COMPARISON OF COCYCLES OF MEASURED EQUIVALENCE RELATION AND LIFTING PROBLEMS

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Abstract. Let $\mathcal{R}$ be an ergodic discrete equivalence relation on a Lebesgue space and α its cocycle with values in a locally compact group G. We say that an automorphism of $\mathcal{R}$ is compatible with α if it preserves the cohomology class of α. We introduce a quasiorder relation on the set of all cocycles of $\mathcal{R}$ by means of comparison of the corresponding groups of all automorphisms being compatible with them. We find simple necessary and sufficient conditions under which two cocycles of a hyperfinite measure-preserving equivalence relation with values in compact (possibly different) groups are connected by this relation. Next, given an ergodic subrelation $\mathcal{S}$ of $\mathcal{R}$, we investigate the problem of extending $\mathcal{S}$-cocycles up to $\mathcal{R}$-cocycles and improve the recent results of Gabriel-Lemańczyk-Schmidt. As an application we study the problem of lifting of automorphisms of $\mathcal{R}$ up to automorphisms of the skew product $\mathcal{R} \times_\alpha G$, provide new short proofs of some known results and answer several questions from [ALV, ALMN].

0. Introduction

Let $\mathcal{R}$ be an ergodic discrete equivalence relation on a standard probability space $(X, \mathfrak{B}, \mu)$ and $N[\mathcal{R}]$ the group of nonsingular automorphisms of $\mathcal{R}$. Given a (1-) cocycle α of $\mathcal{R}$ with values in a locally compact second countable (l.c.s.c.) group G, we say that an element $\theta \in N[\mathcal{R}]$ is compatible with α if $\alpha \circ \theta$ is cohomologous to α [GDB]. For two cocycles α, β with values in (possibly, different) l.c.s.c. groups we write $\alpha \prec \beta$ or $\alpha \asymp \beta$ if the group of α-compatible automorphisms contains the group of β-compatible ones or if these groups coincide respectively. Let $Z^1(\mathcal{R})$ stand for the set of all $\mathcal{R}$-cocycles. Then $(Z^1(\mathcal{R})/\asymp, \prec/\asymp)$ is an upper semilattice, where $/\asymp$ denotes the factorization by modulo the equivalence relation $\asymp$. A problem arises to describe the order relation $\prec/\asymp$ in a more explicit fashion. A particular case of this problem was considered in [D1], where the structure of cocycles of a hyperfinite equivalence relation with values in Abelian groups and compatible with the whole $N[\mathcal{R}]$ was described. In Theorem 3.4 of the present paper we find simple necessary and sufficient conditions when two cocycles α, β satisfy $\alpha \prec \beta$ in terms of the Mackey action associated to $\alpha \times \beta$ provided that this action is transitive and $\mathcal{R}$ is measure preserving (=Krieger's type II) and hyperfinite.

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As it was shown in [GDB], the study of α-compatible automorphisms is intimately related to the problem of extending of α to some new relation containing $\mathcal{R}$ as a normal subrelation in the sense of [FSZ]. So, given an arbitrary pair of ergodic hyperfinite equivalence relations $\mathcal{S} \subset \mathcal{R}$, it seems natural to ask the following questions [GLS]:

(1) is the set of those $\mathcal{S}$-cocycles which have extensions to $\mathcal{R}$-cocycles of first category in the set of $\mathcal{S}$-cocycles?
(2) is the set of those $\mathcal{R}$-cocycles whose restrictions to $\mathcal{S}$ are cocycles generating the ergodic skew products (i.e. with dense Mackey range) residual in the set of $\mathcal{R}$-cocycles?

The affirmative answers were obtained in [GLS] for the case when $\mathcal{S}$ is normal in $\mathcal{R}$ with $\mathcal{R}/\mathcal{S}$ being a cyclic group. We generalize these to arbitrary quasinormal pair $\mathcal{S} \subset \mathcal{R}$ in Theorem 4.2. The notion of quasinormal subrelation is introduced in Section 4. Remark that all normal subrelations and subrelations of finite index are quasinormal.

In the sequel of the paper we deal with the lifting problems and apply the above results to them. Let T be an ergodic nonsingular transformation of $(X, \mathfrak{B}, \mu)$, $C(T)$ the group of nonsingular transformations commuting with T and α a cocycle of T with dense range in G which means that the Anzai skew product transformation T_α of $(X \times G, \mu \times \lambda)$ is ergodic, where λ is the Haar measure on G. Notice that T_α preserves the (σ-finite) measure $\mu \times \lambda$ provided that T preserves μ. We investigate the possibility of lifting $C(T)$-elements to $C(T_\alpha)$. We also ask: when T_α is squashable, i.e. $C(T_\alpha)$ contains non-$(\mu \times \lambda)$-preserving transformations (similar problems can be formulated also for a countable transformation group instead of T alone). These problems were studied by several of authors (see [Aa, ALV, ALMN, LLT, Ne, GLS] and references therein) with the classical tools of ergodic theory being applied, such as Rokhlin's lemma. Our approach is quite different—we consider their orbital analogues, i.e. replace T and $C(T)$ by an ergodic equivalence relation $\mathcal{R}$ and the normalizer $N[\mathcal{R}]$ of the full group $[\mathcal{R}]$ respectively. As it turns out, in the framework of the orbit theory many of these problems can be understood better. Moreover, the use of the remarkable theorem of Existence and Uniqueness of Cocycles of hyperfinite relations ([GS1–GS3]) leads to simple and explicit proofs of the known results and enable us to solve several open problems stated in [ALMN, ALV]. Denote by $L(\mathcal{R}, \alpha)$ the group of those elements of $N[\mathcal{R}]$ which can be lifted to $N[\mathcal{R}(\alpha)]$, where $\mathcal{R}(\alpha)$ is the skew product equivalence relation on $X \times G$. We obtain useful criteria for an element of $N[\mathcal{R}]$ to be in $L(\mathcal{R}, \alpha)$ (Corollary 5.5). In particular, if $\mathcal{R}$ is hyperfinite and of type II and G is compact, then $\theta \in L(\mathcal{R}, \alpha)$ if and only if $\alpha \prec \alpha \circ \theta$. Next, the structure of those automorphisms from $N[\mathcal{R}(\alpha)]$ which are the lifts of some elements of $L(\mathcal{R}, \alpha)$, is described explicitly (Theorem 5.3). We introduce the relevant Polish topology on $L(\mathcal{R}, \alpha)$ and a canonical homomorphism $\pi_\alpha : L(\mathcal{R}, \alpha) \to \mathrm{Out}G = \mathrm{Aut}G/\mathrm{Inn}G$, whose kernel is the group of α-compatible automorphisms. If $\mathcal{R}$ is hyperfinite, π_α is onto (Proposition 5.7).

We also show that if θ is an outer automorphism of $\mathcal{R}$, then it is a generic property in the set of all G-valued cocycles of $\mathcal{R}$ endowed with the (Polish) topology of convergence in measure that θ can not be lifted to $N[\mathcal{R}(\alpha)]$ (Theorem 5.9).

Now return to the classical point of view to observe that if $\mathcal{R}_T$ is the T-orbital equivalence relation, then every lift of an automorphism $\theta \in C(T) \subset N[\mathcal{R}_T]$ to $N[\mathcal{R}_T(\alpha)]$ lies in $C(T_\alpha)$, and $C(T_\alpha)$ endowed with the weak topology is closed in

$N[\mathcal{R}_T]$ endowed with the normal one (to be defined in Section 1). This enables us to deduce as simple corollaries some previously known results, see [Me, Theorems 1–3], [GLS, Theorem 5.1, Proposition 5.4], [ALMN, Proposition 1.1], [ALV, Continuous embedding Lemma], etc. Moreover, these results are generalized broadly, since normally we do neither assume that G is compact or Abelian (as it used to be in the cited papers) nor that T is finite measure preserving. Furthermore, T can be replaced everywhere by a countable transformation group. Remind that if T is an ergodic transformation with discrete spectrum, then every element $\theta \in C(T_\alpha)$ is a lift of some $\zeta \in C(T)$ [Ne, Me, ALMN]. In Example 6.2 we extend this fact to ergodic Abelian dynamical systems whose L^∞-eigenfunctions generate the entire σ-algebra $\mathfrak{B}$ (modulo 0). Other generalizations (including dynamical systems with quasidiscrete spectrum) are obtained in Propositions 6.1 and 6.3.

We prove that if G is amenable and admits an automorphism which multiplies the Haar measure, then there is a cocycle β of T with dense range in G such that T_β is squashable (Proposition 6.5). In the case where $G = \mathbb{R}$ and T has pure point spectrum this gives an answer to the question from [ALMN] (Corollary 6.6).

Next we note that the lifting problems are studied also in the noninvertible case [LLT, ALV]. Let $EC(T)$ be the semigroup of nonsingular endomorphisms of (X, μ) and $\operatorname{End} G$ the semigroup of continuous group endomorphisms of G. Then the canonical homomorphism $\pi(\alpha)$ can be extended naturally to the semigroup of those elements of $EC(T)$ which can be lifted to $EC(T_\alpha)$. This extended map takes values in $\operatorname{End} G / \operatorname{Inn} G$ (in $\operatorname{End} G$ if G is Abelian) and is a semigroup homomorphism. The following statement is one of the main results of [ALV]:

Theorem 0.1. *Let G be an Abelian group, T an ergodic probability preserving transformation, and $S_1, \dots, S_d \in C(T)$ $(d \leq \infty)$ are such that $(T, S_1, \dots, S_d)$ generate a free $\mathbb{Z}^{d+1}$-action. If $w_1, \dots, w_d \in \operatorname{End} G$ commute (i.e. $w_i w_j = w_j w_i$), then there is a cocycle α with dense range in G such that $S_1, \dots, S_d$ can be lifted to $EC(T_\alpha)$ and $\pi_\alpha(S_i) = w_i$, $1 \leq i \leq d$.*

We give a short proof of a generalization of Theorem 0.1 to G amenable under the additional assumption that $w_1, \dots, w_d \in \operatorname{Aut} G$ in Corollary 6.9 (our argument can be adapted easily to another particular case of Theorem 0.1: $d = 1$). Moreover, since the argument of Aaronson-Lemańczyk-Volný is based on the generalized Rokhlin's lemma which holds only for amenable actions, they ask whether there is an ergodic probability preserving transformation T and its cocycle α with dense range in the two-dimensional torus with $\pi_\alpha(C(T) \cap L(\mathcal{R}_T, \alpha)) = SL(2, \mathbb{Z})$? We answer in positive demonstrating Example 6.10.

Remark that the noninvertible case of lifting problems can be considered also in the framework of orbit theory. To this end we first need to find an orbit analogue of $EC(T)$ in such a way that most of the results of Sections 5 and 6 can be generalized easily to the noninvertible case. This will be done in our subsequent paper. Besides, an "orbital" proof of Theorem 0.1 in more general situation—G is arbitrary amenable—will be also contained there.

The outline of this paper is as follows. Section 1 contains background on measured equivalence relations and their cocycles. In Section 2 we extend to the non-Abelian case some facts from [GDB] related to automorphisms compatible with cocycles. Section 3 is devoted to the comparison of cocycles by means of comparing the groups of automorphisms compatible with them. The main result here is Theorem 3.4, where pairs of cocycles α, β with $\alpha \prec \beta$ are classified in terms of the corresponding sets of essential values. In Section 4 we introduce the idea of quasinormal subrelations of an ergodic equivalence relation. Our purpose here is to generalize the results from [GLS] about genericity of extensions of cocycles to quasinormal pairs. The orbital versions of the lifting problems are discussed in Section 5 and their applications to the classical ones (where only the centralizers of dynamical systems involved) in Section 6.

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1. Notation and introductory results

Let $(X, \mathfrak{B}, \mu)$ be a standard probability space. Denote by $\mathrm{Aut}(X, \mathfrak{B}, \mu)$ the group of its automorphisms, i.e. Borel, one-to-one, onto, μ-nonsingular transformations; we do not distinguish between two of them which agree on a μ-conull subset. Given a Borel discrete μ-nonsingular equivalence relation $\mathcal{R} \subset X \times X$, we endow it with the induced Borel structure and the σ-finite measure $\mu_{\mathcal{R}}$, $d\mu_{\mathcal{R}}(x, y) = d\mu(x)$, $(x, y) \in \mathcal{R}$. Write also

$$[\mathcal{R}] = \{\gamma \in \mathrm{Aut}(X, \mu) \mid (\gamma x, x) \in \mathcal{R} \text{ for } \mu\text{-a.a. } x \in X\},$$
$$N[\mathcal{R}] = \{\theta \in \mathrm{Aut}(X, \mu) \mid (\theta x, \theta y) \in \mathcal{R} \text{ for } \mu_{\mathcal{R}}\text{-a.a. } (x, y) \in \mathcal{R}\}$$

for the *full group* of $\mathcal{R}$ and the *normalizer* of $[\mathcal{R}]$ respectively. For a countable subgroup Γ of $\mathrm{Aut}(X, \mu)$ we denote by $\mathcal{R}_\Gamma$ the Γ-orbital equivalence relation (and it is known that each $\mathcal{R}$ is of the form $\mathcal{R}_\Gamma$ [FM]). $\mathcal{R}$ is called *hyperfinite* if it can be generated by a single automorphism. We assume from now on that $\mathcal{R}$ is ergodic, i.e. every $\mathcal{R}$-saturated Borel subset is μ-null or μ-conull.

Given $\theta \in N[\mathcal{R}]$, we set up $i(\theta)(x, y) = (\theta x, \theta y)$ for all $(x, y) \in \mathcal{R}$. Then $i(\theta)$ is a nonsingular transformation of $(\mathcal{R}, \mu_{\mathcal{R}})$ and the map $N[\mathcal{R}] \ni \theta \mapsto i(\theta) \in \mathrm{Aut}(\mathcal{R}, \mu_{\mathcal{R}})$ is a one-to-one homomorphism. Moreover, $i(N[\mathcal{R}])$ is closed in $\mathrm{Aut}(\mathcal{R}, \mu_{\mathcal{R}})$ with respect to the weak topology [D2, §3.1]. The induced (Polish) topology on $N[\mathcal{R}]$ is called *normal*. As it was shown in [D2] this topology is compatible with the distance d introduced in [HO] as follows. Let $\mathcal{R} = \mathcal{R}_\Gamma$ for a countable group $\Gamma = \{\gamma_k\}_{k=1}^\infty \subset \mathrm{Aut}(X, \mu)$. Denote by d_u the uniform distance on $[\mathcal{R}]$ given by

$$d_u(\gamma, \delta) = \mu(\{x \in X \mid \gamma^{-1}x \neq \delta^{-1}x\}) + \mu(\{x \in X \mid \gamma x \neq \delta x\})$$

and set up

$$d(\theta_1, \theta_2) = d_w(\theta_1, \theta_2) + \sum_{k=1}^\infty \frac{1}{2^k} \frac{d_u(\theta_1\gamma_k\theta_1^{-1}, \theta_2\gamma_k\theta_2^{-1})}{1 + d_u(\theta_1\gamma_k\theta_1^{-1}, \theta_2\gamma_k\theta_2^{-1})}$$

for $\theta_1, \theta_2 \in N[\mathcal{R}]$, where d_w is the metric which determines the weak topology on $\mathrm{Aut}(X, \mu)$.

Let G be a locally compact second countable (l.c.s.c.) group, 1_G the identity of G, and λ_G the right Haar measure on G. A Borel map $\alpha : \mathcal{R} \to G$ is a (1-)*cocycle* of $\mathcal{R}$ if

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z) \qquad \text{for a.a. } (x, y), (y, z) \in \mathcal{R}.$$

We do not distinguish between two cocycles if they agree $\mu_{\mathcal{R}}$-a.e. Two cocycles, $\alpha, \beta : \mathcal{R} \to G$, are *cohomologous* ($\alpha \approx \beta$), if

$$\alpha(x, y) = \phi(x)^{-1}\beta(x, y)\phi(y) \qquad \text{for } \mu_{\mathcal{R}}\text{-a.a. } (x, y),$$

where $\phi : X \to G$ is a Borel function (we call it a *transfer* function from α to β). A cocycle is a *coboundary* if it is cohomologous to the trivial one. The set of cocycles of $\mathcal{R}$ with values in G will be denoted by $Z^1(\mathcal{R}, G)$.

Let $\mathcal{R} = \mathcal{R}_\Gamma$. There is a cocycle $\rho \in Z^1(\mathcal{R}, \mathbb{R})$ such that $\rho(x, \gamma x) = \log \frac{d\mu\circ\gamma}{d\mu}(x)$ for all $\gamma \in \Gamma$ at μ-a.e. $x \in X$. It is called the *Radon-Nikodym* cocycle of $\mathcal{R}$. If $\rho \equiv 1$ then μ is $[\mathcal{R}]$-invariant. $\mathcal{R}$ is said to be of type II_1 (II_∞) if there exists a finite (infinite, σ-finite) $[\mathcal{R}]$-invariant measure on X equivalent to μ.

Let $\mathcal{R}$ and $\mathcal{S}$ be two equivalence relations on measure spaces $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{G}, \nu)$ respectively, and some cocycles $\alpha \in Z^1(\mathcal{R}, G)$ and $\beta \in Z^1(\mathcal{S}, G)$ be given. The pairs $(\mathcal{R}, \alpha)$ and $(\mathcal{S}, \beta)$ (or, simply, α and β) are *weakly equivalent* if there is a Borel isomorphism $\theta : X \to Y$ such that $\mu \sim \nu \circ \theta$, $\theta \times \theta(\mathcal{R}) = \mathcal{S}$, and $\alpha \approx \beta \circ \theta$, where the cocycle $\beta \circ \theta \in Z^1(\mathcal{R}, G)$ is defined by $\beta \circ \theta(x, z) = \beta(\theta x, \theta z)$.

For every $\alpha \in Z^1(\mathcal{R}, G)$ and a countable group $\Gamma \subset \mathrm{Aut}(X, \mathfrak{B}, \mu)$ with $\mathcal{R} = \mathcal{R}_\Gamma$ we define the α-*skew product* action of Γ on the space $(X \times G, \mathfrak{P}, \mu \times \lambda_G)$, where $\mathfrak{P}$ is the product Borel field, by

$$\gamma_\alpha(x, g) = (\gamma x, g\alpha(x, \gamma x)), \qquad x \in X, g \in G, \gamma \in \Gamma.$$

We observe that if γ preserves μ then γ_α preserves $\mu \times \lambda_G$. Let $\Gamma_\alpha = \{\gamma_\alpha \mid \gamma \in \Gamma\}$. Notice that the Γ_α-orbital equivalence relation on $X \times G$ does not depend on the particular choice of Γ with $\mathcal{R}_\Gamma = \mathcal{R}$. It will be denoted by $\mathcal{R}(\alpha)$. We also define a Borel action V_α of G on the same space as follows

$$V_\alpha(h)(x, g) = (x, hg), \qquad x \in X, \quad g, h \in G.$$

Since V_α commutes with Γ_α, it induces an action W_α of G on the measure space of Γ_α-ergodic components. W_α is called the *Mackey action* of G associated to α. It is ergodic, since $\mathcal{R}$ is. If two cocycles α, β are weakly equivalent, then the associated Mackey actions W_α and W_β defined on measure spaces (Y, λ) and (Z, κ) respectively are *conjugate*, i.e. there is a nonsingular map $\xi : Y \to Z$ such that $\xi W_\alpha(g)\xi^{-1} = W_\beta(g)$ for each $g \in G$.

A cocycle $\alpha \in Z^1(\mathcal{R}, G)$ is *transient* if $\mathcal{R}(\alpha)$ is of type I, i.e. the $\mathcal{R}(\alpha)$-partition of $X \times G$ is measurable. Otherwise α is called *recurrent*. α has *dense range* in G if W_α is the trivial action on a one-point set, i.e. the α-skew product action is ergodic. More generally, α is *regular* if W_α is (essentially) transitive. Then there exists a closed subgroup $s(\alpha) \subset G$ such that W_α is conjugate to the G-action on the homogeneous space $G/s(\alpha)$ via translations. Notice that $s(\alpha)$ is determined up to conjugacy in G. It is well known that the properties of cocycles to be transient, recurrent, or regular are invariant under the weak equivalence. Moreover, if α and β are weakly equivalent regular cocycles, then $s(\alpha)$ and $s(\beta)$ are conjugate.

Uniqueness Theorem for cocycles. [GS1, GS3, Theorem 3.1] *Let $\mathcal{R}$ and $\mathcal{S}$ be two ergodic hyperfinite equivalence relations on standard probability spaces $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{C}, \nu)$ respectively, and $\alpha \in Z^1(\mathcal{R}, G)$, $\beta \in Z^1(\mathcal{S}, G)$ recurrent cocycles. If $\mathcal{R}$ and $\mathcal{S}$ are both of type II_1 or II_∞ and the Mackey actions W_α and W_β of G are conjugate, then α and β are weakly equivalent.*

It follows that if $\mathcal{R}$ is hyperfinite and of type II and $\alpha, \beta \in Z^1(\mathcal{R}, G)$ are regular cocycles then they are weakly equivalent if and only if $s(\alpha)$ and $s(\beta)$ are conjugate. (Remark that each regular cocycle is recurrent).

An element $g \in G$ is an *essential value* of α if for every $A \in \mathfrak{B}$ of positive measure and every open neighborhood O of g there exists $\gamma \in [\mathcal{R}]$ with

$$\mu(\gamma^{-1}A \cap \{x \in A \mid \alpha(x, \gamma x) \in O\}) > 0.$$

The set $r(\alpha)$ of essential values of α is a closed subgroup of G.

Lemma 1.1[S1, S2]**.** *The following are equivalent*

(i) *α is regular,*

(ii) *there exist $\beta \in Z^1(\mathcal{R}, G)$ and a closed subgroup $H \subset G$ such that $\beta \approx \alpha$, $\beta(\mathcal{R}) \subset H$ (i.e. β takes values in H), and $r(\beta) = H$.*

If (ii) *is satisfied then the conjugate class of $s(\alpha)$ in G contains H.*

It is easy to deduce from this lemma

Corollary 1.2. *Let α, β be two cocycles of $\mathcal{R}$ with values in l.c.s.c. groups G and F respectively. If the cocycle $\alpha \times \beta \in Z^1(\mathcal{R}, G \times F)$ is regular, then so are both α and β. Moreover, if $\alpha \times \beta(\mathcal{R}) \subset r(\alpha \times \beta)$ then $r(\alpha)$ (resp. $r(\beta)$) is the closure of the image of the first (resp. second) coordinate projection of $r(\alpha \times \beta) \subset G \times F$.*

We also need several auxiliary facts.

Lemma 1.3 [GS3, Lemma 1.6]**.** *Let $\mathcal{R}$ be hyperfinite. Then if α has dense range in G, there is a cocycle $\beta \approx \alpha$ such that the subrelation $\mathcal{S} \stackrel{\text{def}}{=} \{(x, y) \in \mathcal{R} \mid \beta(x, y) = 1_G\}$ is ergodic.*

Lemma 1.4. *Let $\mathcal{R}$ be hyperfinite.*

(i) [Z1, GS1, Is] *If G is amenable then there exists a cocycle α of $\mathcal{R}$ with dense range in G,*

(ii) [Z2] *If there exists a cocycle of $\mathcal{R}$ with dense range in G then G is amenable.*

For more detailed exposition of these concepts we refer to [BG2, HO, S1, S2, Z3].

2. Automorphisms compatible with cocycles

The concept of an automorphism compatible with a cocycle was introduced in [BGD]. Here we briefly outline some definitions and facts related to it. See also [GDB] and [GS3, Section 2].

Let $\alpha \in Z^1(\mathcal{R}, G)$. An automorphism $\theta \in N[\mathcal{R}]$ is *compatible* with α if $\alpha \circ \theta \approx \alpha$. The set of transfer functions from $\alpha \circ \theta$ to α will be denoted by $\alpha(\theta)$ and the group of α-compatible automorphisms by $D(\mathcal{R}, \alpha)$. We see that $[\mathcal{R}] \subset D(\mathcal{R}, \alpha)$, since $\alpha \circ \gamma(x, y) = \alpha(x, \gamma x)^{-1}\alpha(x, y)\alpha(y, \gamma y)$ at a.e. $(x, y) \in \mathcal{R}$ for each transformation $\gamma \in [\mathcal{R}]$. Next, if $\alpha \approx \beta$, then $D(\mathcal{R}, \alpha) = D(\mathcal{R}, \beta)$. Notice also that $D(\mathcal{R}, \alpha \circ \theta) = \theta^{-1}D(\mathcal{R}, \alpha)\theta$ for each transformation $\theta \in N[\mathcal{R}]$.

Let $M(X, G)$ stand for the set of μ-measurable functions from X to G. We endow it with the (Polish) topology of convergence in measure. Then

$$C(\mathcal{R}, \alpha) \stackrel{\text{def}}{=} \{(\theta, \phi) \mid \theta \in D(\mathcal{R}, \alpha),\ \phi \in \alpha(\theta)\}$$

is a closed subset of $N[\mathcal{R}] \times M(X, G)$ (remind that $N[\mathcal{R}]$ is furnished with the normal topology). For a hyperfinite equivalence relation this assertion was proved in [GDB], but in the general case only a minor and evident modification of that argument is required. Furthermore, $C(\mathcal{R},\alpha)$ is a Polish group if we define the multiplication by setting

$$(\theta_1,\phi_1)\cdot(\theta_2,\phi_2)=(\theta_1\theta_2,\phi_3),\qquad (\theta_i,\phi_i)\in C(\mathcal{R},\alpha),$$

where $\phi_3(x)=\phi_2(x)\phi_1(\theta_2 x)$, $x\in X$ [GDB, GS3]. It is obvious that

$$\{1\}\to\alpha(\mathrm{Id})\xrightarrow{i}C(\mathcal{R},\alpha)\xrightarrow{p}D(\mathcal{R},\alpha)\to\{1\}$$

is an exact sequence of groups, where p is the first coordinate projection and the embedding i is given by $\alpha(\mathrm{Id})\ni\phi\mapsto i(\phi)=(\mathrm{Id},\phi)\in C(\mathcal{R},\alpha)$. Since $\alpha(\mathrm{Id})$ is closed in $C(\mathcal{R},\alpha)$, we have that $D(\mathcal{R},\alpha)$ endowed with the p-quotient topology—we call it the *α-topology*—is a Polish group.

Given $(\theta,\phi)\in C(\mathcal{R},\alpha)$, we define an automorphism θ_ϕ of $(X\times G,\mathfrak{P},\mu\times\lambda_G)$ by setting

$$\theta_\phi(x,g)=(\theta x,g\phi(x)),\qquad x\in X,\ g\in G.$$

Notice that $\theta_\phi\in N[\mathcal{R}(\alpha)]\cap C(V_\alpha)$, where

$$C(V_\alpha)=\{\zeta\in\mathrm{Aut}(X\times G,\mu\times\lambda_G)\mid \zeta V_\alpha(h)=V_\alpha(h)\zeta \text{ for all } h\in G\}$$

is the centralizer of V_α. Hence θ_ϕ induces an automorphism $\Psi_\alpha(\theta,\phi)$ of the space of $[\mathcal{R}(\alpha)]$-ergodic components and $\Psi_\alpha(\theta,\phi)\in C(W_\alpha)$. Moreover,

$$\Psi_\alpha: C(\mathcal{R},\alpha)\ni(\theta,\phi)\mapsto\Psi_\alpha(\theta,\phi)\in C(W_\alpha)$$

is a continuous homomorphism, provided that $C(W_\alpha)$ is furnished with the weak topology.

Given a closed subgroup H of G, we write $Z_G(H)=\{g\in G\mid gh=hg \text{ for all } h\in H\}$ and $N_G(H)=\{g\in G\mid gH=Hg\}$ for the centralizer and the normalizer of H in G.

Lemma 2.1. *Let $\mathcal{R}$ be hyperfinite, α a regular cocycle with $\alpha(\mathcal{R})\subset r(\alpha)$, and θ an α-compatible automorphism. Then*

(i) *for every pair $\phi,\psi\in\alpha(\theta)$ there is $a\in Z_G(r(\alpha))$ with $\phi(x)\psi(x)^{-1}=a$ for a.e. $x\in X$;*
(ii) *for every $\phi\in\alpha(\theta)$ there are $b\in N_G(r(\alpha))$ and a function $\sigma: X\to r(\alpha)$ such that $\phi(x)=b\sigma(x)$ for a.e. $x\in X$.*

Proof. It follows from Lemma 1.3 that there is a Borel function $\vartheta: X\to r(\alpha)$ and an ergodic subrelation $\mathcal{S}\subset\mathcal{R}$ such that the cocycle $\beta(x,y)=\vartheta(x)^{-1}\alpha(x,y)\vartheta(y)$ is trivial on $\mathcal{S}$. It is straightforward that the functions $\widetilde{\phi}(x)=\vartheta(x)^{-1}\phi(x)\vartheta(x)$ and $\widetilde{\psi}(x)=\vartheta(x)^{-1}\psi(x)\vartheta(x)$ belongs to $\beta(\theta)$. We set $\chi(x)=\widetilde{\phi}(x)\widetilde{\psi}(x)^{-1}$, $x\in X$. Since

$$\chi(x)^{-1}\beta(x,y)\chi(y)=\beta(x,y)\qquad \text{for a.e. } (x,y)\in\mathcal{R}, \tag{2-1}$$

χ is $[\mathcal{S}]$-invariant and hence equals a.e. to some $a\in G$. It is easy to deduce from the definition of essential values of cocycle that $\beta(\mathcal{R})$ is dense in $r(\alpha)$. Therefore (2-1) implies $a \in Z_G(r(\alpha))$. Then we have $\phi(x)\psi(x)^{-1} = \vartheta(x)\chi(x)\vartheta(x)^{-1} = a$ for a.e. $x \in X$, as claimed in (i).

Now we observe that $\beta(\theta x, \theta y) = \widetilde{\phi}(x)^{-1}\widetilde{\phi}(y)$ for a.e. $(x, y) \in \mathcal{S}$. Hence $\widetilde{\phi}(x)r(\alpha) = \widetilde{\phi}(y)r(\alpha)$ for a.e. $(x, y) \in \mathcal{S}$. Since $\mathcal{S}$ is ergodic and the homogeneous space $G/r(\alpha)$ is countably separated, $\widetilde{\phi}(x) = b\sigma(x)$ for some $b \in G$ and a Borel function $\sigma : X \ni x \mapsto \sigma(x) \in r(\alpha)$. Now we have

$$\sigma(x)\,\beta \circ \theta(x, y)\,\sigma(y)^{-1} = b\beta(x, y)b^{-1} \qquad \text{for a.e. } (x, y) \in \mathcal{R}.$$

Notice that the left hand side is a cocycle of $\mathcal{R}$ with dense range in $r(\alpha)$ and—like β—takes a dense subset of values in $r(\alpha)$. It follows that $b\,r(\alpha)\,b^{-1} = r(\alpha)$ as desired for (ii). $\square$

Thus if α is a regular cocycle of a hyperfinite equivalence relation, with $\alpha(\mathcal{R}) \subset r(\alpha)$ then $\alpha(\mathrm{Id})$ consists of a.e. constant functions with values in with $Z_G(r(\alpha))$.

Example 2.2. Let $\mathcal{R}$ be a hyperfinite equivalence relation on (X, μ) and α its cocycle with dense range in the multiplicative group $\mathbb{R}^*$. Such a cocycle exists because $\mathbb{R}^*$ is an Abelian group. Let also $G = GL(2, \mathbb{R})$ and H be the diagonal subgroup of G. Consider the product equivalence relation $\mathcal{R} \times \mathcal{R}$ on $(X \times X, \mu \times \mu)$ and its cocycle $\beta \in Z^1(\mathcal{R} \times \mathcal{R}, G)$ given by

$$\beta(x_1, y_1, x_2, y_2) = \begin{pmatrix} \alpha(x_1, x_2) & 0 \\ 0 & \alpha(y_1, y_2) \end{pmatrix}, \qquad (x_1, x_2),\, (y_1, y_2) \in \mathcal{R}.$$

It is clear that β is regular and $r(\beta) = H$. Denote by θ the Sakai flip, i.e. $\theta(x, y) = (y, x)$ for all $(x, y) \in X \times X$. Then $\theta \in N[\mathcal{R} \times \mathcal{R}]$ and

$$\beta \circ \theta(x_1, y_1, x_2, y_2) = \begin{pmatrix} \alpha(y_1, y_2) & 0 \\ 0 & \alpha(x_1, x_2) \end{pmatrix}, \qquad (x_1, x_2),\, (y_1, y_2) \in \mathcal{R}.$$

Hence $\theta \in D(\mathcal{R} \times \mathcal{R}, \beta)$ and the function $\phi(x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $(x, y) \in X \times X$, is in $\beta(\theta)$. Notice that since $\beta(\mathcal{R} \times \mathcal{R}) \subset H$, β can be regarded as a cocycle from $Z^1(\mathcal{R} \times \mathcal{R}, H)$. But in this case θ is not compatible with β, since otherwise there exists a function $\psi : X \times X \to H$ which belongs to $\beta(\theta)$. By Lemma 2.1(i) $\phi\psi^{-1} \in Z_G(H) = H$, a contradiction.

This effect is impossible for cocycles with values in Abelian groups [GDB].

Let H be a closed subgroup of G. There are two natural continuous group actions on the homogeneous space $Y = G/H$ endowed with the quotient topology. The first one is the left shiftwise action of G and the second one is the right shiftwise action of $N_G(H)$ (it is well defined). Notice that H equals to the stability group of this right action at each point of Y. Hence Y is a free right $N_G(H)/H$-space. Denote by λ a quasiinvariant (for both actions) measure on G/H.

Lemma 2.3. *The centralizer of the G-action in* $\mathrm{Aut}(Y, \lambda)$ *is* $N_G(H)/H$ (*or, more precisely, the image of* $N_G(H)/H$ *in* $\mathrm{Aut}(Y, \lambda)$).

Proof. Let $s : Y \to G$ be a Borel cross-section of the natural projection $G \to G/H$ and $h_s(g, y) = s(y)^{-1}g^{-1}s(gy)$, $g \in G$, $y \in Y$, the corresponding H-valued cocycle for the G-action, i.e. $h_s(g, y)h_s(k, gy) = h_s(kg, y)$ for all $k, g \in G$, $y \in Y$ (do not confuse h_s with cocycles of equivalence relations which we have considered so far). Given an automorphism θ from the centralizer of the G-action, we can write

$$s(\theta y) = s(y)f(y), \qquad y \in Y, \tag{2-2}$$

for some Borel function $f : Y \to G$. Applying [R1, Theorem 3.5] we can assume without loss of generality that $\theta gy = g\theta y$ for all $g \in G$ and $y \in Y$. It follows that

$$\begin{aligned} gs(y)h_s(g,y)f(gy) &= s(gy)f(gy) = s(\theta gy) = s(g\theta y) \\ &= gs(\theta y)h_s(g,\theta y) = gs(y)f(y)h_s(g,\theta y) \end{aligned} \tag{2-3}$$

for all $g \in G$, $y \in Y$. Divide out $gs(y)$ from each side and substitute $y = H$ and $g = s(y')$ to deduce that there are two Borel functions $l_1, l_2 : Y \to H$ and some $a \in G$ with

$$f(y') = l_1(y')al_2(y'), \qquad y' \in Y. \tag{2-4}$$

It follows from (2-3) and (2-4) that

$$l_2(y)h_s(g,\theta y)l_2(gy)^{-1} = a^{-1}l_1(y)^{-1}h_s(g,y)l_1(gy)a \tag{2-5}$$

for all $g \in G$, $y \in Y$. Take an arbitrary $y_0 \in Y$ and denote by S the stability group of y_0, i.e. $S = \{g \in G \mid gy_0 = y_0\}$. Notice that S coincides with the stability group of θy_0. Since

$$\{h_s(g,y_0) \mid g \in S\} = \{h_s(g,\theta y_0) \mid g \in S\} = H,$$

it follows from (2-5) that $a \in N_G(H)$. Hence (2-2) and (2-4) entail $\theta y = ya$ for all y, as desired. □

Remark that the particular case of the above statement—G is compact—was considered in [Ge] with a different approach.

It follows from Lemma 2.3 that if $\alpha \in Z^1(\mathcal{R}, G)$ is regular with $\alpha(\mathcal{R}) \subset r(\alpha)$ then $C(W_\alpha)$ is isomorphic to $N_G(r(\alpha))/r(\alpha)$. Furthermore, the weak topology on $C(W_\alpha)$ coincides with the locally compact quotient topology on $N_G(r(\alpha))/r(\alpha)$. Next, under the condition (ii) of Lemma 2.1 we have $\Psi_\alpha(\theta,\phi) = br(\alpha)$.

Lemma 2.4. *Let $\mathcal{R}$ be an ergodic hyperfinite type II equivalence relation on $(X, \mathfrak{B}, \mu)$, ν an $[\mathcal{R}]$-invariant μ-equivalent measure on X, and $\alpha \in Z^1(\mathcal{R}, G)$ a regular cocycle with $\alpha(\mathcal{R}) \subset r(\alpha)$. Then for each $a \in N_G(r(\alpha))$ there is a ν-preserving automorphism $\theta \in D(\mathcal{R}, \alpha)$ and a function $\phi \in \alpha(\theta)$ with $\Psi_\alpha(\theta, \phi) = ar(\alpha)$.*

Proof. Let $r(\alpha)Ⓢ\mathbb{Z}$ be the semidirect product of locally compact groups with the multiplication low given by

$$(h_1, n_1)(h_2, n_2) = (h_1 a^{n_1} h_2 a^{-n_1}, n_1 + n_2), \qquad h_1, h_2 \in r(\alpha),\ n_1, n_2 \in \mathbb{Z}.$$

Denote by v the natural continuous homomorphism of $r(\alpha)Ⓢ\mathbb{Z}$ into G:

$$r(\alpha)Ⓢ\mathbb{Z} \ni (h, n) \mapsto v(h, n) = ha^n \in G.$$

Since $r(\alpha)$ is amenable by Lemma 1.4(ii) and a semidirect product of amenable groups is amenable itself, there exists a cocycle β of $\mathcal{R}$ with dense range in $r(\alpha)Ⓢ\mathbb{Z}$

(see Lemma 1.4(i)). Moreover, in view of Lemma 1.3 one can pass to a cohomologous cocycle if necessary to assure that the subrelation

$$\mathcal{S} = \{(x,y) \in \mathcal{R} \mid \beta(x,y) = (1_G, 0)\} \subset \mathcal{R}$$

is ergodic. It is routine to deduce from the definition of $r(\beta)$ that there is an automorphism $\theta \in [\mathcal{R}]$ with $\beta(x,\theta x) = (f(x), 1)$ for a.a. $x \in X$, where $f : X \to r(\alpha)$ is a Borel function. We let

$$\mathcal{P} = \{(x,y) \in \mathcal{R} \mid \beta(x,y) \in r(\alpha) \times \{0\}\}.$$

Then $\mathcal{P}$ is an ergodic type II equivalence relation, because of $\mathcal{P} \supset \mathcal{S}$. Moreover, if $\mathcal{R}$ is of type II_1 or II_∞ then so is $\mathcal{P}$. Since $r(\alpha) \times \{0\}$ is an open subgroup of $r(\beta)$, we obtain that $r(\beta \restriction \mathcal{P}) = r(\alpha) \times \{0\}$ and hence $\beta \restriction \mathcal{P}$ is regular by Lemma 1.1.

Since $\beta(\theta x, \theta y) = \beta(x, \theta x)^{-1}\beta(x,y)\beta(y,\theta y) = (.,-1)(.,0)(.,1) = (.,0) \in r(\alpha) \times \{0\}$ for all $(x,y) \in \mathcal{P}$, we have $(\theta x, \theta y) \in \mathcal{P}$, i.e. $\theta \in N[\mathcal{P}]$. Moreover, since $\theta \in [\mathcal{R}] \subset D(\mathcal{R},\beta)$, it follows that $\theta \in D(\mathcal{P}, \beta \restriction \mathcal{P})$ and the function $\psi : X \ni x \mapsto (f(x), 1) \in r(\alpha) \circledS \mathbb{Z}$ lies in $(\beta \restriction \mathcal{P})(\theta)$. From this we deduce that $\Psi_\beta(\theta,\psi) = 1 \in \mathbb{Z}$ (notice that $N_{r(\alpha)\circledS\mathbb{Z}}(r(\alpha))/r(\alpha)$ is isomorphic to $\mathbb{Z}$ in a natural way). Since v is a continuous group homomorphism with $v(r(\alpha) \times \{0\}) = r(\alpha)$ and $v((1_G, 1)) = a$, we obtain that the composition $\delta = v \circ \beta \restriction \mathcal{P}$ is a regular cocycle of $\mathcal{P}$ with values in G, $r(\delta) = r(\alpha)$, and $\delta(\mathcal{P}) \subset r(\delta)$. Next, θ is compatible with δ, the function $\phi : X \ni x \overset{\text{def}}{\mapsto} v(\psi(x)) = f(x)a \in G$ lies in $\delta(\theta)$, and $\Psi_\delta(\theta,\phi) = ar(\alpha)$. By the Uniqueness Theorem for cocycles the pairs $(\mathcal{R},\alpha)$ and $(\mathcal{P},\delta)$ are weakly equivalent and we are done. $\square$

Denote by N_α the subgroup of G generated (algebraically) by $r(\alpha)$ and $Z_G(r(\alpha))$. We see that N_α is a normal subgroup of $N_G(r(\alpha))$.

Let $(\mathcal{R},\alpha)$ be as in Lemma 2.4. Suppose that N_α is closed in $N_G(r(\alpha))$. Then the map

$$\Phi_\alpha : D(\mathcal{R},\alpha) \ni \theta \mapsto \Phi_\alpha(\theta) = \Psi_\alpha(\theta,\phi)N_\alpha \in N_G(r(\alpha))/N_\alpha$$

is a well defined continuous epimorphism. We shall call it the *α-fundamental homomorphism*. The closure of $[\mathcal{R}]$ in $D(\mathcal{R},\alpha)$ is called the group of *α-approximately inner* automorphisms. We denote it by $\mathrm{Cl}_\alpha([\mathcal{R}])$.

Lemma 2.5. $\mathrm{Cl}_\alpha([\mathcal{R}]) = \Phi_\alpha^{-1}(1)$.

Proof. The inclusion $\subset$ is obvious. Now let $\theta \in D(\mathcal{R},\alpha)$ with $\Phi_\alpha(\theta) = 1$. Then there are $\phi \in \alpha(\theta)$ and $a \in Z_G(r(\alpha))$ with $\Psi_\alpha(\theta,\phi) = ar(\alpha)$. Remind that Ψ_α takes values in $N_G(r(\alpha))/r(\alpha)$. We set up $\psi(x) = a^{-1}\phi(x)$ for $x \in X$. It is easy to verify that $\psi \in \alpha(\theta)$ and $\Psi_\alpha(\theta,\psi) = r(\alpha)$. It follows from Lemma 2.1(ii) and the remark before Lemma 2.4 that $\psi(x) \in r(\alpha)$ for a.e. x. In virtue of [GDB, Theorem 3.4, GS3, Lemma 2.7] θ is α-approximately inner. $\square$

So the following short sequence of topological groups is exact:

$$\{1\} \to \mathrm{Cl}_\alpha([\mathcal{R}]) \to D(\mathcal{R},\alpha) \to N_G(r(\alpha))/N_\alpha \to \{1\}. \tag{2-6}$$

3. Comparison of cocycles

Let $\mathcal{R}$ be an ergodic equivalence relation on $(X, \mathfrak{B}, \mu)$. We need two relations on the set $Z^1(\mathcal{R})$ of all cocycles of $\mathcal{R}$ with values in l.c.s.c. groups.

Definition 3.1. Let α and β be two cocycles of $\mathcal{R}$ with values in (possibly different) l.c.s.c. groups G and H respectively We write

$$\alpha \prec \beta \text{ if } D(\mathcal{R}, \alpha) \supset D(\mathcal{R}, \beta);$$
$$\alpha \asymp \beta \text{ if } D(\mathcal{R}, \alpha) = D(\mathcal{R}, \beta).$$

Notice that if $\alpha' \approx \alpha \circ \theta$ and $\beta' \approx \beta \circ \theta$ for an automorphism $\theta \in N[\mathcal{R}]$, then $\alpha \prec \beta$ implies $\alpha' \prec \beta'$ as well as $\alpha \asymp \beta$ implies $\alpha' \asymp \beta'$. We see that both relations, $\prec$ and $\asymp$, can be lifted to $Z^1(\mathcal{R})/\approx$. It is obvious that $\asymp$ is an equivalence relation on both spaces $Z^1(\mathcal{R})$ and $Z^1(\mathcal{R})/\approx$ while $\prec/\asymp$ is an order relation on $Z^1(\mathcal{R})/\asymp$. Notice that the $\asymp$-class of the trivial cocycle is the least element for $\prec/\asymp$. In [GDB] an example of α was demonstrated with $D(\mathcal{R}, \alpha) = [\mathcal{R}]$. Hence the $\asymp$-class of α is the largest element for $\prec/\asymp$. Notice that for every pair α, β of cocycles of $\mathcal{R}$ there exists a supremum of their $\asymp$-classes in $Z^1(\mathcal{R})/\asymp$: $\sup([\alpha], [\beta]) = [\alpha \times \beta]$.

Proposition 3.2. *If $\alpha \prec \beta$ then the inclusion map $D(\mathcal{R}, \beta) \to D(\mathcal{R}, \alpha)$ is continuous.*

Proof. It is straightforward that $(\alpha \times \beta) \asymp \beta$. Moreover, the $(\alpha \times \beta)$-topology is clearly stronger than the β-topology. It follows from the Banach open mapping theorem for Polish groups that these topologies are equivalent. Since the $(\alpha \times \beta)$-topology is stronger than the α-topology induced on $D(\mathcal{R}, \beta)$, so is the β-topology, as desired. □

Corollary 3.3. *Let $\alpha \in Z^1(\mathcal{R}, G)$ and $\beta \in Z^1(\mathcal{R}, H)$ be regular cocycles of an ergodic hyperfinite equivalence relation $\mathcal{R}$, $\alpha(\mathcal{R}) \subset r(\alpha)$, $\beta(\mathcal{R}) \subset r(\beta)$, N_α and N_β closed in G and H respectively, and $\alpha \prec \beta$. Then there is a continuous homomorphism $l_{\beta,\alpha} : N_H(r(\beta))/N_\beta \to N_G(r(\alpha))/N_\alpha$. Moreover, if $\alpha \asymp \beta$ then $l_{\beta,\alpha}$ is an isomorphism.*

Proof. It follows from Proposition 3.2 that $\mathrm{Cl}_\beta([\mathcal{R}]) \subset \mathrm{Cl}_\alpha([\mathcal{R}])$. Hence the canonical inclusion map $D(\mathcal{R}, \beta) \to D(\mathcal{R}, \alpha)$ can be lifted up to a continuous homomorphism $D(\mathcal{R}, \beta)/\mathrm{Cl}_\beta([\mathcal{R}]) \to D(\mathcal{R}, \alpha)/\mathrm{Cl}_\alpha([\mathcal{R}])$. We denote it by $l_{\beta,\alpha}$ and use (2-5). The first assertion follows. To prove the second one we observe that if $\alpha \asymp \beta$, the above inclusion map is a topological isomorphism. □

Now we formulate the main result of this section

Theorem 3.4. *Let $\mathcal{R}$ be a hyperfinite ergodic equivalence relation of type II, G, H are l.c.s.c. groups, and $\alpha \in Z^1(\mathcal{R}, G), \beta \in Z^1(\mathcal{R}, H)$. Suppose that the cocycle $\alpha \times \beta$ is regular, $\alpha \times \beta(\mathcal{R}) \subset r(\alpha \times \beta)$, $r(\alpha)$ admits a metric ρ_α which is compatible with the topology and invariant under the group of inner automorphisms of $r(\alpha)$, and N_β is closed. Then $\alpha \prec \beta$ if and only if*

(i) *$r(\alpha \times \beta) = \{(l(h), h) \mid h \in r(\beta)\}$ for some continuous homomorphism $l : r(\beta) \to r(\alpha)$ with $l(r(\beta))$ being dense in $r(\alpha)$;*
(ii) *the homomorphism $l_{\alpha\times\beta,\beta}$ is onto (see Corollary 3.3 for its definition).*

Proof. We restrict ourselves to the case $\mathcal{R}$ is of type II_1 (the II_∞-case can be studied similarly) and proceed in several steps.
(**A**) First "only if" part will be proved. Assume that $\alpha \prec \beta$. Let $p_2 : r(\alpha\times\beta) \to H$ be the second coordinate projection. Note that in fact $p_2 : r(\alpha\times\beta) \to r(\beta)$.
(**A1**) We shall show that p_2 is one-to-one. Suppose that there is an element $(a, 1_H) \in r(\alpha\times\beta)$ with $a \neq 1_G$. Since $\mathcal{R}$ is hyperfinite, $r(\alpha\times\beta)$ is an amenable group by Lemma 1.4(ii). Like it was done in [Is, Lemma 10] one can deduce from the Reiter criteria of amenability for locally compact groups [Gr, 3.7.3] that there exists a sequence of probability measures $\{\tau_n\}_{n=1}^\infty$ on $r(\alpha\times\beta)$ such that

(1°) the support of τ_n is finite and $\tau_n(A) \in \mathbb{Q}$ for every subset $A \subset r(\alpha\times\beta)$;
(2°) $\|f * \tau_i * \dots * \tau_n\|_1 \to 0$ as $n \to \infty$ for all $i \in \mathbb{N}$ and $f \in L^1(r(\alpha\times\beta), \lambda_r)$ with $\|f\|_1 = 0$, where $*$ stands for the convolution and λ_r for a Haar measure on $r(\alpha\times\beta)$.

By δ_w we denote the Dirac measure concentrated at $w \in r(\alpha\times\beta)$. It follows from (1°) that there are finite sets $Y_n = \{0, 1, \dots, m_n - 1\}$ and functions $f_n : Y_n \ni i \mapsto f_n(i) = (f_n^{(1)}(i), f_n^{(2)}(i)) \in r(\alpha\times\beta))$ such that τ_n can be written in the form $\tau_n = \sum_{i=0}^{m_n-1} m_n^{-1}\delta_{f_n(i)}$ (unlike [Is] we do admit f_n to be noninjective). Without loss in generality we can also assume that

(3°) there is a sequence $n_1 < n_2 < \dots$ with $m_{n_k} = m_{n_k+1} = 2$, $f_{n_k}(0) = f_{n_k+1}(0) = f_{n_k+1}(1) = (1_G, 1_H)$ and $f_{n_k}(1) = (a, 1_H)$ (see the proof of [Is, Lemma 10]).

We identify Y_n with the cyclic group of order m_n in a usual way (with addition mod m_n) and set $Y = \prod_{n=1}^\infty Y_n$. Let λ_Y be the probability Haar measure on Y. Denote by Γ the countable subgroup of Y generated by γ_n, where $\gamma_n = (\underbrace{0, \dots, 0, 1}_{n}, 0, \dots)$, $n \in \mathbb{N}$. The action of Γ on (Y, λ_Y) via translations is free, ergodic, and λ_Y-preserving. We define a cocycle $\widetilde{\alpha}\times\widetilde{\beta} \in Z^1(\mathcal{R}_\Gamma, G\times H)$ by setting

$$\widetilde{\alpha}\times\widetilde{\beta}(y, y+\gamma_n) = f_1(y_1)^{-1}\dots f_n(y_n)^{-1} f_n(y_n+1) f_{n-1}(y_{n-1})\dots f_1(y_1).$$

It follows from (1°), (2°), and [Is, Corollary 4] that $\widetilde{\alpha}\times\widetilde{\beta}$ is regular and $r(\widetilde{\alpha}\times\widetilde{\beta}) = r(\alpha\times\beta)$. By Uniqueness Theorem for cocycles the pairs $(\mathcal{R}_\Gamma, \widetilde{\alpha}\times\widetilde{\beta})$ and $(\mathcal{R}, \alpha\times\beta)$ are weakly equivalent and hence $\widetilde{\alpha} \prec \widetilde{\beta}$. Denote by θ_k the automorphism of (Y, λ_Y) associated to the transposition (n_k, n_k+1), i.e.

$$(\theta_k y)_j = \begin{cases} y_{n_k+1}, & \text{if } j = n_k \\ y_{n_k}, & \text{if } j = n_k + 1 \\ y_j, & \text{otherwise.} \end{cases}$$

Since $\theta_k y = y$ for each y with $y_{n_k} = y_{n_k+1}$ and $\theta_k y = y + \gamma_{n_k} + \gamma_{n_k+1}$ otherwise, we have $\theta_k \in [\mathcal{R}_\Gamma]$. Use (3°) to calculate that $\widetilde{\beta}(y, \theta_k y) = 1_H$ for all $k \in \mathbb{N}$ and $y \in Y$. One can also verify that the infinite product $\prod_{k=1}^\infty \theta_k$ converges in $N[\mathcal{R}_\Gamma]$ to some automorphism θ. (Remark that in view of (3°) $n_k + 1 \neq n_{k+1}$ and hence θ_k commutes with θ_{k+1}.) Pass to the limit in the equality

$$\widetilde{\beta}\left(\prod_{k=1}^{j}\theta_k y, \left(\prod_{k=1}^{j}\theta_k\right)(y+\gamma)\right) = \widetilde{\beta}(y, y+\gamma), \qquad \gamma \in \Gamma,\ y \in Y,$$

to obtain $\theta \in D(\mathcal{R}_\Gamma, \widetilde{\beta})$ and $\prod_{k=1}^j \theta_k \to \theta$ in the $\widetilde{\beta}$-topology as $j \to \infty$. It follows from Proposition 3.2 that the infinite product converges also in the $\widetilde{\alpha}$-topology and hence $\theta_k \to 1$ in $D(\mathcal{R}_\Gamma, \widetilde{\alpha})$. Next, it is easy to verify that

$$\widetilde{\alpha}(y, \theta_k y) = \begin{cases} 1_G, & \text{for each } y \in Y \text{ with } y_{n_k} = y_{n_k+1} \\ \widetilde{\alpha}(y, y+\gamma_{n_k}), & \text{otherwise .} \end{cases}$$

Since for each $k \in \mathbb{N}$ the function $Y \ni y \to \widetilde{\alpha}(y, \theta_k y) \in G$ equals to 1_G on a subset of measure $1/2$, we deduce from the definition of the $\widetilde{\alpha}$-topology that the sequence $\alpha(., \theta_k.)$ must converge in measure to the constant function 1_G as $k \to \infty$. But then

$$\begin{aligned} &\rho_\alpha(1_G, \widetilde{\alpha}(y, \theta_k y)) \\ &= \rho_\alpha(1_G, {f_1}^{(1)}(y_1)^{-1} \cdots {f_{n_k}}^{(1)}(y_{n_k})^{-1} {f_{n_k}}^{(1)}(y_{n_k}+1) {f_{n_k-1}}^{(1)}(y_{n_k-1}) \cdots {f_1}^{(1)}(y_1)) \\ &= \rho_\alpha(1_G, {f_{n_k}}^{(1)}(y_{n_k})^{-1} {f_{n_k}}^{(1)}(y_{n_k}+1)) \geq \min(\rho_\alpha(1_G, a), \rho_\alpha(1_G, a^{-1})) \end{aligned}$$

for all $y \in Y$ with $y_{n_k} \neq y_{n_k+1}$, i.e. on a subset of measure $1/2$, a contradiction. Hence $a = 1_G$ and p_2 is one-to-one.

(**A2**) Next we shall show that p_2 is onto. To this end we use a similar approach. Suppose that there is a sequence $(a_n, b_n) \in r(\alpha \times \beta)$ with $b_n \to 1_H$ as $n \to \infty$ but $\rho_\alpha(a_n, 1_G) > \epsilon$ for some positive ϵ. As above we choose a sequence of probability measures $\{\tau_n\}_{n=1}^\infty$ on $r(\alpha \times \beta)$ satisfying (1°), (2°), and

(3°)' there is a sequence $n_1 < n_2 < \ldots$ with $m_{n_k} = m_{n_k+1} = 2$, $f_{n_k}(0) = f_{n_k+1}(0) = f_{n_k+1}(1) = (1_G, 1_H)$ and $f_{n_k}(1) = (a_n, b_n)$.

Let θ_k, θ be as in (**A1**). We can assume that $b_n \to 1_H$ fast enough that $\widetilde{\beta}(., \prod_{k=1}^j \theta_k.)$ converges in $r(\beta)$ as $j \to \infty$ for all $y \in Y$. It remains to apply the argument of (**A1**) to obtain a contradiction.

Now suppose that $b \in r(\beta) - p_2(r(\alpha \times \beta))$. In view of Corollary 1.2 there exists a sequence $(a_n, b_n) \in r(\alpha \times \beta)$ with $b_n \to b$ as $n \to \infty$ and $\rho_\alpha(a_n, a_{n+1}) > \epsilon_0$ for some positive ϵ_0. Since $b_{n+1}b_n^{-1} \to 1_H$, it follows that $a_{n+1}a_n^{-1} \to 1_G$ as $n \to \infty$, a contradiction.

(**A3**) We deduce from (**A1**) and (**A2**) that $r(\alpha \times \beta) = \{(l(h), h) \mid h \in r(\beta)\}$ for some homomorphism $l : r(\beta) \to r(\alpha)$. Since $r(\alpha \times \beta)$ is closed in $G \times H$, l is continuous. The density of $l(r(\beta))$ in $r(\alpha)$ follows from Corollary 1.2. Thus (i) is proved. We deduce from (i) that

$$\begin{aligned} N_{G\times H}(r(\alpha \times \beta)) &= \{(a, b) \in N_G(r(\alpha)) \times N_H(r(\beta)) \mid l(bhb^{-1}) \\ &\qquad = al(h)a^{-1} \text{ for all } h \in r(\beta)\}, \\ Z_{G\times H}(r(\alpha \times \beta)) &= Z_G(r(\alpha)) \times Z_H(r(\beta)), \\ p^{-1}(r(\beta)) &= r(\alpha \times \beta) \cdot (Z_G(r(\alpha)) \times 1_H), \end{aligned} \tag{3-1}$$

where $p : N_{G\times H}(r(\alpha \times \beta)) \to N_H(r(\beta))$ is the second coordinate projection. One can easily deduce from this that the subgroup $N_{\alpha\times\beta}$ is closed in $G \times H$ whenever N_β is closed in H. Hence the homomorphism $l_{\alpha\times\beta,\beta}$ is well defined, and Corollary 3.3 implies (ii).

(**B**) Now we prove the "if" part. Take an arbitrary $(\theta, \phi) \in C(\mathcal{R}, \beta)$. By Lemma 2.1 $\phi(x) = b\psi(x)$ at a.a. x for some $b \in N_H(r(\beta))$ and a Borel function $\psi : X \to r(\beta)$.

Since $\alpha\times\beta \subset r(\alpha\times\beta)$, we deduce from (i) that $\alpha(x,y) = l(\beta(x,y))$ for all $(x,y) \in \mathcal{R}$. Hence

$$\text{(3-2)} \qquad \alpha \circ \theta(x,y) = l(\psi(x))^{-1} l(b^{-1}\beta(x,y)b) l(\psi(y)) \qquad \text{for a.a. } (x,y) \in \mathcal{R}.$$

It follows from (3-1) that $N_{\alpha\times\beta} = p^{-1}(N_\beta)$. Therefore p induces a one-to-one continuous homomorphism $\widetilde{p}_N : N_{G\times H}(r(\alpha\times\beta))/N_{\alpha\times\beta} \to N_H(r(\beta))/N_\beta$. Moreover, it is straightforward that $\widetilde{p}_N$ is $l_{\alpha\times\beta,\beta}$ exactly. Hence applying (ii) we deduce that $\widetilde{p}_N$ is onto. Then there is $a \in N_G(r(\alpha))$ with $(a,b) \in N_{G\times H}(r(\alpha\times\beta))$ and we deduce from (3-1) and (3-2) that

$$\alpha \circ \theta(x,y) = l(\psi(x))^{-1} a^{-1}\alpha(x,y) a l(\psi(y)) \qquad \text{for a.a. } (x,y) \in \mathcal{R},$$

i.e. $\theta \in D(\mathcal{R},\alpha)$. Thus, $\alpha \prec \beta$, as desired. □

Corollary 3.5. *Let $\mathcal{R}$, G, H, α, and β be as in Theorem* 3.4. *Then $\alpha \asymp \beta$ implies $r(\alpha)$ and $r(\beta)$ are isomorphic as topological groups. In particular, if $D(\mathcal{R},\alpha) = N[\mathcal{R}]$, then α is a coboundary* (*cf.* [D1]).

Corollary 3.6. *Let $\mathcal{R}$ be an ergodic hyperfinite equivalence relation of type II, α, β cocycles of $\mathcal{R}$ with dense ranges in compact groups G and H respectively. Then $\alpha \prec \beta$ if and only if $s(\alpha\times\beta) = \{(l(h),h) \mid h \in H\}$ for some continuous epimorphism $l : H \to G$. Furthermore, $\alpha \asymp \beta$ if and only if l is an isomorphism.*

Remind that $s(\alpha\times\beta)$ is the stability group of the (essentially transitive) Mackey action of $G\times H$ associated to $\alpha\times\beta$; this group is determined up to conjugacy in $G\times H$ (see §1).

4. Quasinormal subrelations and extensions of cocycles

We begin this section with a brief exposition of the basic notions of *measurable index theory* which is intimately related to Jones index theory in operator algebras.

Let $\mathcal{R}$ be an ergodic equivalence relation on $(X,\mathfrak{B},\mu)$ and $\mathcal{S}$ an ergodic subrelation of $\mathcal{R}$. Then there exist $N \in \mathbb{N}\cup\{\infty\}$ and Borel functions $\phi_j : X \to X$ so that $\{\mathcal{S}[\phi_j(x)] \mid 0 \le j < N\}$ is a partition of $\mathcal{R}[x]$, where $\mathcal{R}[x]$ (resp. $\mathcal{S}[x]$) stands for the $\mathcal{R}$- (resp. $\mathcal{S}$-) class of x [FSZ]. N is called the *index* of $\mathcal{S}$ in $\mathcal{R}$ and $\{\phi_j\}_j$ the *choice functions* for the pair $\mathcal{S} \subset \mathcal{R}$. We may assume without loss in generality that $\phi_0(x) = x$ for all $x \in X$. Denote by $\Sigma(J)$ the full permutation group on the set $J \overset{\text{def}}{=} \{0,1,\dots,N-1\}$ for $N < \infty$ or $J \overset{\text{def}}{=} \{0,1,2,\dots\}$ for $N = \infty$. We define the *index cocycle* $\sigma \in Z^1(\mathcal{R},\Sigma(J))$ by setting $\sigma(x,y)(i) = j$ if $\mathcal{S}[\phi_i(y)] = \mathcal{S}[\phi_j(x)]$. Notice that although the choice functions are nonunique, the cohomological class of σ is independent of their particular choice and is an invariant of $\mathcal{S} \subset \mathcal{R}$.

Let $\mathcal{S}_1 \subset \mathcal{R}_1$ be a pair of ergodic equivalence relations on a standard measure space $(X_1,\mathfrak{B}_1,\mu_1)$. Remind that $\mathcal{S}_1 \subset \mathcal{R}_1$ is said to be *isomorphic* to $\mathcal{S} \subset \mathcal{R}$ if there is a Borel isomorphism $\psi : X_1 \to X$ with $\mu\circ\psi \sim \mu_1$, $\psi\times\psi(\mathcal{R}_1) = \mathcal{R}$ and $\psi\times\psi(\mathcal{S}_1) = \mathcal{S}$. From now on we assume that $\mathcal{R}$ (and hence $\mathcal{S}$) is of type II. Then in view of [FSZ, Theorem 1.6 and the comment at the end of §1] $\mathcal{S} \subset \mathcal{R}$ is isomorphic to $\mathcal{S}_1 \subset \mathcal{R}_1$ if and only if their indices are the same and the index cocycles are weakly equivalent.

Let σ be the index cocycle of $\mathcal{S} \subset \mathcal{R}$. Then $\mathcal{S}$ is said to be *normal* in $\mathcal{R}$ if the restriction $\sigma \restriction \mathcal{S}$ of σ to $\mathcal{S}$ is a coboundary. If, in addition, $\mathcal{R}$ is hyperfinite, then by [FSZ, §2] there is a countable amenable group $Q \subset N[S]$ with $Q \cap [\mathcal{S}] = 1_Q$ and such that $\mathcal{R}$ is generated by $\mathcal{S}$ and Q. Notice that this group is determined uniquely (up to an isomorphism) by $\mathcal{S} \subset \mathcal{R}$, and we shall write $Q = \mathcal{R}/\mathcal{S}$. Remark also that for each countable amenable group Q there is an ergodic normal subrelation $\mathcal{S} \subset \mathcal{R}$ with $\mathcal{R}/\mathcal{S} = Q$.

Definition 4.1. A subrelation $\mathcal{S} \subset \mathcal{R}$ is quasinormal if there are an ergodic subrelation $\mathcal{P} \subset \mathcal{S}$, a countable group $Q \subset N[\mathcal{P}]$ and a subgroup $H \subset Q$ such that the following properties are satisfied: $Q \cap [\mathcal{P}] = 1_Q$, H contains no nontrivial normal subgroups of Q, and $\mathcal{R}$ (respectively $\mathcal{S}$) is generated by $\mathcal{P}$ and Q (respectively by $\mathcal{P}$ and H).

For a quasinormal subrelation we can identify J with the H-coset space Q/H and "zero" of J with $H \in Q/H$. Since H contains no nontrivial normal subgroups of Q, the natural action of Q on J via translations induces an embedding of Q in $\Sigma(J)$. Now we set up $\sigma(qx, y) = q$ for all $(x, y) \in \mathcal{P}$, $q \in Q$. Then σ is an index cocycle of $\mathcal{S} \subset \mathcal{R}$ and $\mathcal{S} = \{(x, y) \in \mathcal{R} \mid \sigma(x, y)(0) = 0\}$. Notice that σ as a cocycle with values in the discrete group Q is regular and $r(\sigma) = Q$, $r(\sigma \restriction \mathcal{S}) = H$. Hence if $\mathcal{R}$ is hyperfinite then both groups Q, H are amenable. Each normal subrelation is quasinormal with H being trivial. It follows from [Ha] that every subrelation of finite index is also quasinormal. Moreover, let $\mathcal{S}_i$ be a quasinormal subrelation of a (type II) equivalence relation $\mathcal{R}_i$ on a standard probability space $(X_i, \mathfrak{B}_i, \mu_i)$, $i = 1, 2$. Then, clearly, $\mathcal{S}_1 \times \mathcal{S}_2$ is a quasinormal subrelation of $\mathcal{R}_1 \times \mathcal{R}_2$. A similar statement is also true for any countable Cartesian sums (do not confuse with the Cartesian product) of type II_1 pairs of relation-subrelation.

Let G be a l.c.s.c. group and ν a probability measure on $\mathcal{R}$ equivalent to $\mu_{\mathcal{R}}$. We endow the set $Z^1(\mathcal{R}, G)$ with the topology of convergence in ν (it is independent of the particular choice of ν inside its equivalence class). Thus $Z^1(\mathcal{R}, G)$ becomes a Polish space [FM]. If two measured equivalence relations $\mathcal{R}_1$, $\mathcal{R}_2$ are isomorphic, then $Z^1(\mathcal{R}_1, G)$ and $Z^1(\mathcal{R}_2, G)$ are canonically homeomorphic.

Given a pair $\mathcal{S} \subset \mathcal{R}$ of ergodic equivalence relations, we say that a cocycle $\alpha \in Z^1(\mathcal{S}, G)$ *can be extended* to $\mathcal{R}$ if there is $\beta \in Z^1(\mathcal{R}, G)$ with $\beta \restriction \mathcal{S} = \alpha$. The problem of extension for cocycles was studied in [GDB, GLS] in the case of normal $\mathcal{S}$ with $\mathcal{R}/\mathcal{S}$ being a cyclic group. We observe that if $\mathcal{S}$ is normal in $\mathcal{R}$ and a cocycle $\alpha \in Z^1(\mathcal{S}, G)$ can be extended to $\mathcal{R}$, then each automorphism $q \in Q = \mathcal{R}/\mathcal{S}$ is compatible with α and the function $X \ni x \mapsto \beta(x, qx) \in G$ lies in $\alpha(q)$, where β stands for the extended cocycle. However, the converse is not valid even for Q being a finite cyclic group—we refer to [GDB, pp. 78–79]. The following statement is a generalization of [GLS, Theorem 1.2] to arbitrary quasinormal subrelations.

Theorem 4.2. *Let $\mathcal{R}$ be an ergodic hyperfinite equivalence relation of type II and $\mathcal{S}$ an ergodic quasinormal subrelation of $\mathcal{R}$. Then for each amenable G*

(i) *the subset of cocycles in $Z^1(\mathcal{S}, G)$ which can not be extended to $\mathcal{R}$ is residual in $Z^1(\mathcal{S}, G)$;*
(ii) *the subset of cocycles in $Z^1(\mathcal{R}, G)$ whose restriction to $\mathcal{S}$ define cocycles with dense ranges in G is a dense G_δ in $Z^1(\mathcal{R}, G)$.*

Proof. (ii) We first show that the set $Z_* = \{\delta \in Z^1(\mathcal{R}, G) \mid r(\delta \restriction \mathcal{S}) = G\}$ is nonempty. Since $\mathcal{S}$ is quasinormal, we use Q, H, σ in the same meaning as above. By Lemma 1.4(i) there exist cocycles $\beta \in Z^1(\mathcal{R}, Q)$ and $\alpha \in Z^1(\mathcal{R}, G)$ such that

$\beta\times\alpha$ has dense range in $Q\times G$. Moreover, it follows from the Uniqueness Theorem for cocycles that σ and β are weakly equivalent as cocycles with values in Q and hence with values in $\Sigma(J)$, since $Q\subset\Sigma(J)$. Hence the equivalence relation $\mathcal{R}\times_\beta J$ on $X\times J$ defined by

$$\mathcal{R}\times_\beta J=\{((x,i),(y,j))\mid (x,y)\in\mathcal{R} \text{ and } \beta(x,y)(j)=i\}$$

is ergodic because so is $\mathcal{R}\times_\sigma J$ (which is defined in a similar way) [FSZ, Proposition 1.5]. By [FSZ, Theorem 1.6] β is the index cocycle of the ergodic subrelation $\widetilde{\mathcal{S}}\overset{\text{def}}{=}\{(x,y)\in\mathcal{R}\mid \beta(x,y)(0)=0\}$. Let us prove that $\alpha\restriction\widetilde{\mathcal{S}}$ has dense range in G. Since $W=\{\pi\in\Sigma(J)\mid\pi(0)=0\}$ is an open subgroup of $\Sigma(J)$ and $\{1_Q\}\times G\subset r(\beta\times\alpha)$, for each open subset $O\subset G$ and Borel subset $A\subset X$ of positive measure there are $B\subset A$ and $\gamma\in[\mathcal{R}]$ with $\mu(B)>0$, $B\cup\gamma B\subset A$, and $(\beta(x,\gamma x),\alpha(x,\gamma x))\in W\times O$ for all $x\in B$. Hence we may assume that $\gamma\in[\widetilde{\mathcal{S}}]$. Thus using the definition of essential values of cocycle once more we deduce that $r(\alpha\restriction\widetilde{\mathcal{S}})=G$, as desired. Since σ and β are weakly equivalent, the pairs $\mathcal{S}\subset\mathcal{R}$ and $\widetilde{\mathcal{S}}\subset\mathcal{R}$ are isomorphic and hence $Z_*\neq\emptyset$.

Next, since $\mathcal{R}$ is hyperfinite, we apply Rokhlin's lemma to see that the cohomological class of every cocycle in $Z^1(\mathcal{R},G)$ is dense. Hence Z_* is dense and it remains to show that Z_* is G_δ in $Z^1(\mathcal{R},G)$. We need an auxiliary

Lemma 4.3. *Let $\mathcal{R}$ be an equivalence relation on (X,μ) and $\gamma\in[\mathcal{R}]$. Then the map $Z^1(\mathcal{R},G)\ni\delta\mapsto\gamma_\delta\in\mathrm{Aut}(X\times G,\mu\times\lambda_G)$ is continuous if the automorphism group is endowed with the weak topology (see Section 1 for the definition of the skew product transformation γ_δ).*

We omit the proof of this standard statement, since it is routine (see, for instance, [CHP, Theorem 1]).

Now let us continue the proof of Theorem 4.2. Take an ergodic transformation $\tau\in[\mathcal{S}]\subset[\mathcal{R}]$ generating $\mathcal{S}$. Since the set of ergodic transformations in $\mathrm{Aut}(X\times G,\mu\times\lambda_G)$ is a dense G_δ [CK] and

$$Z_*=\{\delta\in Z^1(\mathcal{R},G)\mid \tau_\delta \text{ is an ergodic automorphism of } (X\times G,\mu\times\lambda_G)\},$$

we deduce from Lemma 4.3 that Z_* is a G_δ in $Z^1(\mathcal{R},G)$. Thus, (ii) is proved completely.

(i) Suppose first that $\mathcal{R}$ (and hence $\mathcal{S}$) is of type II_1. Let $\mathcal{P}$, Q, H, σ be as above. Suppose that μ is the $\mathcal{P}$- (and hence $\mathcal{R}$-) invariant probability measure on $(X,\mathfrak{B})$ and $\beta\in Z^1(\mathcal{R},G)$ is a cocycle with dense range in G. Let $N=\mathrm{Card}(H\backslash Q)$ and set up $(Y,\mathfrak{C},\kappa)=(X,\mathfrak{B},\mu)^N$. Thus each element of Y is a sequence of X-points indexed by elements of the homogeneous space $H\backslash Q$. We define an equivalence relation $\widetilde{\mathcal{P}}$ on Y by setting $(y_k)\sim_{\widetilde{\mathcal{P}}}(y_k{}')$ if there is a finite subset $F\subset H\backslash Q$ such that $y_k\sim_{\mathcal{P}}y'_k$ for all $k\in F$ and $y_k=y'_k$ otherwise. Then $\widetilde{\mathcal{P}}$ is ergodic and hyperfinite, and κ is $[\mathcal{P}]$-invariant. There is a natural action of Q on Y:

$$(qy)_k=y_{kq},\qquad y\in Y,k\in H\backslash Q,q\in Q.$$

We see that $q\in N[\widetilde{\mathcal{P}}]-[\widetilde{\mathcal{P}}]$ for all $q\neq 1_Q$. Denote by $\widetilde{\mathcal{R}}$ (respectively $\widetilde{\mathcal{S}}$) the equivalence relation generated by $\widetilde{\mathcal{P}}$ and Q (respectively by $\widetilde{\mathcal{P}}$ and H). Then $\widetilde{\mathcal{S}}$ is a quasinormal subrelation of $\widetilde{\mathcal{R}}$ and the associated index cocycle $\widetilde{\sigma} \in Z^1(\widetilde{\mathcal{R}}, Q)$ is weakly equivalent to σ. Hence the pairs $\mathcal{S} \subset \mathcal{R}$ and $\widetilde{\mathcal{S}} \subset \widetilde{\mathcal{R}}$ are isomorphic. Now we determine a cocycle $\alpha \in Z^1(\widetilde{\mathcal{P}}, G)$ by setting $\alpha(y, z) = \beta(y_0, z_0)$, where 0, as above stands for $H \in H\backslash Q$. Then α can be extended to $\widetilde{\mathcal{S}}$, since $\alpha \circ h = \alpha$ and we can set $\alpha(y, hy) = 1_G$ for all $y \in Y$, $h \in H$. We also observe that the cocycle $\alpha \times (\alpha \circ q) \in Z^1(\widetilde{\mathcal{P}}, G \times G)$ has dense range in $G \times G$ if $q \in Q - H$—use the definition of essential values of a cocycle and the fact that α and $\alpha \circ q$ are "independent". It follows that $q \notin D(\widetilde{\mathcal{P}}, \alpha)$, since otherwise $s(\alpha \times (\alpha \circ q))$ would be conjugate to the diagonal subgroup of $G \times G$. Hence α can not be extended to $\widetilde{\mathcal{R}}$. It is easy to see that the restriction map

$$Z^1(\widetilde{\mathcal{S}}, G) \ni \delta \mapsto \delta \restriction \widetilde{\mathcal{P}} \in Z^1(\widetilde{\mathcal{P}}, G)$$

is continuous. A minor modification of the proof of (i) shows that the set

$$\{\delta \in Z^1(\widetilde{\mathcal{S}}, G) \mid r((\delta \restriction \widetilde{\mathcal{P}}) \times (\delta \restriction \widetilde{\mathcal{P}}) \circ q) = G \times G \text{ for all } q \in Q - H\}$$

is a dense G_δ in $Z^1(\widetilde{\mathcal{S}}, G)$. Hence the set of $\widetilde{\mathcal{S}}$-cocycles which can not be extended to $\widetilde{\mathcal{R}}$ is residual in $Z^1(\widetilde{\mathcal{S}}, G)$. Since $\mathcal{S} \subset \mathcal{R}$ is isomorphic to $\widetilde{\mathcal{S}} \subset \widetilde{\mathcal{R}}$, we are done.

Now we remark that the II_∞-case can be easily reduced to the II_1-case, if one considers the reduction of $\mathcal{S} \subset \mathcal{R}$ to a subset $A \subset X$ of positive finite measure ν, where ν is a σ-finite, $[\mathcal{R}]$-invariant, μ-equivalent measure. Actually, if $\mathcal{R}_A = \mathcal{R} \cap (A \times A)$, $\mathcal{S}_A = \mathcal{S} \cap (A \times A)$, then $\mathcal{S}_A$ is a quasinormal subrelation of $\mathcal{R}_A$ and the canonical restriction map $Z^1(\mathcal{R}, G) \ni \delta \mapsto \delta \restriction \mathcal{R}_A \in Z^1(\mathcal{R}_A, G)$ is continuous and onto. To complete the proof, it remains to notice that $r(\delta) = r(\delta \restriction \mathcal{R}_A)$ for each cocycle $\delta \in Z^1(\mathcal{R}, G)$ and use the above argument. $\square$

Remark 4.4. In fact, we have proved a stronger statement than that from Theorem 4.2. Actually, let us say that α can be extended to $\mathcal{R}$ *in the generalized sense* if there is a locally compact group G' containing G as a closed subgroup and a cocycle $\beta \in Z^1(\mathcal{R}, G')$ with $\beta \restriction \mathcal{S} = \alpha$. Notice that if an automorphism $\theta \in N[\mathcal{S}]$ is compatible with α then α can be extended to $\mathcal{R}$ in the generalized sense (with G being of finite index in G') [GDB, GS3]. It is easy to see that Theorem 4.2(i) holds if the word "extended" is replaced by "extended in the generalized sense" with the same argument.

5. Lifting problems

Let $\mathcal{R}$ be an ergodic equivalence relation on $(X, \mathfrak{B}, \mu)$, G a l.c.s.c. group, and $\alpha \in Z^1(\mathcal{R}, G)$ a cocycle with dense range.

Definition 5.1. We say that an automorphism $\theta \in N[\mathcal{R}]$ can be lifted to $N[\mathcal{R}(\alpha)]$ if there is $\widetilde{\theta} \in N[\mathcal{R}(\alpha)]$ of the form

$$\widetilde{\theta}(x, g) = (\theta x, .) \qquad \text{for a.a. } (x, g) \in X \times G. \tag{5-1}$$

We denote by $L(\mathcal{R}, \alpha)$ the group of automorphisms from $N[\mathcal{R}]$ which can be lifted to $N[\mathcal{R}(\alpha)]$ and by $\widetilde{L}(\mathcal{R}(\alpha))$ the group of automorphisms from $N[\mathcal{R}(\alpha)]$ of the form (5-1). Notice that

$$\widetilde{L}(\mathcal{R}(\alpha)) = \{\zeta \in N[\mathcal{R}(\alpha)] \mid \zeta^{-1}(\mathfrak{B}') = \mathfrak{B}'\},$$

where $\mathfrak{B}' = \{B \times G \mid B \in \mathfrak{B}\}$. It follows that $\widetilde{L}(\mathcal{R}(\alpha))$ is closed in $N[\mathcal{R}(\alpha)]$, since the normal topology on the normalizer of a full group is stronger than the weak one (see Section 1). Clearly, the natural epimorphism $p : \widetilde{L}(\mathcal{R}(\alpha)) \ni \widetilde{\theta} \mapsto \theta \in L(\mathcal{R}, \alpha) \subset N[\mathcal{R}]$ is continuous. Let us find its kernel.

Lemma 5.2. $\operatorname{Ker} p = \{\theta \in N[\mathcal{R}(\alpha)] \mid \theta(x,g) = (x, hg)$ at a.a. $(x,g) \in X \times G$ for some $h \in G\}$.

Proof. We can write an automorphism $\theta \in \operatorname{Ker} p$ in the form $\theta(x,g) = (x, f(x,g)g)$ for some Borel function $f : X \times G \to G$. Then for $\mu_{\mathcal{R}}$-a.a. $(x,y) \in \mathcal{R}$ we have $\mathcal{R}(\alpha)[(y, f(x,g)g\alpha(x,y))] = \mathcal{R}(\alpha)[(x, f(x,g)g)] = \mathcal{R}(\alpha)[(y, f(x, g\alpha(x,y))g\alpha(x,y))]$ and hence $f(y, g\alpha(x,y)) = f(x,g)$. Since $\mathcal{R}(\alpha)$ is ergodic, $f(x) = h$ at a.a. x for some $h \in G$. $\square$

It follows that $\operatorname{Ker} p$ is algebraically isomorphic to G. We obtain a short exact sequence of groups

$$G \to \widetilde{L}(\mathcal{R}(\alpha)) \to L(\mathcal{R}, \alpha).$$

Notice that the first map is continuous and its image is closed in $\widetilde{L}(\mathcal{R}(\alpha))$ with respect to the normal topology induced from $N[\mathcal{R}(\alpha)]$. Hence $L(\mathcal{R}, \alpha)$ endowed with the quotient topology, say the L-topology, is a Polish group. Moreover, p passes through the quotient map and determines a continuous embedding $L(\mathcal{R}, \alpha) \to N[\mathcal{R}]$. It follows, in particular, that $L(\mathcal{R}, \alpha)$ is a Borel subset of $N[\mathcal{R}]$. Next, we describe the structure of automorphisms from $\widetilde{L}(\mathcal{R}(\alpha))$.

Denote by $\mathfrak{B}_G$ the Borel σ-algebra on G and by $\operatorname{Aut} G$ the group of continuous group automorphisms of G. Notice that $\operatorname{Aut} G$ is embedded naturally into $\operatorname{Aut}(G, \mathfrak{B}_G, \lambda_G)$ as a closed subgroup with respect to the weak topology. The induced (Polish) topology coincides with that of uniform convergence on compact subsets of G.

Theorem 5.3. *Every lift $\widetilde{\theta}$ of an element $\theta \in N[\mathcal{R}]$ to $N[\mathcal{R}(\alpha)]$ is of the form*

$$\widetilde{\theta}(x,g) = \widetilde{\theta}_{l,a} \stackrel{\text{def}}{=} (\theta x, l(g)a(x)) \qquad \text{at a.a. } (x,g) \in X \times G, \tag{5-2}$$

for some Borel map $a : X \to G$ and some automorphism $l \in \operatorname{Aut} G$ satisfying

$$\alpha \circ \theta(x,y) = a(x)^{-1} l(\alpha(x,y)) a(y), \qquad \text{a.a. } (x,y) \in \mathcal{R} \tag{5-3}$$

Conversely, every triplet θ, l, a satisfying (5-3) *defines a lift of θ to $N[\mathcal{R}(\alpha)]$ by* (5-2).

Proof. We write $\widetilde{\theta}$ in the form $\widetilde{\theta}(x,g) = (\theta x, f(x,g))$ for some Borel function $f : X \times G \to G$. Notice that $\widetilde{\theta} \in N[\mathcal{R}(\alpha)]$ if and only if

$$f(y, g\alpha(x,y)) = f(x,g)\, \alpha \circ \theta(x,y) \text{ for } \mu_{\mathcal{R}} \times \lambda_G\text{-a.e. } ((x,y), g). \tag{5-4}$$

Use Fubini theorem and the G-quasiinvariance of λ_G to deduce from this that

$$f(y, hg\alpha(x,y)) = f(x, hg) \alpha \circ \theta(x,y) \text{ for } \mu_{\mathcal{R}} \times \lambda_G \times \lambda_G\text{-a.e. } ((x,y), g, h). \tag{5-5}$$

We set up $F(x,g,h) \stackrel{\text{def}}{=} f(x,hg)f(x,g)^{-1}$. It follows from (5-4) and (5-5) that F is $\mathcal{R}(\alpha)$-invariant for a.e. fixed $h \in G$. Since $\mathcal{R}(\alpha)$ is ergodic, there is a Borel function $l : G \to G$ with

(5-6) $$F(x,g,h) = l(h) \qquad \text{for a.a. } (x,g,h) \in X \times G \times G.$$

By Fubini theorem there exists $g_0 \in G$ such that $F(x, g_0, h) = l(h)$ for $\mu \times \lambda_G$-a.a. (x, h). Thus we can write

(5-7) $$f(x,g) = l(gg_0^{-1})l(g_0)a(x) \qquad \text{at a.e. } (x,g) \in X \times G.$$

for some Borel function $a : X \to G$. It follows from (5-6) that $l(hgg_0^{-1})l(gg_0^{-1})^{-1} = l(h)$ and hence $l(hg) = l(h)l(g)$ for $\lambda_G \times \lambda_G$-a.a. (h, g).

Lemma 5.4. *Let $v : G \to G$ be a Borel function with $v(hg) = v(h)v(g)$ for $\lambda_G \times \lambda_G$-a.a. (h, g). Then there is a continuous homomorphism $v' : G \to G$ such that $v' = v$ almost everywhere.*

This lemma can be readily deduced from [R1] and [Z2, Appendix B]. We leave details to the reader.

Continue the proof of Theorem 5.3. We can view l as a continuous homomorphism by virtue of Lemma 5.4. Then (5-7) implies (5-2). Since $\widetilde{\theta} \in \mathrm{Aut}(X \times G, \mathfrak{B} \times \mathfrak{B}_G, \mu \times \lambda_G)$, it follows from (5-2) that l is one-to-one. Then $l(G)$ is a Borel λ_G-conull subgroup of G. As it was shown in [Ma, R2] this implies $l(G) = G$. Finally, (5-3) follows from (5-4). The second part of Theorem 5.3 is established by a straightforward verification. □

Corollary 5.5. *Let α has dense range in G. The following statements are equivalent:*

(i) *$\theta \in N[\mathcal{R}]$ can be lifted to $N[\mathcal{R}(\alpha)]$;*
(ii) *$\alpha \circ \theta \approx l \circ \alpha$ for some $l \in \mathrm{Aut}\, G$;*
(iii) *$\alpha \times (\alpha \circ \theta)$ is regular and the conjugacy class of $s(\alpha \times (\alpha \circ \theta))$ in $G \times G$ contains the subgroup $\{(l(g), g) \mid g \in G\}$ for some $l \in \mathrm{Aut}\, G$ (l is the same as in* (ii)*).*

Moreover, (i)–(iii) *imply*

(iv) $\alpha \prec \alpha \circ \theta$.

If, in addition, $\mathcal{R}$ is hyperfinite, type II and G is compact, then (i)–(iii) *is equivalent to* (iv).

Proof. (i) $\Longleftrightarrow$ (ii) was established in Theorem 5.3;
(ii) $\Longleftrightarrow$ (iii) follows from Lemma 1.1 and the definition of $s(\alpha \times (\alpha \circ \theta))$;
(ii)$\Longrightarrow$(iv) by a straightforward verification;
(iv)$\Longrightarrow$(iii) by Corollary 3.6. □

We introduce an auxiliary topological space $\mathcal{G} = N[\mathcal{R}] \times \mathrm{Aut} G \times M(X, G)$, where $M(X, G)$ is the set of measurable functions from X to G endowed with the topology of convergence in measure. We also define two maps, $\bullet : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and $* : Z^1(\mathcal{R}, G) \times \mathcal{G} \to Z^1(\mathcal{R}, G)$ by setting

$$(\theta, l, f) \bullet (\theta', l', f') = (\theta\theta', ll', (l' \circ f) \cdot (f' \circ \theta))$$
$$(\beta * (\theta, l, f))(x, y) = l^{-1}(f(x)\beta \circ \theta(x, y) f(y)^{-1}),$$

for all $(\theta, l, f), (\theta', l', f') \in \mathcal{G}$, $\beta \in Z^1(\mathcal{R}, G)$, and $(x, y) \in \mathcal{R}$.

Lemma 5.6. *$(\mathcal{G}, \bullet)$ is a Polish group and $*$ defines a continuous action of $\mathcal{G}$ on $Z^1(\mathcal{R}, G)$.*

We omit the proof of this standard statement, since it is routine. Notice that $(\theta, l, f)^{-1} = (\theta^{-1}, l^{-1}, l^{-1} \circ ((f \circ \theta^{-1})^{-1}))$.

Given a cocycle $\alpha \in Z^1(\mathcal{R}, G)$ with dense range in G, we denote by $S(\alpha) \subset \mathcal{G}$ the stability group of α with respect to $*$. It is easy to deduce from Theorem 5.3 that

$$S(\alpha) \ni (\theta, l, f) \mapsto \widetilde{\theta}_{l,f} \in \widetilde{L}(\mathcal{R}(\alpha))$$

is an (algebraic) group isomorphism. It is a routine to verify that it is continuous. Hence by the Banach open mapping theorem for Polish groups $S(\alpha)$ and $\widetilde{L}(\mathcal{R}(\alpha))$ are isomorphic as topological groups. Then we can reformulate Lemma 5.2 as follows

$$\operatorname{Ker} p = \{(1, \mathrm{Ad}_h, f_h) \mid h \in G\}, \tag{5-8}$$

where $\mathrm{Ad}_h(g) = hgh^{-1}$, $g \in G$, and $f_h(x) = h$, $x \in X$. Denote by $\operatorname{Inn} G \subset \operatorname{Aut} G$ the normal subgroup of inner automorphisms of G, i.e. $\operatorname{Inn} G = \{\mathrm{Ad}_h \mid h \in H\}$ and set $\operatorname{Out} G = \operatorname{Aut} G / \operatorname{Inn} G$. Since

$$p_2 : \widetilde{L}(\mathcal{R}(\alpha)) \ni \widetilde{\theta}_{l,f} \mapsto l \cdot \operatorname{Inn} G \in \operatorname{Out} G$$

is an (algebraic) group homomorphism, it follows from (5-8) that there exists an (algebraic) group homomorphism $\pi_\alpha : L(\mathcal{R}, \alpha) \to \operatorname{Out} G$ with $p_2 = \pi_\alpha \circ p$. Notice that $\pi_\alpha^{-1}(1) = D(\mathcal{R}, \alpha)$ and the α-topology on $D(\mathcal{R}, \alpha)$ (see Section 2) is stronger than the induced L-topology. In fact, consider a continuous one-to-one group homomorphism

$$N[\mathcal{R}] \times M(X, G) \ni (\theta, f) \mapsto (\theta, 1, f) \in \mathcal{G}.$$

It maps the closed subgroup $C(\mathcal{R}, \alpha) \subset N[\mathcal{R}] \times M(X, G)$ (see §2 for its definition) onto a closed normal subgroup of $S(\alpha)$. Hence we may think that $C(\mathcal{R}, \alpha)$ is a closed subgroup of $\widetilde{L}(\mathcal{R}(\alpha))$ (we do not distinguish between $\widetilde{L}(\mathcal{R}(\alpha))$ and $S(\alpha)$). Remind that $\widetilde{L}(\mathcal{R}(\alpha))$ contains also G as a kernel of the canonical projection $\widetilde{L}(\mathcal{R}(\alpha)) \to L(\mathcal{R}, \alpha)$. Since α has dense range in G, we deduce from Lemma 2.1(i) that the group $\alpha(\mathrm{Id})$ is isomorphic to $Z_G(G)$ which means that $C(\mathcal{R}, \alpha) \cap G = Z_G(G)$. It follows that there is a continuous embedding of $C(\mathcal{R}, \alpha)/Z_G(G)$ into $\widetilde{L}(\mathcal{R}(\alpha))/G$, where the two groups are furnished with the quotient topologies. It remains to notice that the first one is (topologically) isomorphic to $D(\mathcal{R}, \alpha)$ and the second one to $L(\mathcal{R}, \alpha)$. In general, $D(\mathcal{R}, \alpha)$ may be nonclosed in $L(\mathcal{R}, \alpha)$. However, if G is compact or Abelian then $\operatorname{Inn} G$ is closed in $\operatorname{Aut} G$ and hence $\operatorname{Out} G$ is a Polish group and π_α is continuous. Therefore $D(\mathcal{R}, \alpha)$ is closed in $L(\mathcal{R}, \alpha)$ and the α-topology on $D(\mathcal{R}, \alpha)$ coincides with the induced L-topology by the open mapping theorem for Polish groups.

Proposition 5.7. *If $\mathcal{R}$ is hyperfinite and of type II, π_α is onto.*

Proof. It is easy to see that for an arbitrary $l \in \operatorname{Aut} G$ the cocycle $l \circ \alpha$ has dense range in G because of α so is. By Lemma 1.4(ii) G is amenable. It follows from the Uniqueness Theorem for cocycles that there exists $\theta \in N[\mathcal{R}]$ with $\alpha \circ \theta \approx l \circ \alpha$. This means that $\theta \in L(\mathcal{R}, \alpha)$ and $\pi_\alpha(\theta) = l \cdot \operatorname{Inn} G$. □

Theorem 5.8. *Let G be amenable, $\mathcal{R}$ a hyperfinite type II equivalence relation, $\mathcal{S}$ its normal ergodic subrelation with $Q = \mathcal{R}/\mathcal{S}$, and $v : Q \ni q \mapsto v_q \in \operatorname{Aut} G$ a group homomorphism. Then there is a cocycle $\alpha \in Z^1(\mathcal{S}, G)$ with dense range in G such that $Q \subset L(\mathcal{S}, \alpha)$ and $\pi_\alpha(q) = v_q \operatorname{Inn} G$ for every $q \in Q$.*

Proof. Denote by $G \circledS_v Q$ the semidirect product of G and Q. Remind that the multiplication low in $G \circledS_v Q$ is given by

$$(g, q)(h, r) = (g v_q(h), qr), \qquad (g, q), (h, r) \in G \times Q.$$

Notice that G is contained in $G \circledS_v Q$ as a closed normal subgroup. Since G and Q are amenable, so is $G \circledS_v Q$. Let $\beta \in Z^1(\mathcal{R}, G \circledS_v Q)$ be a cocycle with dense range in $G \circledS_v Q$. We set up

$$\mathcal{S} = \{(x, y) \in \mathcal{R} \mid \beta(x, y) \in G\}$$

and choose automorphisms $\gamma_q \in [\mathcal{R}]$ with $\beta(\gamma_q x, x) \in G \times \{q\}$. Clearly $\mathcal{S}$ is an ergodic subrelation of $\mathcal{R}$, $\gamma_q \in N[\mathcal{S}]$, $\gamma_q \gamma_r \in \gamma_{qr}[\mathcal{S}]$ for all $q, r \in Q$, and $\gamma_q \in [\mathcal{S}]$ if and only if $q = 1_Q$. That is, $Q \ni q \mapsto \gamma_q$ is an *outer near action* of Q. By [BG1] there is $\zeta \in N[\mathcal{S}]$ such that $\zeta \gamma_q \zeta^{-1} \in q[\mathcal{S}]$ for all $q \in Q$. Now the cocycle $\alpha = (\beta \restriction \mathcal{S}) \circ \zeta \in Z^1(\mathcal{S}, G)$ is as desired. □

Theorem 5.9. *Let $\mathcal{R}$ be an ergodic hyperfinite equivalence relation of type II and G an amenable l.c.s.c. group. Then for each automorphism $\theta \in N[\mathcal{R}] - [\mathcal{R}]$ there is a residual set of cocycles $\alpha \in Z^1(\mathcal{R}, G)$ with dense range in G such that $\theta \notin L(\mathcal{R}, \alpha)$.*

Proof. Denote by $\mathcal{T}$ the equivalence relation generated by $\mathcal{R}$ and θ. Then $\mathcal{R}$ is a nontrivial normal subrelation of $\mathcal{T}$. It follows from the proof of Theorem 4.2(i) that the set of cocycles $\alpha \in Z^1(\mathcal{R}, G)$ such that $r(\alpha \times (\alpha \circ \theta)) = G \times G$ is a dense G_δ in $Z^1(\mathcal{R}, G)$. By Corollary 5.5 $\theta \notin L(\mathcal{R}, \alpha)$. □

Notice that the above theorem is a sharpening of Theorem 4.2(i) for normal pairs $\mathcal{S} \subset \mathcal{R}$ with $\mathcal{R}/\mathcal{S}$ being a cyclic group, since by Proposition 5.7 $D(\mathcal{R}, \alpha)$ is a proper subset of $L(\mathcal{R}, \alpha)$ if $\operatorname{Out} G$ is nontrivial.

Proposition 5.10. *Let $\mathcal{R}$ be as above, G compact or Abelian and $\alpha \in Z^1(\mathcal{R}, G)$ have dense range in G. Then $L(\mathcal{R}, \alpha) \neq N[\mathcal{R}]$.*

Proof. Without loss in generality we may assume that $\mathcal{R}$ is of type II_1. Suppose that $L(\mathcal{R}, \alpha) = N[\mathcal{R}]$. Then the L-topology on $L(\mathcal{R}, \alpha)$ coincides with the normal one by the open mapping theorem for Polish groups. It follows that $[\mathcal{R}]$ is dense in $N[\mathcal{R}]$ with respect to the L-topology. Since G is compact or Abelian, $\pi_\alpha :$ $N[\mathcal{R}] \to \operatorname{Out} G$ is continuous. Clearly, $\pi_\alpha([\mathcal{R}]) = \operatorname{Inn} G \in \operatorname{Out} G$ and we deduce that $\pi_\alpha(N[\mathcal{R}]) = \operatorname{Inn} G$. Since $\pi_\alpha^{-1}(\operatorname{Inn} G) = D(\mathcal{R}, \alpha)$, it follows that $N[\mathcal{R}] = D(\mathcal{R}, \alpha)$. By Corollary 3.5 α is a coboundary. Since α has dense range in G, we obtain that G is trivial, a contradiction. □

6. Centralizers of ergodic skew product transformation groups

Let Γ be a countable ergodic subgroup of $\operatorname{Aut}(X, \mathfrak{B}, \mu)$. Clearly, $C(\Gamma)$ is contained in $N[\mathcal{R}_\Gamma]$ as a closed subgroup. Moreover, the normal topology on $N[\mathcal{R}_\Gamma]$ induces the weak one on $C(\Gamma)$. Let α be a cocycle of $\mathcal{R}$ taking values and having dense range in a l.c.s.c. group G. It follows readily from (5-2) and (5-3) that every lift $\widetilde{\theta}$ of an element $\theta \in C(\Gamma)$ to $N[\mathcal{R}_\Gamma(\alpha)]$ lies in $C(\Gamma_\alpha) \cap \widetilde{L}(\mathcal{R}(\alpha))$. Thus we arrive at the (restricted) well-known problem of lifting $C(\Gamma)$-elements to $C(\Gamma_\alpha)$ (see

[GLS], [Me], [Aa], [ALMN], [ALV] and references therein). The first application of the above results to this problem is a new version of Corollary 5.5 with

(i)' $\theta \in C(\Gamma)$ can be lifted to $C(\Gamma_\alpha)$

instead of (i) there. The next one is

Proposition 6.1. *Let $(X, \mathfrak{B})$ be a standard Borel group, μ a σ-finite measure on $(X, \mathfrak{B})$ and Γ a countable subgroup of X whose action on X via left translations is μ-nonsingular and ergodic. If a cocycle $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ has dense range in G then for each element $\widetilde{\theta} \in C(\Gamma_\alpha)$ there are an automorphism $\theta \in C(\Gamma)$, a function $f : X \to G$, and a group automorphism $l \in \mathrm{Aut}\, G$ satisfying* (5-2) *and* (5-3). *In other words, $C(\Gamma_\alpha) \subset \widetilde{L}(\mathcal{R}(\alpha))$.*

Proof. We write $\widetilde{\theta}$ in the form $\widetilde{\theta}(x, g) = (xf(x, g), .)$ for some Borel function $f : X \times G \to G$. Since $\widetilde{\theta}$ commutes with every element $\gamma \in \Gamma$, we deduce that f is Γ_α-invariant and hence a constant a.e., say h. It follows that the right h-shift on X is μ-nonsingular and, evidently, lies in $C(\Gamma)$. It remains to apply Theorem 5.3. □

It should be noticed that in the particular case when G is compact, X a compact Abelian monothetic group, μ the Haar measure on X, and Γ is generated by a single shift—i.e. $(X, \mathfrak{B}, \mu, \Gamma)$ is a dynamical system with pure point spectrum—Proposition 6.1 is exactly [Me, Theorem 3]. However our argument is quite different.

Example 6.2. Let a countable Abelian group Γ act ergodically on a standard measure space $(X, \mathfrak{C}, \nu)$ in such a way that the L^∞-eigenfunctions of this action generate the whole σ-algebra $\mathfrak{C}$ (modulo ν-null subsets). Remind that a Borel map $f : X \to \mathbb{C}$ is an L^∞-*eigenfunction* of the Γ-action if $|f(x)| = 1$ and $f(\gamma x) = \chi(\gamma) f(x)$ at a.a. x for some character χ of Γ. Then the Γ-action is conjugate to some translation-type action on a compact group endowed with a Borel Γ-nonsingular measure (which is nonequivalent, in general, to the Haar measure)—this assertion is an evident modification of [AN, Theorem 1.2]. By Proposition 6.1 for every cocycle $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ with dense range in G we have $C(\Gamma_\alpha) \subset \widetilde{L}(\mathcal{R}(\alpha))$.

We can combine the arguments of Proposition 6.1 with Lemma 2.3 to obtain the following generalization of this proposition—for simplicity's sake we restrict ourselves to the case of Polish groups instead of standard Borel ones.

Proposition 6.3. *Let L be a Polish group, H a closed subgroup of L with the double coset space $H \backslash L / H$ being countably separated, and Γ a countable normal subgroup of L with $\Gamma \cap H$ being dense in H. Suppose that the natural shiftwise action of Γ on the homogeneous space $X \stackrel{\text{def}}{=} L/H$ is ergodic with respect to some Borel measure μ on X, and $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ is a cocycle with dense range in G. Then for each element $\widetilde{\theta} \in C(\Gamma_\alpha)$ the statement of Proposition* 6.1 *holds.*

Proof. We write $\widetilde{\theta}$ in the form

$$\widetilde{\theta}(x, g) = (A(x, g), .) \tag{6-1}$$

for some Borel function $A : X \times X \to X$ and denote by $s : X \to G$ a Borel cross-section of the natural projection $L \to L/H$ [Di]. Then

$$s(A(x, g)) = s(x) f(x, g) \tag{6-2}$$

for some Borel function $f : X \times G \to L$. Denote by $h_s : L \times X \to H$ the Borel cocycle corresponding to s (see the proof of Lemma 2.3). Then, like it was done in Lemma 2.3, we deduce that for each $\gamma \in \Gamma$

$$f(x,g)h(\gamma, A(x,g)) = h(\gamma,x)f(\gamma x, g\alpha(x,\gamma x)) \qquad \text{for a.a. } (x,g) \in X \times G.$$

Since $H\backslash L/H$ is countably separated and Γ_α is ergodic, it follows that there are two Borel maps $l_1, l_2 : X \times G \to H$, element $a \in L$, and a $\mu \times \lambda_G$-conull Γ_α-invariant subset $B \subset X \times G$ with

$$\text{(6-3)} \qquad f(x,g) = l_1(x,g)al_2(x,g),$$

$$\text{(6-4)} \qquad l_2(x,g)h(\gamma, A(x,g))l_2(\gamma x, g\alpha(x,\gamma x))^{-1} = a^{-1}l_1(x,g)^{-1}h(\gamma,x)l_1(\gamma x, g\alpha(x,\gamma x))a$$

Take an element $(x_0, g_0) \in B$. Notice that the group $\Gamma_{x_0} = \{\gamma \in \Gamma : \gamma x_0 = x_0\}$ is dense in the L-stability group of x_0. We deduce from this fact that $\{h_s(\gamma, x_0) \mid \gamma \in \Gamma_{x_0}\}$ and $\{h_s(\gamma, A(x_0, g_0)) \mid \gamma \in \Gamma_{x_0}\}$ are dense subgroups of H. Then it follows from (6-4) that $H = a^{-1}Ha$, i.e. $a \in N_L(H)$. Now use (6-1)–(6-3) to obtain $A(x,g) = x \cdot \widetilde{a}$ for a.a. (x,g), where $\widetilde{a} = aH \in N_L(H)/H$. It follows that the right $\widetilde{a}$-shift on X is μ-nonsingular and, clearly, commutes with the (left) Γ-action. $\square$

Theorem 6.4. *Let Γ be an Abelian countable ergodic type II transformation group on $(X, \mathfrak{B}, \mu)$ and G an amenable l.c.s.c. group. Suppose that there is $\theta \in C(\Gamma) - \Gamma$. Then there is a residual subset of cocycles $\beta \in Z^1(\mathcal{R}_\Gamma, G)$ with dense ranges in G such that θ can not be lifted to $C(\Gamma_\beta)$. In particular, for such β, $L(\mathcal{R}_\Gamma, \beta) \cap C(\Gamma)$ is a proper subset of $C(\Gamma)$.*

Proof. It suffices to observe that $C(\Gamma) \cap [\Gamma] = \Gamma$ and apply Theorem 5.8. $\square$

Remark that a particular case of Theorem 6.4 where $\Gamma = \mathbb{Z}$, G is compact, and $C(\Gamma)$ contains a nontrivial connected Polish group (or Γ is rigid instead) was established in [GLS, Theorem 5.1, (or Proposition 5.4 respectively)].

Let Σ be an ergodic countable type II_∞ transformation group on $(Y, \mathfrak{C}, \nu)$ with ν being Σ-invariant (and hence infinite). It is easy to see that for each $\theta \in N[\mathcal{R}_\Sigma]$ there is certain $c \in \mathbb{R}_+^*$ with $\mu \circ \theta = c\mu$. It is called the *modulus* of θ and denoted by modθ. If Σ is amenable or, more generally, $\mathcal{R}_\Sigma$ is hyperfinite, then each positive real is a modulus of some $\theta \in N[\mathcal{R}_\Sigma]$ (see, for example, [HO]). It is known also that this fact is no longer valid for nonhyperfinite $\mathcal{R}_\Sigma$ [GG]. A similar problem can also be considered for $C(\Sigma)$ instead of $N[\mathcal{R}_\Sigma]$. We set

$$\Delta(\Sigma) = \{c \in \mathbb{R}_+^* \mid c = \operatorname{mod}\theta \text{ of some } \theta \in C(\Sigma)\}.$$

Clearly, the multiplicative subgroup $\Delta(\Sigma)$ of $\mathbb{R}_+^*$ is an invariant for the conjugacy class of Σ. Like in [Aa, ALMN, ALV] we call Σ *squashable* if $\Delta(\Sigma)$ is nontrivial. In this paper we restrict ourselves to the case when Σ is a skew product extension of a type II_1 transformation group. Given a unimodular l.c.s.c. group G, we put

$$\Delta_G = \{c \in \mathbb{R}_+^* \mid \lambda_G \circ l = c\lambda_G \text{ for some } l \in \operatorname{Aut} G\}.$$

Proposition 6.5. *Let Γ be a type II_1 transformation group on $(X, \mathfrak{B}, \mu)$ generating the hyperfinite orbital equivalence relation $\mathcal{R}_\Gamma$ and G an amenable unimodular l.c.s.c. group. Then for each $c \in \Delta_G$, $c \neq 1$, and every $\theta \in C(\Gamma)$ with $\theta^n \notin [\Gamma]$ for all $n \in \mathbb{Z} - \{0\}$ (i.e. θ is $\mathcal{R}_\Gamma$-outer aperiodic* [CoK]*) there is a cocycle $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ with dense range in G such that $\theta \in L(\mathcal{R}_\Gamma, \alpha)$ and for every lift $\widetilde{\theta}$ of θ to $C(\Gamma_\alpha)$ one has $c = \operatorname{mod} \widetilde{\theta}$. Hence Γ_α is squashable and $c \in \Delta(\Gamma_\alpha)$.*

Proof. Let $\lambda_G \circ l = c\lambda_G$ for some automorphism $l \in \operatorname{Aut} G$ and $\beta \in Z^1(\mathcal{R}_\Gamma, G)$ a cocycle with dense range in G. By Proposition 5.7 there is $\theta_1 \in L(\mathcal{R}_\Gamma, \beta)$ with $\pi_\beta(\theta_1) = l \cdot \operatorname{Inn} G$. Since $c \neq 1$, l is aperiodic and hence θ_1 is outer aperiodic. Then by [CoK] there are automorphisms $\zeta \in N[\mathcal{R}_\Gamma]$ and $\gamma \in [\Gamma]$ with $\zeta^{-1}\theta_1\gamma\zeta = \theta$. We define $\alpha = \beta \circ \zeta$. It is clear that α has dense range in G, $\theta \in L(\mathcal{R}_\Gamma, \alpha)$, and $\pi_\alpha(\theta) = \pi_\beta(\theta_1)$. Finally, it follows from Theorem 5.3 and the fact that inner automorphisms of G preserve λ_G that for every lift $\widetilde{\theta}$ of θ we have $c = \operatorname{mod} \widetilde{\theta}$, as desired. □

If Γ is generated by a single transformation with pure point spectrum, then a family of "product-type" cocycles $\alpha \in Z^1(\mathcal{R}_\Gamma, \mathbb{R})$ such that Γ_α is ergodic and nonsquashable was demonstrated in [ALMN]. However, the problem of existence of squashable Γ_α was marked there as open. We solve it positively (see also [ALV]) in the following

Corollary 6.6. *Let T be an ergodic transformation with pure point spectrum. Then for each $c > 0$ there exists a cocycle $\alpha \in Z^1(\mathcal{R}_T, \mathbb{R})$ such that the skew product transformation T_α is ergodic and squashable and $c \in \Delta(T_\alpha)$.*

Proof. It suffices to notice that $\Delta_\mathbb{R} = \mathbb{R}_+^*$ and there is an $\mathcal{R}_T$-outer aperiodic automorphism $\theta \in C(T)$. The latter follows from the structure of monothetic Abelian l.c.s.c. groups [HR]. □

Remark 6.7. Notice that Proposition 6.5 (and hence Corollary 6.6) can be refined in the following way: if a countable subgroup $\Lambda \subset \Delta_G$ can be embedded in $C(\Gamma)$, then there is $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ with dense range in G such that $\Delta(\Gamma_\alpha) \supset \Lambda$. The proof of this statement is an obvious modification of that of Proposition 6.5 but with reference to [BG1] instead of [CoK].

Remark 6.8. Let Γ be an ergodic type II_1 transformation group on $(X, \mathfrak{B}, \mu)$, $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ a cocycle with dense range in G and $C(\Gamma_\alpha) \subset \widetilde{L}(\mathcal{R}_\Gamma(\alpha))$. Examples of such pairs (Γ, α) are given in Propositions 6.1 and 6.3. Then in view of Theorem 5.3

$$\Delta(\mathcal{R}_\Gamma(\alpha)) = \{c \in \mathbb{R}_+^* \mid \lambda_G \circ l = c\lambda_G \text{ for some } l \in \pi_\alpha(\theta) \in \operatorname{Out} G \\ \text{and certain } \theta \in C(\Gamma)\}.$$

In particular, $\Delta(\mathcal{R}_\Gamma(\alpha)) \subset \Delta_G$. Thus if Δ_G is trivial, Γ_α is nonsquashable. For example, if G is countable and amenable and $(X, \mathfrak{B}, \mu, \Gamma)$ is a dynamical system with pure point spectrum, then Γ_α is nonsquashable for every cocycle α with dense range in G (cf. with [Aa, Theorem 3.4], where G is assumed to have no large finite normal subgroups, i.e. $G = \mathbb{Z}^k \times \mathbb{Q}^l$).

The following statement is a direct consequence of Theorem 5.8 (cf. with Theorem 0.1).

Corollary 6.9. *Let G be an amenable l.c.s.c. group, T an ergodic measure preserving transformation on $(X, \mathfrak{B}, \mu)$, Q a countable amenable subgroup of $C(T)$ with $Q \cap [T] = \mathrm{Id}$ and $v : Q \ni q \mapsto v_q \in \mathrm{Aut}G$ be a group homomorphism. Then there is a cocycle $\alpha \in Z^1(\mathcal{R}_T, G)$ with dense range in G such that $Q \subset L(\mathcal{R}_T, \alpha)$ and $\pi_\alpha(q) = v_q \mathrm{Inn}G$ for every $q \in Q$.*

Our argument permits one to answer a question from [ALW, remark after Theorem 2]: if there is an ergodic type II_1 transformation T and its cocycle α with dense range in the two-dimensional torus with $SL(2, \mathbb{Z}) \subset \pi_\alpha(C(T) \cap L(\mathcal{R}, \alpha))$?

Example 6.10. Let H, G be two compact metrizable Abelian groups, $T \in \mathrm{Aut}H \subset \mathrm{Aut}(H, \lambda_H)$ an ergodic group automorphism, and $p : H \to G$ a continuous group epimorphism. We define a cocycle $\alpha \in Z^1(\mathcal{R}_T, G)$ by setting $\alpha(h, Th) = p(h)$ for all $p \in H$. If $l \in \mathrm{Aut}\, G$, then we look for $\theta_l \in \mathrm{Aut}H$ so that $l \circ p = p \circ \theta_l$. Let us apply this to $H = G^{\mathbb{Z}}$, T the Bernoulli shift on H, and p the projection onto the zero coordinate and set up $(\theta_l h)_n = l(h_n)$ for all $h = (h_n) \in H$, $n \in \mathbb{Z}$. It follows from the theory of random walks on compact groups (see for example [Z1, Theorem 3]) that α has dense range in G. Furthermore, $\theta_l \in L(\mathcal{R}_T, \alpha) \cap C(T)$ and $\pi_\alpha(\theta_l) = l$.

Thus for each compact second countable Abelian group G there is a probability preserving transformation $(X, \mathfrak{B}, \mu, T)$ and a cocycle α of T with dense range in G such that $\pi_\alpha(C(T) \cap L(\mathcal{R}_T, \alpha))$ is the biggest possible, i.e. equals to $\mathrm{Aut}\, G$. Remind that $\mathrm{Aut}(\mathbb{R}^2/\mathbb{Z}^2) = SL(2, \mathbb{Z})$.

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