$$\alpha(x, x') = a_1(x_1) \dots a_n(x_n) a_n(x_n')^{-1} \dots a_1(x_1')^{-1}$$

for all $(x, x') \in \mathcal{R}^{(n)}$, $n \in \mathbb{N}$. It is easy to verify that α is well defined. We call it a *product cocycle*. Notice that for each group homomorphism $l : G \to H$ the composition $l \circ \alpha$ is also a product cocycle.

Lemma 2.9. *Let $\alpha, \beta \in Z^1(\mathcal{R}, G)$ be two product cocycles generated by the functions $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ respectively. If there are an integer N and an element $g \in G$ such that $a_k(x_k) = g b_k(x_k) g^{-1}$ for all $x_k \in X_k$ and $k > N$ then $\alpha \approx \beta$.*

Proof. We define a Borel function $f : X \to G$ by setting

$$f(x) = b_1(x_1) \dots b_N(x_N) g^{-1} a_N(x_N)^{-1} \dots a_1(x_1)^{-1}.$$

It is a routine to verify that $f(x)\alpha(x, x') f(x')^{-1} = \beta(x, x')$ for all $(x, x') \in \mathcal{R}$. □

The following criterion is not difficult. We leave its proof to the reader.

Lemma 2.10. *Let $\alpha \in Z^1(\mathcal{R}, G)$ be a product cocycle and $0 < c < 1$. Then α has dense range in G if and only if for every open $O \subset G$ and every cylinder $I \subset X$ there are a positive integer n, a n-cylindric subset I_1 and a transformation $\gamma \in [[\mathcal{R}^{(n)}]]$ such that $I_1 \cup \gamma I_1 \subset I$, $\mu(I_1) > c\mu(I)$ and $\alpha(x, \gamma x) \in O$ for all $x \in I_1$.*

The next statement is a refinement of Lemma 2.5.

Lemma 2.11 ([GS1], [Is]**).** *Let G be an amenable l.c.s.c. group. Then there exists a product cocycle with dense range in G.*

We extend this result to countable projective limits of amenable l.c.s.c. groups. To this end we modify an approach of Golodets and Sinelshchikov [GS1, Theorem 4.1].

Lemma 2.12. *Let $1 \to H \to G \xrightarrow{p} F \to 1$ be an exact short sequence of Polish groups, α and β product cocycles with dense ranges in H and F respectively. Then there exists a product cocycle δ with dense range in G.*

Proof. Let $X = \prod_{k=1}^\infty X_k$, $Y = \prod_{k=1}^\infty Y_k$, $\mathcal{R}_X = \bigcup_{k=1}^\infty \mathcal{R}_X^{(k)}$, $\mathcal{R}_Y = \bigcup_{k=1}^\infty \mathcal{R}_Y^{(k)}$, μ_X and μ_Y be the invariant product measures on X and Y respectively, and the product cocycles $\alpha \in Z^1(\mathcal{R}_X, H)$ and $\beta \in Z^1(\mathcal{R}_Y, F)$ generated by some sequences of functions $a_k : X_k \to H$ and $b_k : Y_k \to F$, $k \in \mathbb{N}$, have dense ranges in H and F respectively.

Denote by $s : F \to G$ a normalized cross-section of p. We shall construct a Borel product space $Z = \prod_{k=1}^\infty Z_k$ and functions $c_k : Z_k \to G$ in such a way that the corresponding product cocycle will have dense range in G. To this end we use an inductive process. Let $\{O_n\}_{n=1}^\infty$ be a countable base of the topology on H.

Step 1. We first put $Z_1 = Y_1$ and $c_1 = s \circ b_1$. Given $y \in Y$ and $k \in \mathbb{N}$, we define a function $a_{y,k}^{(1)} : X_k \to H$ by setting

$$a_{y,k}^{(1)}(x_k) = c_1(y_1) a_k(x_k) c_1(y_1)^{-1}.$$

Then the sequence $\{a_{y,k}^{(1)}\}_{k=1}^\infty$ determines a cocycle $\alpha_y^{(1)} \in Z^1(\mathcal{R}_X, H)$. Clearly, for each $y \in Y$ we have $\alpha_y^{(1)} \approx \alpha$ and hence $\alpha_y^{(1)}$ has dense range in H. Remark that there are only finitely many of these cocycles, since $\alpha_y^{(1)}$ depends, in fact, only on the first coordinate of y. By Lemma 2.10 there are $m(1) \in \mathbb{N}$, a $m(1)$-cylindric subset I_y and a transformation $\gamma_y \in [[\mathcal{R}^{(m(1))}]]$ (for each $y \in Y$) such that $\mu_X(I_y) > 0.9$, and $\alpha_y^{(1)}(x, \gamma_y x) \in O_1$ for all $x \in I_y$. Moreover, without loss of generality we may assume that I_y and γ_y depends only on y_1. Now we let $Z_k = X_{k-1}$ and $c_k = a_{k-1}$ for $k = 2, \dots, m(1) + 1$.

Step n. In a similar way, we first put $Z_{m(n-1)+1} = Y_n$ and $c_{m(n-1)+n} = s \circ b_n$. Given $y \in Y$ and $k \in \mathbb{N}$, we define a function $a_{y,k}^{(n)} : X_k \to H$ by setting

$$a_{y,k}^{(n)} = \begin{cases} a_{y,k}^{(n-1)}(x_k), & \text{for} \quad k = 1, \dots, m(n-1) \\ d^{(n)}(y) a_k(x_k) d^{(n)}(y)^{-1}, & \text{for} \quad k > m(n-1), \end{cases}$$

where $d^{(n)}(y) = s(b_1(y_1)) s(b_2(y_2)) \dots s(b_n(y_n))$. The sequence $\{a_{y,k}^{(n)}\}_{k=1}^\infty$ determines a cocycle $\alpha_y^{(n)} \in Z^1(\mathcal{R}_X, H)$. It is easy to deduce from Lemma 2.9 that for each $y \in Y$ we have $\alpha_y^{(n)} \approx \alpha$ and hence $\alpha_y^{(n)}$ has dense range in H. There are only finitely many of these cocycles, since in fact $\alpha_y^{(n)}$ depends only on $y_1, \dots, y_n$. By Lemma 2.10 there exists $m(n) > m(n-1)$ such that for every $m(n-1)$-cylinder I and $y \in Y$ there are an $m(n)$-cylindric subset I_y and a transformation $\gamma_y \in [[\mathcal{R}^{(m(n-1))}]]$ with $\mu_X(I_y) > 0.9\mu_X(I)$, $I_y \cup \gamma_y I_y \subset I$ and

$$\alpha_y^{(n)}(x, \gamma_y x) \in O_n \text{ for all } x \in I_y. \tag{2-1}$$

Without loss of generality one may assume that I_y and γ_y depend only on $y_1, \dots, y_n$. We complete Step n by setting $Z_k = X_{k-n}$ and $c_k = a_{k-n}$ for $k = m(n-1) + n + 1, \dots, m(n) + n$.

Let δ be the G-valued product cocycle of $\mathcal{R}_Z$ generated by $\{c_k\}_{k=1}^{\infty}$. Define a map $\theta : Z \to Y \times X$ by setting $\theta z = (y, x)$, where $z = (z_1, z_2, \dots)$, $y = (z_1, z_{m(1)+2}, z_{m(2)+3}, \dots)$, and x the subsequence of z which is complementary to y. Clearly, θ is a Borel isomorphism, $\mu_Z \circ \theta^{-1} = \mu_Y \times \mu_X$, $(\theta \times \theta)\mathcal{R}_Z^{(m(n)+n)} = \mathcal{R}_Y^{(n)} \times \mathcal{R}_X^{(m(n))}$ for each $n \in \mathbb{N}$ and hence $(\theta \times \theta)\mathcal{R}_Z = \mathcal{R}_Y \times \mathcal{R}_X$. We set $\widehat{\delta} = \delta \circ \theta^{-1}$. For each $y \in Y$, we define a cocycle $\widehat{\delta}_y \in Z^1(\mathcal{R}_X, H)$ by setting

$$\widehat{\delta}_y(x, x') = \widehat{\delta}((y, x), (y, x')).$$

It is a routine to verify that

$$p \circ \widehat{\delta}((y, x), (y', x')) = \beta(y, y') \text{ for all } (x, x') \in \mathcal{R}_X,\ (y, y') \in \mathcal{R}_Y, \tag{2-2}$$

$$\widehat{\delta}_y \restriction \mathcal{R}_X^{(m(n))} = \alpha_y^{(n)} \restriction \mathcal{R}_X^{(m(n))} \text{ for all } n \in \mathbb{N}. \tag{2-3}$$

We claim that $\widehat{\delta}$ (and hence δ) has dense range in G. Take an n-cylinder I_1 in Y, a $m(n-1)$-cylinder I_2 in X and an open subset U in G. Since β has dense range in F, by Lemma 2.10 there are $l > n$, an l-cylindric subset I_3, and a transformation $\gamma_1 \in [[\mathcal{R}_Y^{(l)}]]$ such that $\mu_Y(I_3) > 0.9\mu_Y(I_1)$, $I_3 \cup \gamma_1 I_3 \subset I_1$ and $\beta(y, \gamma_1 y) \in p(U)$ for all $y \in I_3$. Since δ is a product cocycle, it follows from (2-2) that for each l-cylinder $I_5 \subset I_3$ and each $m(l-1)$-cylinder $I_4 \subset I_2$ there is $g_0 \in G$ with $\widehat{\delta}((y, x), (\gamma_1 y, x)) = g_0$ for all $y \in I_5$ and $x \in I_4$. Notice that $p(g_0) \in p(U)$. Hence there is $d > l$ with $O_d \subset g_0^{-1}U$. We deduce from (2-1) and (2-3) that there are a $m(d)$-cylindric subset I_5' and a transformation $\gamma_1' \in [[\mathcal{R}_X^{(m(d))}]]$ such that $I_5' \cup \gamma_1' I_5' \subset I_4$ $\mu_X(I_5') > 0.9\mu_X(I_4)$, and $\widehat{\delta}_y(x, \gamma_1' x) \in O_d$ for all $x \in I_5'$ and $y \in \gamma_1 I_5$. It follows that

$$\widehat{\delta}((y, x), (\gamma_1 y, \gamma_1' x)) = \widehat{\delta}((y, x), (\gamma_1 y, x))\widehat{\delta}_{\gamma_1 y}(x, \gamma_1' x) \in g_0 O_d \subset U$$

for all $y \in I_5$ and $x \in I_5'$. Bringing up all I_5 and I_4 contained in I_3 and I_2 respectively, we let I be the (finite) union of all $I_5 \times I_5'$ and denote by γ the "concatenation" of all $\gamma_1 \times \gamma_1'$. Then $\theta^{-1}I$ is a cylindric subset of Z, $\mu_Z(\theta^{-1}I) > 0.8\mu_Z(\theta^{-1}(I_1 \times I_2))$ and we are done by virtue of Lemma 2.10. □

Theorem 2.13. *Let $G = \operatorname{proj\,lim}_{\lambda \in \Lambda} G_\lambda$, where G_λ is an amenable l.c.s.c. group and Λ a countable directed set. Then there exists a product cocycle with dense range in G.*

Proof. Without loss of generality we may assume that $\Lambda = \mathbb{N}$ with the natural ordering. Let H_n be the kernel of the canonical projection $p_n : G_{n+1} \to G_n$. Then for each $n \in \mathbb{N}$ we have a short exact sequence of amenable l.c.s.c. groups

$$1 \to H_n \to G_{n+1} \to G_n \to 1.$$

By virtue of Lemma 2.11 there are product cocycles with values in G_1 and H_n, $n \in \mathbb{N}$. Therefore one can apply Lemma 2.12 subsequently countably many times and modify the inductive process described there in order to construct

(i) a product space $X = \prod_{k=1}^{\infty} X_k$,
(ii) a "filtration" of $\mathbb{N}$, i.e. an increasing sequence of infinite subsets of positive integers $J_1 \subset J_2 \subset \dots$ with $\bigcup_{n=1}^{\infty} J_n = \mathbb{N}$,
(iii) a family of cocycles $\alpha_n \in Z^1(\mathcal{R}_{X^{(n)}}, G_n)$ with dense ranges in G_n respectively such that $\alpha_n \circ \pi_n = p_n \circ \alpha_{n+1}$, where $X^{(n)}$ stands for $\prod_{k \in J_n} X_k$ and $\pi_n : X^{(n+1)} \to X^{(n)}$ is the natural projection, $n \in \mathbb{N}$.

Denote by $p^{(n)} : G \to G_n$ and $\pi^{(n)} : X \to X^{(n)}$ the natural quotient maps. Then

$$\pi_n \circ \pi^{(n+1)} = \pi^{(n)} \quad \text{and } p_n \circ p^{(n+1)} = p^{(n)}$$

for all $n \in \mathbb{N}$. Now we are ready to define a product cocycle $\beta \in Z^1(\mathcal{R}_X, G)$ with dense range in G. Given $n > 0$, set $\beta_n(x, x') = \alpha_n \circ \pi^{(n)}(x, x')$, $(x, x') \in \mathcal{R}_X$. Clearly, $\beta_n \in Z^1(\mathcal{R}, G_n)$. From (iii) we deduce that $p_n \circ \beta_{n+1} = \beta_n$, $n \in \mathbb{N}$. Hence the sequence $\{\beta_n\}_{n=1}^{\infty}$ well defines a cocycle $\beta \in Z^1(\mathcal{R}_X, G)$. It is easy to see that β is a product cocycle. Since β_n has dense range in G_n, it follows that β has dense range in G. □

The following corollary will be applied to the proof of the main result of the paper.

Corollary 2.14. *Let $1 \to G \to H \to A \to 1$ be a short exact sequence of Polish groups, where A is amenable l.c.s.c. and G as above. Then there exists a product cocycle with dense range in H.*

Proof. Apply Lemma 2.12 and Theorem 2.13. □

Remark 2.15. One can prove also that the subset of H-valued cocycles of an ergodic hyperfinite measure preserving equivalence relation with dense range in H is a dense G_δ in $Z^1(\mathcal{R}, H)$ with the topology of convergence in measure. To this end one should slightly modify the argument from [PS] (see also [CHP]) and use Corollary 2.14.

Remark 2.16. In fact we have proved more than it is claimed in Lemma 2.12 and Theorem 2.13: one can replace the words "l.c.s.c. amenable group" with "Polish group admitting a product cocycle with dense range in it" everywhere.

3. Lifting problems

Orbital approach. Let $\mathcal{R}$ be an ergodic λ-preserving equivalence relation on X, G a l.c.s.c. group and α a cocycle of $\mathcal{R}$ with dense range in G.

Definition 3.1. We say that an endomorphism $S \in \mathcal{N}^*[\mathcal{R}]$ can be lifted to $\mathcal{N}[\mathcal{R}(\alpha)]$ if there exists a transformation $\widetilde{S} \in \mathcal{N}[\mathcal{R}]$ of the form

$$\widetilde{S}(x, g) = (Sx, .) \tag{3-1}$$

for a.e. $(x,g)\in X\times G$.

Notice that $\widetilde{S}$ is class-bijective. Denote by $\mathcal{L}(\mathcal{R},\alpha)$ the monoid of all endomorphisms from $\mathcal{N}^*[\mathcal{R}]$ which can be lifted to $\mathcal{N}[\mathcal{R}(\alpha)]$ and by $\widetilde{\mathcal{L}}(\mathcal{R}(\alpha))$ the monoid of all endomorphisms from $\mathcal{N}^*[\mathcal{R}(\alpha)]$ of the form (3-1). Notice that the natural projection

$$p:\widetilde{\mathcal{N}}(\mathcal{R}(\alpha))\ni\widetilde{S}\mapsto S\in\mathcal{L}(\mathcal{R},\alpha)\subset\mathcal{N}^*[\mathcal{R}]$$

is a monoid homomorphism. We find its kernel.

Lemma 3.2. $\operatorname{Ker}p=\{R(h)\mid h\in G\}$, *where* $R(h)\in\operatorname{Aut}(X\times G,\lambda\times\lambda_G)$ *is given by* $R(h)(x,g)=(x,hg)$.

Proof. is an almost literal repetition of [Da3, Lemma 5.2]. □

Let $\operatorname{End}G$, $\operatorname{Aut}G$, and $M(X,G)$ stand for the monoid of continuous endomorphisms of G, the subgroup of invertible ones, and the group of measurable functions from X to G respectively. Denote by $\mathcal{L}^\bullet(\mathcal{R},\alpha)$ the submonoid of invertible transformations from $\mathcal{L}(\mathcal{R},\alpha)$ and set $\widetilde{\mathcal{L}}^\bullet(\mathcal{R}(\alpha))=p^{-1}(\mathcal{L}^\bullet(\mathcal{R},\alpha))$.

Remark that given $S\in\mathcal{L}^\bullet(\mathcal{R},\alpha)$, it may very well happen that $S^{-1}\notin\mathcal{L}(\mathcal{R},\alpha)$. Thus $\mathcal{L}^\bullet(\mathcal{R},\alpha)$ is not a group in general.

Theorem 3.3. *Every lift of an element* $S\in\mathcal{L}(\mathcal{R},\alpha)$ *to* $\mathcal{N}[\mathcal{R}(\alpha)]$ *is of the form*

$$S_{l,f}(x,g)=(Sx,l(g)f(x))\qquad\text{at a.e. }(x,g)\in X\times G\tag{3-2}$$

for some $f\in M(X,G)$ *and some continuous group homomorphism* $l:G\to G$ *satisfying*

$$\alpha\circ S(x,y)=f(x)^{-1}l(\alpha(x,y))f(y).\tag{3-3}$$

If S *is invertible then* l *is onto. Given two lifts* $S_{l,f}$ *and* $S_{l'f'}$ *of* S, *there is* $h\in G$ *with* $l(g)=hl'(g)h^{-1}$ *and* $f(x)=hf'(x)$, *i.e.* $S_{l,f}=R(h)S_{l',f'}$. *Conversely, every triplet* $(S,l,f)\in N[\mathcal{R}]\times\operatorname{End}G\times M(X,G)$ *satisfying* (3-3) *defines a lift of* S *to* $\mathcal{N}[\mathcal{R}(\alpha)]$ *by* (3-2). *Moreover,* $S_{l,f}$ *is invertible if and only if so is* l.

Proof. See [Da3, Theorem 5.3]. We only need to prove the third statement. It is easy to see that the cocycle

$$\mathcal{R}\ni(x,y)\mapsto(l\circ\alpha(x,y),l'\circ\alpha(x,y))\in G\times G$$

takes values and has dense range in the subgroup $H=\{(l(g),l'(g))\mid g,g'\in G\}$ of $G\times G$. On the other hand we deduce from (3-3) that the cocycle

$$\mathcal{R}\ni(x,y)\mapsto(f(x)^{-1}l\circ\alpha(x,y)f(y),f'(x)^{-1}l'\circ\alpha(x,y)f'(y))\in G\times G$$

takes values and has dense range in the diagonal subgroup of $G\times G$. Since these cocycles are cohomologous, H is conjugate to the diagonal subgroup and we are done. □

It follows that if a lift of an invertible S is invertible then so is every lift of S. We observe also that lifts of S are invertible if and only if $S^{-1}\in\mathcal{L}(\mathcal{R},\alpha)$.

Let M be a monoid and N a normal subgroup of M, i.e. $aN = Na$ for all $a \in M$. Then a quotient monoid multiplication on M/N is well defined. The homomorphism $M \ni a \mapsto aN \in M/N$ will be called a *canonical projection*. We deduce from Theorem 3.3 that the map

$$\widetilde{\mathcal{L}}^\bullet(\mathcal{R}(\alpha)) \ni T \to p(T) \in \mathcal{L}^\bullet(\mathcal{R}, \alpha)$$

is a canonical projection (associated to the normal subgroup $R(G)$).

Set $\mathfrak{G} := N[\mathcal{R}] \times \operatorname{End} G \times M(X,G)$, and $\mathfrak{G}_0 := \{(S,l,f) \in \mathfrak{G} \mid \alpha \circ S(x,y) = f(x)^{-1} l(\alpha(x,y)) f(y)\}$. Define multiplication law on $\mathfrak{G}$ by setting

$$(S,l,f) \cdot (S',l',f') = (SS', ll', (l \circ f')(f \circ S')).$$

Clearly, $\mathfrak{G}$ is a monoid and $\mathfrak{G}_0$ its submonoid. We put

$$J : \widetilde{\mathcal{L}}(\mathcal{R}(\alpha)) \ni S_{l,f} \mapsto (S,l,f) \in \mathfrak{G}.$$

Corollary 3.4. *J is a one-to-one monoid homomorphism and*

(i) $J(\widetilde{\mathcal{L}}(\mathcal{R}(\alpha))) = \mathfrak{G}_0$,
(ii) *if l is invertible then so is (S,l,f) and $(S,l,f)^{-1} = (S^{-1}, l^{-1}, l^{-1} \circ (f \circ S^{-1})^{-1})$.*

Denote by $\operatorname{Inn} G \subset \operatorname{End} G$ the subgroup of inner automorphisms of G, i.e. $l \in \operatorname{Inn} G$ if there exists $h \in G$ such that $l(g) = hgh^{-1}$ for all $g \in G$. Clearly, $\operatorname{Inn} G$ is a normal subgroup of $\operatorname{End} G$. Let $\operatorname{Ond} G$ stand for the quotient monoid $\operatorname{End} G / \operatorname{Inn} G$ and $q : \operatorname{End} G \to \operatorname{Ond} G$ for the canonical projection. Then we have an exact sequence of monoids

$$1 \to Z(G) \to G \to \operatorname{Inn} G \to \operatorname{End} G \xrightarrow{q} \operatorname{Ond} G \to 1, \tag{3-4}$$

where $Z(G)$ is the center of G. It is clear that the map

$$p_2 : \widetilde{\mathcal{L}}^\bullet(\mathcal{R}(\alpha)) \ni S_{l,f} \mapsto l \in \operatorname{End} G$$

is a monoid homomorphism. Given $S \in \mathcal{L}^\bullet(\mathcal{R}, \alpha)$, we set $\pi(S) = q(l) \in \operatorname{Ond} G$, where $l \in \operatorname{End} G$ is determined by $\alpha \circ S \approx l \circ \alpha$. By Theorem 3.3 $\pi : \mathcal{L}^\bullet(\mathcal{R}, \alpha) \to \operatorname{Ond} G$ is a well defined monoid homomorphism. Remark that π can be viewed as the canonical projection with respect to the normal subgroup $D(\mathcal{R}, \alpha) = \{T \in N[\mathcal{R}] \mid \alpha \circ T \approx \alpha\}$, i.e. the subgroup of *$\alpha$-compatible automorphisms* studied in [DG], [GS3], [Da1], [Da3]. Remind that if $T \in D(\mathcal{R}, \alpha)$ then the transfer function from $\alpha \circ T$ to α is determined up to multiplication by a constant from $Z(G)$.

Lemma 3.5 (cf. [Da3, Proposition 5.7]). *If $\mathcal{R}$ is hyperfinite then p_2 and hence π are onto.*

Consider the case of compact G in more details. First we state two auxiliary topological statements

Lemma 3.6. *Let P be a Polish monoid and N a compact normal subgroup of P. Then P/N with the quotient topology is a Polish monoid. Moreover, the canonical projection is an open map.*

Proof. Let t be a complete bounded metric on P compatible with the topology. Given $p_1, p_2 \in P$, we put $t'(p_1, p_2) = \int_N \int_N t(n_1 p_1 n_2, n_1 p_2 n_2)\, d\lambda_N(n_1)\, d\lambda_N(n_2)$, where λ_N is the probability Haar measure on N. Then t' is a (two side) N-invariant complete metric on P, compatible with the topology. Now we define a metric d on the quotient monoid P/N by setting $d(p_1 N, p_2 N) = t'(p_1 N, p_2 N)$, the distance between two cosets in G. It is easy to verify that d is complete and does the trick. The second statement of the lemma is obvious. $\square$

Lemma 3.7 ([Ku], [La]). *Let $t : Y \to Z$ be a continuous map of Polish spaces. If $t^{-1}(z)$ is σ-compact for all $z \in Z$ then $t(X)$ is a Borel subset of Z.*

Since every group endomorphism preserves the probability Haar measure, we obtain $\operatorname{End} G \subset \operatorname{End}_\times(G, \lambda_G)$. Moreover, $\operatorname{End} G$ is a weakly closed submonoid of $\operatorname{End}_\times(G, \lambda_G)$ and the induced topology coincides with the topology of uniform convergence on G. Next, $\operatorname{Inn} G$ is compact in $\operatorname{End} G$ and by Lemma 3.6 $\operatorname{Ond} G$ with the quotient topology is a Polish monoid. Moreover, (3-4) is an exact sequence of Polish monoids.

Every lift $\widetilde{S}$ of $S \in \mathcal{L}^\bullet(\mathcal{R}, \alpha)$ is $(\lambda \times \lambda_G)$-multiplying and $\Delta_\lambda(S) = \Delta_{\lambda \times \lambda_G}(\widetilde{S})$. Remind that Δ denotes the dilation function (see §1). It is easy to see that $\widetilde{\mathcal{L}}^\bullet(\mathcal{R}(\alpha))$ is a Polish submonoid of $\mathcal{N}_\times^*[\mathcal{R}(\alpha)]$ and $p : \widetilde{\mathcal{L}}^\bullet(\mathcal{R}(\alpha)) \ni \widetilde{S} \mapsto S \in \mathcal{N}[\mathcal{R}]$ is continuous. Moreover, the map $R : G \ni h \mapsto R(h) \in \operatorname{Ker} p$ (see Lemma 3.2) is a topological group isomorphism. Hence Lemma 3.6 implies that $\mathcal{L}^\bullet(\mathcal{R}, \alpha)$ endowed with the quotient topology, say the *L-topology*, is a Polish monoid. Clearly, the L-topology is stronger than the normal one. It follows from Lemma 3.7 that $\mathcal{L}^\bullet(\mathcal{R}, \alpha)$ is a Borel subset of $N[\mathcal{R}]$. Equip $M(X, G)$ with the (Polish) topology of convergence in measure. It is easy to see that $\mathfrak{G}$ with the product topology is a Polish monoid and $\mathfrak{G}_0$ is closed in $\mathfrak{G}$. We claim that J is a homeomorphism (see Lemma 3.4). Actually, it is obvious that J^{-1} is continuous. Let us prove that so is J. From the definition we obtain that $\widetilde{\mathcal{L}}^\bullet(\mathcal{R}(\alpha)) \ni S_{l,f} \mapsto S \in \mathcal{L}^\bullet(\mathcal{R}, \alpha)$ is a continuous map. Next, since $S_{l,f} R(h) = R(l(h)) S_{l,f}$ for all $S_{l,f} \in \widetilde{\mathcal{L}}^\bullet(\mathcal{R}(\alpha))$ and $h \in G$, one can deduce in a rather standard way (see, for example, [LLT], [ALV]) that p_2 is continuous. These facts plus (3-2) imply that $S_{l,f} \ni \widetilde{\mathcal{L}}^\bullet(\mathcal{R}(\alpha)) \mapsto f \in \mathcal{L}^\bullet(\mathcal{R}, \alpha)$ is also a continuous map and we are done. It follows that π is continuous.

We summarize the properties of these maps in

Proposition 3.8. *Let $\mathcal{R}$ be hyperfinite, G amenable l.c.s.c., and α have dense* *range in G. Then the following diagram commutes*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z(G) & \longrightarrow & G & \longrightarrow & \operatorname{Inn} G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R} & & \downarrow & & \\
1 & \longrightarrow & \operatorname{Ker} p_2 & \longrightarrow & \widetilde{\mathcal{L}}^\bullet(\mathcal{R}(\alpha)) & \xrightarrow{p_2} & \operatorname{End} G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \\
1 & \longrightarrow & D(\mathcal{R},\alpha) & \longrightarrow & \mathcal{L}^\bullet(\mathcal{R},\alpha) & \xrightarrow{\pi} & \operatorname{Ond} G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Moreover, if G is compact all arrows are continuous maps.

We state a refinement of [Da3, Proposition 5.9]

Proposition 3.9. *Let $\mathcal{R}$ be an ergodic hyperfinite measure preserving equivalence relation and S an outer automorphism of $\mathcal{R}$, i.e. $S \in N[\mathcal{R}] \setminus [\mathcal{R}]$. Then there is a residual subset of cocycles $\alpha \in Z^1(\mathcal{R}, G)$ with dense range in G such that $S \notin \mathcal{L}^\bullet(\mathcal{R},\alpha)$.*

Now we provide an orbital version of Main Theorem.

Theorem 3.10. *Let G be an amenable l.c.s.c. group, $\mathcal{R}$ a hyperfinite measure preserving equivalence relation, (I, I_+) a countable amenable left ordered group. Let $\omega : I_+ \to \operatorname{End} G$ be a monoid homomorphism and $S : I \to N[\mathcal{R}]$ a group homomorphism such that $S(i) \in [\mathcal{R}]$ implies $i = 1_I$. Then there exists a cocycle $\delta : \mathcal{R} \to G$ with dense range in G such that $S(I_+) \subset \mathcal{L}^\bullet(\mathcal{R}, \delta)$ and $\delta \circ S(i) \approx \omega(i) \circ \delta$ for all $i \in I_+$.*

Proof. We set $G_i = G$ for all $i \in I$ and $p_{ij} = \omega(j^{-1}i)$ for all $i \succ j$. Then (G_i, p_{ij}) is a reverse spectrum of amenable l.c.s.c. groups. Denote by $\widehat{G}$ the projective limit $\operatorname{proj\,lim}(G_i, p_{ij})$ and by $p^{(i)} : \widehat{G} \to G_i$ the canonical projections. An element of $\widehat{G}$ can be viewed as a sequence $\{g_i\}_{i\in I}$ of G-elements such that $g_i = \omega(j^{-1}i)g_j$ for all $i \succ j$. Now we define a group homomorphism $\widehat{\omega} : I \to \operatorname{Aut} \widehat{G}$ as follows: $(\widehat{\omega}(k)g)_j = g_{k^{-1}j}$. It is easy to verify that $\widehat{\omega}$ is well defined. Hence we can form the semidirect product $E = \widehat{G} \rtimes_{\widehat{\omega}} I$ with the multiplication law as follows

$$(g, i)(g', i') = (g \cdot \widehat{\omega}(i)g', ii').$$

By Corollary 2.14 there is a cocycle α of $\mathcal{R}$ with dense range in E. Let $\mathcal{S} := \{(x, y) \in \mathcal{R} \mid \alpha(x, y) \in \widehat{G}\}$. Since α has dense range in E and $\widehat{G}$ is an open subgroup of E, it follows that $\mathcal{S}$ is an ergodic subrelation of $\mathcal{R}$. Next, use Lemma 2.3 to produce transformations $R_i \in [\mathcal{R}]$ with $\alpha(R_i x, x) = (1_I, i)$, $i \in I$. Without loss in generality we may assume that each R_i is measure preserving (if μ is finite then this holds automatically). Since

$$\alpha(R_i x, R_i y) = \alpha(R_i x, x)\alpha(x, y)\alpha(x, R_i y) \in (1_I, i) \cdot \widehat{G} \cdot (1_I, i)^{-1} = \widehat{G},$$

for all $(x, y) \in \mathcal{S}$, it follows that $R_i \in N[\mathcal{S}]$. Moreover, the map $I \ni i \mapsto R_i \in [\mathcal{R}]$ is an outer near homomorphism, i.e. $R_i \in [\mathcal{S}]$ if and only if $i = 1_I$ and $R_i R_j \in R_{ij}[\mathcal{S}]$ for all $i, j \in I$. Since I is amenable, it is well known that there exists a group homomorphism $I \ni i \mapsto T(i) \in N[\mathcal{S}]$ such that $T(i)R_i^{-1} \in [\mathcal{S}]$ [FSZ, §3]. By Lemma 2.2 the cocycle $\beta := p^{(1)} \circ (\alpha \restriction \mathcal{S}) \in Z^{(1)}(\mathcal{S}, G)$ has dense range in G. Next,

$$\beta \circ T(i)(x, y) = p^{(1)}(\alpha(T(i)x, x)\alpha(x, y)\alpha(y, T(i)y)) =$$
$$p^{(1)}((1_I, i) \cdot \alpha(x, y) \cdot (1_I, i)^{-1}) = p^{(1)}(\widehat{\omega}(i) \circ \alpha(x, y)) = \alpha(x, y)_{i^{-1}} = \omega(i) \circ \beta(x, y)$$

for all $(x, y) \in \mathcal{S}$. Now we set $Z := X \times X$, $\nu := \lambda \times \lambda$, $\mathcal{P} := \mathcal{S} \times \mathcal{R}$, $\widehat{S}(i) := T(i) \times \mathcal{S}(i)$, and $\widehat{\alpha} := \beta \otimes 1 \in Z^1(\mathcal{P}, G)$. Then $\mathcal{P}$ is a hyperfinite ergodic ν-preserving equivalence relation on Z, the cocycle $\widehat{\alpha}$ has dense range in G, $\widehat{\alpha} \circ \widehat{S}(i) \approx \widehat{\alpha}$, and $\Delta_\nu(\widehat{S}(i)) = \Delta_\mu(S(i))$.

It follows from [BG, Theorem 4.1] that $\widehat{S}$ and S are other conjugate, i.e. there exists an isomorphism $U : (X, \mu) \to (Z, \nu)$ such that $(U \times U)\mathcal{R} = \mathcal{P}$ and $US(i)U^{-1} \in \widehat{S}(i)^{-1}[\mathcal{P}]$. It is easy to see that the cocycle $\delta := \widehat{\alpha} \circ U : \mathcal{R} \to G$ is as desired. □

Remark 3.11. We indicate two particular cases where Theorem 3.10 can be proved without use of Corollary 2.14:

(i) G is compact. If this is the case then $\widehat{G}$ is compact and hence E is amenable l.c.s.c., and one can apply standard Lemma 2.5 instead of Corollary 2.14.

(ii) G and I are solvable. If this is the case then E is solvable. Pick up a dense countable subgroup E_d of E. It is solvable and hence amenable as a discrete group. By Lemma 2.5 there exists a cocycle of an ergodic hyperfinite equivalence relation with dense range in E_d. It is clear that this cocycle viewed as one with values in E has dense range in it.

Applications to the "classical" lifting problems. Let T be an ergodic measure preserving transformation of $(X, \mathfrak{B}, \mu)$. We assume that μ is finite or σ-finite. Put

$$\mathcal{C}(T) = \{R \in \mathrm{End}(X, \mathfrak{B}, \mu) \mid RT = TR\},$$
$$\mathcal{C}_\times(T) = \{S \in C(T) \mid \mu \restriction S^{-1}\mathfrak{B} \text{ is } \sigma\text{-finite}\}.$$

Remind that $C(T) = \mathcal{C}(T) \cap \mathrm{Aut}(X, \mu)$. Denote by $\mathcal{R}$ the T-orbital equivalence relation. It is easy to see that $\mathcal{C}(T)$ is a submonoid of $\mathcal{N}^*[\mathcal{R}]$. From Proposition 1.5 we deduce that $\mathcal{C}_\times(T)$ is a submonoid of $\mathcal{N}^*_\times[\mathcal{R}]$. Moreover, it follows from Lemma 1.7 that $\mathcal{C}_\times(T)$ is closed in $\mathcal{N}^*_\times[\mathcal{R}]$ with respect to the normal topology which induces the weak one on $\mathcal{C}_\times(T)$. Notice that if μ is finite then $\mathcal{N}^*_\times[\mathcal{R}] = \mathcal{N}^*[\mathcal{R}]$ and hence $\mathcal{C}_\times(T) = C(T)$, and the dilation function is trivial.

Now given a l.c.s.c. amenable group G and a measurable function $\phi : X \to G$, one can define a cocycle $\alpha_\phi : \mathcal{R} \to G$ by setting

$$\alpha_\phi(x, Tx) = \phi(x), \qquad x \in X,$$

and then extending α_ϕ to the whole $\mathcal{R}$ via the cocycle identity. Notice that the map $M(X, G) \ni \phi \to \alpha_\phi \in Z^1(\mathcal{R}, G)$ is a homeomorphism if we furnish the two spaces with the topology of convergence in measure. Furthermore, if G is Abelian, then this map is a group homomorphism. We call elements of $\mathcal{M}(X, G)$ cocycles if no confusion arises. Thus two cocycles $\phi, \psi : X \to G$ are cohomologous if $\phi(x) = f(x)\psi(x)f(Tx)^{-1}$ for some function $f \in M(X, G)$. It is easy to see that:

(i) the skew product extension $T_\phi \in \mathrm{Aut}(X \times G, \mu \times \lambda_G)$ of T (see §0) generates $\mathcal{R}(\alpha_\phi)$;

(ii) if ϕ and ψ are cohomologous then T_ϕ and T_ψ are conjugate in a canonical way.

Suppose that ϕ is ergodic, i.e. α_ϕ has dense range in G. We set $\mathcal{L}(T,\phi) := \mathcal{L}(\mathcal{R}, \alpha_\phi) \cap \mathcal{C}(T)$, $\widetilde{\mathcal{L}}(T_\phi) := \widetilde{\mathcal{L}}(\mathcal{R}(\alpha_\phi)) \cap \mathcal{C}(T_\phi)$, $\mathcal{L}^\bullet(T,\phi) := \mathcal{L}^\bullet(\mathcal{R}, \alpha_\phi) \cap \mathcal{C}(T)$, $\widetilde{\mathcal{L}}^\bullet(T_\phi) := \widetilde{\mathcal{L}}^\bullet(\mathcal{R}(\alpha)) \cap \mathcal{C}(T_\phi)$, $D(T,\phi) := D(\mathcal{R}, \alpha_\phi) \cap C(T)$ and $\widetilde{D}(T_\phi) := \mathrm{Ker}\, p_2 \cap C(T_\phi)$. Clearly, the first four objects are monoids, and the last two are groups.

It is worthwhile to remark that each lift of every $S \in \mathcal{L}(T,\phi)$ to $\mathcal{N}[\mathcal{R}(\alpha_\phi)]$ lies in $\widetilde{\mathcal{L}}(T_\phi)$. Thus we deduce from Theorem 3.3

Corollary 3.12 (cf. [JLM, Proposition 6.1])**.** *Every lift of an element $S \in \mathcal{L}(T,\phi)$ is of the form* (3-2) *with l, f satisfying* (3-3)*. If S is invertible, then l is onto. Given two lifts $S_{l,f}$ and $S_{l'f'}$ of S, there is $h \in G$ with $l(g) = hl'(g)h^{-1}$ and $f(x) = hf'(x)$, i.e. $S_{l,f} = R(h)S_{l',f'}$. Conversely, every triplet $(S, l, f) \in C(T) \times \mathrm{End}\, G \times M(X, G)$ satisfying* (3-3) *defines a lift of S to $\mathcal{C}(T_\phi)$ by* (3-2)*. Moreover, $S_{l,f}$ is invertible if and only if so is l.*

Remark that if S is not invertible then it may happen that l is not onto. We give an example where l is trivial.

Example 3.13**.** Let T be a Bernoulli shift, G a compact group and $\phi : X \to G$ a cocycle such that T_ϕ is weakly mixing. Then T_ϕ is conjugate to T [Ru]. Thus there is a measure space isomorphism $R : (X, \mu) \to (X \times G, \mu \times \lambda_G)$ with $R^{-1}T_\phi R = T$. Denote by $P : X \times G \to X$ the first coordinate projection and set $Q := RP$, $\psi := \phi \circ R^{-1}$. It is easy to see that $Q \in \mathcal{C}(T_\phi)$ and ψ is a cocycle of T_ϕ with dense range in G. We have

$$\psi \circ Q(x, g) = (\phi \circ R^{-1}) \circ RP(x, g) = \phi(x) = g^{-1}g\phi(x) = f(x, g)^{-1}f(T_\phi(x, g)),$$

where $f(x, g) = g$. Thus $\psi \circ Q$ is a T_ϕ-coboundary. We see that $Q_{\mathrm{id},f} \in \mathcal{C}((T_\phi)_\psi)$, where $Q_{\mathrm{id},f}(x, g, h) = (Q(x, g), hg)$.

Proposition 3.8 implies

Corollary 3.14 (cf. [LLT, Proposition 2])**.** *The following diagram commutes*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z(G) & \longrightarrow & G & \longrightarrow & \mathrm{Inn}\, G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R} & & \downarrow & & \\
1 & \longrightarrow & \widetilde{D}(T_\phi) & \longrightarrow & \widetilde{\mathcal{L}}^\bullet(T_\phi) & \xrightarrow{p_2} & p_2(\widetilde{\mathcal{L}}^\bullet(T_\phi)) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \\
1 & \longrightarrow & D(T,\phi) & \longrightarrow & \mathcal{L}^\bullet(T,\phi) & \xrightarrow{\pi} & \pi(\widetilde{\mathcal{L}}^\bullet(T_\phi)) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

We conclude this paper with observation that Main Theorem follows directly from Theorem 3.10 (see also Remark 3.11).

References

[ALMN] J. Aaronson, M. Lemańczyk, Ch. Mauduit, and H. Nakada, *Koksma's inequality and group extensions of Kronecker transformations*, in Algorithms, Fractals and Dynamics (Y. Takahashi, ed.), Plenum Press, 1995, pp. 27–50.

[ALV] J. Aaronson, M. Lemańczyk, and D. Volný, *A salad of cocycles*, preprint.

[BG] S. I. Bezuglyi and V. Ya. Golodets, *Outer conjugacy for actions of countable amenable groups on a measure space*, Izv. Akad. Nauk SSSR, Ser. Mat. **50** (1986), 643–660 (Russian); English transl. in Math. USSR-Izv. **29** (1987), 1–18.

[CHP] J. R. Choksi, J. M. Hawkins, and V. S. Prasad, *Abelian cocycles for nonsingular ergodic transformations and genericity of type III_1 transformations*, Mh. Math. **103** (1987), 187–205.

[Da1] A. I. Danilenko, *On cocycles compatible with normalizers of full groups of measure space transformations*, Dokl. Akad. Nauk Ukr., 1994, No 7, 14–17.

[Da2] ———, *The topological structure of Polish groups and groupoids of measure space transformations*, Publ. RIMS Kyoto Univ. **31** (1995), 913–940.

[Da3] ———, *Comparison of cocycles of measured equivalence relations and lifting problems*, Ergod. Th. and Dynam. Sys. **18** (1998), 125–151.

[DG] A. I. Danilenko and V. Ya. Golodets, *Extension of cocycles to normalizer elements, outer conjugacy and related problems*, Trans. Amer. Math. Soc. **348** (1996), 4857–4882.

[Dy] H. Dye, *On groups of measure preserving transformations.* I, Amer. J. Math. **81** (1959), 119–159.

[Ef] E. G. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conf. Ser. in Math., vol. 46, Amer. Math. Soc., Providence, R.I., 1981.

[FM1] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras.* I, Trans. Amer. Math. Soc. **234** (1977), 289–324.

[FM2] ———, *Ergodic equivalence relations, cohomology, and von Neumann algebras.* II, Trans. Amer. Math. Soc. **234** (1977), 325–359.

[FSZ] J. Feldman, C. E. Sutherland, and R. J. Zimmer, *Subrelations of ergodic equivalence relations*, Ergod. Th. and Dynam. Syst. **9** (1989), 239–269.

[GS1] V. Ya. Golodets and S. D. Sinelshchikov, *Existence and uniqueness of cocycles of an ergodic automorphism with dense ranges in amenable groups*, Preprint, Inst. Low Temp. Phys. and Engin., No 19-83, Kharkov, 1983.

[GS2] ———, *Amenable ergodic actions of groups and images of cocycles*, Soviet Math. Dokl. **41** (1990), 523–525.

[GS3] ———, *Classification and structure of cocycles of amenable ergodic equivalence relation*, J. Funct. Anal. **121** (1994), 455–485.

[Go] K. R. Goodearl, *Partially ordered abelian groups with interpolation*, Math. surveys and monogr. ser., vol. 20, Amer. Math. Soc., Providence, R.I., 1986.

[HO] T. Hamachi and M. Osikawa, *Ergodic groups of automorphisms and Krieger's theorems*, Sem. Math. Sci., Keio University, Vol. 3, 1981.

[Is] R. S. Ismagilov, *Application of group algebra to problems on residual σ-algebra of random walks and ergodicity of skew action*, Izv. AN SSSR, Ser. Mat. **51** (1987), 893–907 (Russian); English transl. in Math. USSR-Izv.

[JLM] A. del Junco, M. Lemańczyk, and M. Mentzen, *Semisimplicity, joinings, and group extensions*, Studia Math. **112** (1995), 141–164.

[KW] Y. Katznelson and B. Weiss, *Commuting measure-preserving transformations*, Isr. J. Math. **12** (1972), 161–173.

[Ku] K. Kunugui, *Sur un problème de M. E. Szpilrajn*, Proc. Japan Acad. Ser. A **16** (1940), 70–108.

[La] D. G. Larman, *Projecting and uniformormizing Borel sets with K_σ-sections.* I, Mathematika **19** (1972), 231–244; II, Mathematika **20** (1973), 233–244.

[LLT] M. Lemańczyk, P. Liardet and J-P. Thouvenot, *Coalescence of circle extensions of measure preserving transformations*, Ergod. Th. and Dyn. Syst. **12** (1992), 769–789.

[Ne] D. Newton, *On canonical factors of ergodic dynamical systems*, J. London Math. Soc. **19** (1979), 129–136.

[OW] D. Ornstein and B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, J. d'Analyse Math. **48** (1987), 1–142.

[PS] K. R. Parthasarathy and K. Schmidt, *On the cohomology of a hyperfinite action*, Mh. Math. **84** (1977), 37–48.

[Ru] D. Rudolph, *Classifying the isometric extensions of a Bernoulli shift*, J. d'Analyse Math. **34** (1978), 36–60.

[Sc] K. Schmidt, *Lectures on cocycles of ergodic transformation groups*, Lect. Notes in Math. Vol.1, Macmillan Co. of India, 1977.

[Z1] R. Zimmer, *Random walks on compact groups and existence of cocycles*, Isr. J. Math. **26** (1977), 84–90.

[Z2] ———, *Amenable ergodic group actions and an application to Poisson boundaries of random walks*, J. Funct. Anal. **27** (1978), 350–372.

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