FUNNY RANK-ONE WEAK MIXING FOR NONSINGULAR ABELIAN ACTIONS

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Abstract. We construct fanny rank-one infinite measure preserving free actions T of a countable Abelian group G satisfying each of the following properties: (1) $T_{g_1}\times\cdots\times T_{g_k}$ is ergodic for each finite sequence $g_1,\dots,g_k$ of G-elements of infinite order, (2) $T\times T$ is nonconservative, (3) $T\times T$ is nonergodic but all k-fold Cartesian products are conservative, and the L^∞-spectrum of T is trivial, (4) for each g of infinite order, all k-fold Cartesian products of T_g are ergodic, but $T_{2g}\times T_g$ is nonconservative.

A topological version of this theorem holds. Moreover, given an AT-flow W, we construct nonsingular G-actions T with the similar properties and such that the associated flow of T is W. Orbit theory is used in an essential way here.

0. Introduction

The goal of this work is to construct infinite measure preserving and nonsingular fanny rank-one free actions of countable Abelian groups with various dynamical properties. The construction of these actions is based on a common idea: every one appears as an inductive limit of some partially defined actions associated to certain two sequences (C_n) and (F_n) of finite subsets in the group. We call them (C,F)**-actions**. It is worthwhile to remark that the (C,F)-actions appear as minimal topological actions on locally compact totally disconnected spaces. Moreover, they are uniquely ergodic, i.e. they admit a unique (up to scaling) invariant σ-finite Radon measure (Borel measure which is finite on the compact subsets).

Now we record our main result about infinite measure preserving actions.

Theorem 0.1. *Let G be a countable Abelian group. Given $i\in\{1,\dots,5\}$, there exists a funny rank one infinite measure preserving free (C,F)-action $T=\{T_g\}_{g\in G}$ of G such that the property (i) of the following list is satisfied:*

(1) *for every $g\in G$ of infinite order, the transformation T_g has infinite ergodic index, i.e. all its k-fold Cartesian products are ergodic,*

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(2) *for each finite sequence* $g_1, \dots, g_n$ *of* G*-elements of infinite order, the transformation* $T_{g_1} \times \cdots \times T_{g_n}$ *is ergodic,*
(3) *for each* $g \in G$ *of infinite order,* T_g *has infinite ergodic index but* $T_{2g} \times T_g$ *is nonconservative,*
(4) *the Cartesian square of* T *is nonconservative,*
(5) T *has trivial* L^∞*-spectrum, nonergodic Cartesian square but all* k*-fold Cartesian products conservative.*

When proving this theorem we obtain automatically a topological version of (a part of) it as follows

Theorem 0.2. *Given* $i \in \{1, 2\}$*, there exists a minimal uniquely ergodic* (C, F)*-action* $T = \{T_g\}_{g \in G}$ *of* G *on a locally compact non-compact totally disconnected metrizable space without isolated points such that the property* (i) *of the following list is satisfied:*

(1) *for every* $g \in G$ *of infinite order, the transformation* T_g *has infinite topologically transitive index, i.e. all its* k*-fold Cartesian products are topologically transitive,*
(2) *for each finite sequence* $g_1, \dots, g_n$ *of* G*-elements of infinite order, the transformation* $T_{g_1} \times \cdots \times T_{g_n}$ *is topologically transitive.*

For other topological properties of (C, F)-transformations we refer to [Da]. The third main result of this paper is a nonsingular counterpart of Theorem 0.1.

Theorem 0.3. *Let* W *be an AT-flow (see the comment below). Given* $i \in \{1, \dots, 5\}$*, there is a funny rank one nonsingular free* (C, F)*-action* T *of* G *whose associated flow is* W *and the property* (i) *of Theorem 0.1 is valid.*

After the main results being formulated let us make some comments. We recall that for finite measure preserving actions the following properties are equivalent: (a) T has trivial L^∞ (=L^2)- spectrum, (b) $T \times T$ is ergodic, (c) T has infinite ergodic index. In general—for arbitrary nonsingular actions—we have only $(c) \Rightarrow (b) \Rightarrow (a)$. The first counterexamples to $(b) \Rightarrow (c)$ and $(a) \Rightarrow (b)$ for infinite measure preserving actions of $\mathbb{Z}$ were given in [KP] and [ALW] respectively. Those transformations are infinite Markov shifts. They possess "strong" stochastic properties and are quite different from our (C, F)-actions. Moreover, as it was noticed in [AFS1] it is impossible to construct Markov shifts satisfying Theorem 0.1(5). Another sort of counterexamples which are similar to our ones were demonstrated in [AFS1], [AFS2], [DGMS] and [M–Z]. A particular case of Theorem 0.1 where $G = \mathbb{Z}$ was proved there: the examples (1), (4), (5) appear in [AFS1], the example (2) in [DGMS], and the example (3) in [AFS2]. Moreover, for $G = \mathbb{Z}^d$, the example (2) appears in [M–Z].

Before we pass to Theorem 0.3 let us remind that one can associate a measurable flow (i.e. an action of $\mathbb{R}$) to every nonsingular action of G [Sc], [HO]. By the celebrated Dye-Krieger theorem there is a bijective correspondence between the orbit equivalent classes of ergodic G-actions and the conjugacy classes of ergodic

$\mathbb{R}$-flows. An important class of ergodic flows, AT-flows, was isolated in [CW]. They are exactly the associated flows of product odometers. For example, the transitive flows and the finite measure preserving flows with discrete spectra are AT.

By the way, we mention a natural problem concerning AT-flows. Since a product odometer is a rotation on a compact group, its L^∞-spectrum is **large**: the eigenfunctions separate points. So, given G and an AT-flow W, is it possible to find a free action of G with trivial L^∞-spectrum and whose associated flow is W? For the moment, the answer was not known even for $G = \mathbb{Z}$. Now the positive solution to this problem follows from Theorem 0.3(2).

A very particular case of Theorem 0.3 was proved for the moment. If $G = \mathbb{Z}$, W is transitive and its stabilizer is $(\log \lambda)\mathbb{Z}$, $0 < \lambda < 1$, then the example (5) was constructed in [AFS1] and the example (2) in [AFS2]. These assumptions on W mean that the corresponding G-action is of type III_λ. As concern to $\mathbb{Z}^d$-actions only a weak version of (1) is demonstrated in [M–Z]: there is an action of type $III_\lambda, 0 < \lambda \leq 1$ such that the generators of $\mathbb{Z}^d$ have infinite ergodic index, and there is an action of type III_0 such that the generators are ergodic and have trivial L^∞-spectrum (the associated flow is not specified there).

Now we specify the main point of difference between our work and those papers. The transformations from [AFS1], [AFS2], [DGMS] and [M–Z] are constructed via the well-known "cutting and stacking" techniques [Fr]. It has a clear geometrical nature and is very convenient in the case $G = \mathbb{Z}$ or even $G = \mathbb{Z}^d$. However for more "complicated" groups like $\mathbb{Z}_0^\infty, \mathbb{Q}$ or groups with torsions it does not appear quite transparent especially when constructing nonsingular actions. That is why we develop an alternative approach replacing "cutting and stacking" with the "more algebraic" (C, F)-construction. The latter is rather universal and does not "feel" much difference between $\mathbb{Z}$ and $\mathbb{Q}$ or between measure preserving and nonsingular actions. Moreover, it is especially well suited for applying the measurable orbit theory which is used in an essential way here.

Remark also that the transformations from [AFS1], [AFS2], [DGMS] and [M–Z] are rank-one. However, it is not quite clear what "rank one" is for actions of Abelian groups other than $\mathbb{Z}^d$. But a concept of "fanny rank one"—introduced by J.-P. Thouvenot—is generalized naturally to nonsingular actions of arbitrary groups [So]. We show that the (C, F)-actions have fanny rank one. Moreover, in the case $G = \mathbb{Z}^d$, the assertion of Theorems 0.1–0.3 can be a bit strengthened by claim that T has **rank one** ("by cubes") and not only funny rank one. This is because the sets F_n in the (C, F)-construction in our theorems are rather "flexible" and it is always possible to choose them in the form of cubes (see Remark 2.6(ii)).

The outline of the paper is as follows. Section 1 contains a background material mainly from orbit theory. In Section 2 we introduce and study the (C, F)-construction of σ-finite measure preserving actions and prove Theorems 0.1, 0.2 and related results. A particular case of this construction, where C_n and F_n are "well balanced" is considered in Section 3. It results to what we call **generalized**

Hajian-Kakutani actions (cf. [HK], [EHI], [M–Z]). They are of **finite type** in the sense of [EHI]. The problems related to tilings, weakly wandering subsets, nonsingular disjointness are under discussion here. In the final § 4 we adapt the (C, F)-construction to the non-singular case and prove Theorem 0.3.

After this work has been already done the author learned that the (C, F)-construction appeared initially in [Ju]. However A. del Junco studied there only finite measure preserving actions while our paper is devoted to infinite measure preserving and nonsingular ones. Moreover, the problems considered here and in [Ju] are quite different and have no "common" part. I thank C.E. Silva for drawing my attention to [AFS2], [M–Z] and [Ju].

1. Preliminaries

Measured equivalence relations and their cocycles. For a detailed account of the discussion in this section we refer the reader to [FM], [Sc], [HO].

Let $(X, \mathfrak{B})$ be a standard Borel space and $\mathcal{R}$ a Borel countable equivalence relation on it [FM]. Consider a Borel bijection γ of a Borel subset A onto a Borel subset B. If $(x, \gamma x) \in \mathcal{R}$ for each $x \in A$ then γ is a **partial $\mathcal{R}$-transformation** with the domain A and the range B (we shall write $D(\gamma) = A$, $R(\gamma) = B$). The groupoid of partial $\mathcal{R}$-transformations is denoted by $[[\mathcal{R}]]$. The **full group** $[\mathcal{R}]$ is the subset of partial $\mathcal{R}$-transformations whose domain and range are the entire X. Given $A \in \mathfrak{B}$, we denote by $\mathcal{R} \restriction A$ the restriction of $\mathcal{R}$ to A, i.e. $\mathcal{R} \restriction A := \mathcal{R} \cap (A \times A)$ with the induced Borel structure. The product of two equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ is denoted by $\mathcal{R} \otimes \mathcal{R}'$.

Let μ be a σ-finite measure on X. $\mathcal{R}$ is said to be μ-**nonsingular** if $\mu \circ \gamma$ is equivalent to μ for each $\gamma \in [\mathcal{R}]$. It is known that every Borel equivalence relation is the orbit equivalence relation for a countable group Γ of Borel automorphisms of X (see [FM]) (this group is not unique). $\mathcal{R}$ is μ-nonsingular if every transformation $\gamma \in \Gamma$ so is. $\mathcal{R}$ is **ergodic** if every Borel $\mathcal{R}$-saturated subset (i.e. a union of $\mathcal{R}$-classes) is either μ-null or μ-conull. Two nonsingular equivalence relations $\mathcal{R}$ on $(X, \mathfrak{B}, \mu)$ and $(X', \mathfrak{B}', \mu')$ are **isomorphic** if there is a Borel bijection $\phi\colon X \to X'$ such that $\mu' \circ \phi \sim \mu$ and $\phi \times \phi(\mathcal{R} \restriction X_0) = \mathcal{R}' \restriction X_0'$ for conull subsets $X_0 \subset X$ and $X_0' \subset X'$. An ergodic equivalence relation is **hyperfinite** if it is isomorphic to the orbit equivalence relation of a single (ergodic) transformation. If Γ is an ergodic Abelian transformation group then its orbit equivalence relation is hyperfinite.

Let G be a locally compact second countable group. A Borel map $\alpha\colon \mathcal{R} \to G$ is a **cocycle** if

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z)$$

for all $(x, y), (y, z) \in \mathcal{R}$. We define the α-skew product equivalence relation $\mathcal{R} \times_\alpha G$ on $X \times G$ equipped with the product Borel structure by setting

$$(x, g) \sim (y, h) \text{ if } (x, y) \in \mathcal{R} \text{ and } h = g\alpha(x, y).$$

Suppose that $\mathcal{R}$ is μ-nonsingular. Then $\mathcal{R}\times_\alpha G$ is $\mu\times\lambda_G$-nonsingular, where λ_G is right Haar measure on G. If $\mathcal{R}\times_\alpha G$ is ergodic then α is said to **have dense range in** G. α is said to be **transient** if $\mathcal{R}\times_\alpha G$ is nonconservative, i.e. its "orbit partition" is measurable.

Remark that the natural G-action on $X\times G$ by left translations along the second coordinate induces a nonsingular G-action on the quotient measure space of $\mathcal{R}\times_\alpha G$-ergodic components. It is called the **action associated to** $(\mathcal{R},\alpha)$ or the **Mackey action**. It is ergodic if and only if $\mathcal{R}$ so is.

Let $\mathcal{R}$ be an ergodic equivalence relation generated by a countable transformation group Γ and $\rho_\mu\colon G\to\mathbb{R}_+$ the Radon-Nikodym cocycle, i.e.

$$\rho_\mu(x,\gamma x)=\log\frac{d\mu\circ\gamma}{d\mu}(x) \text{ at a.e. } x \text{ for each } \gamma\in\Gamma.$$

The corresponding Mackey action $W=\{W_t\}_{t\in\mathbb{R}}$ is called the **associated flow** of $(\mathcal{R},\mu)$ (or the **associated flow** of Γ). There are several cases:

— W is (essentially) transitive and free,
— W is (essentially) transitive, nonfree. Its stabilizer is $(\log\lambda)\mathbb{Z}$ for some $\lambda\in(0,1)$,
— W is trivial (on a singleton),
— W is free and nontransitive.

$\mathcal{R}$ is said to be of (Krieger's) type II, III_λ, III_1, III_0 respectively. Remark that $\mathcal{R}$ is of type II if and only if there exists a measure $\mu'\sim\mu$ which is $\mathcal{R}$-invariant, i.e. $\mu'\circ\gamma=\mu'$ for each $\gamma\in[\mathcal{R}]$. The $\mathcal{R}$-invariant measure in the class of μ is unique up to scaling. If it is finite then $\mathcal{R}$ is **of type** II_1, otherwise $\mathcal{R}$ is **of type** II_∞. If A is a subset of positive measure then the associated flows of $(\mathcal{R},\mu)$ and $(\mathcal{R}\restriction A,\mu\restriction A)$ are conjugate.

Theorem 1.1(Dye-Krieger). *Two ergodic hyperfinite equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ are isomorphic if and only if one of the following is fulfilled:*

(i) *they are both of type II_1,*
(ii) *they are both of type II_∞,*
(iii) *they are both of type III and the flows associated to them are conjugate.*

We also need the following simple fact. Let $\mathcal{R}$ and $\mathcal{S}$ be two ergodic equivalence relations. If $\mathcal{S}$ is of type II then the associated flow of $\mathcal{R}\otimes\mathcal{S}$ is conjugate to that of $\mathcal{R}$.

Tail equivalence relations. Let $(V_n)_{n=1}^\infty$ be a sequence of finite nonempty sets. Put $V=\prod_{n=1}^\infty V_n$ and endow it with the product of the discrete topologies. Then V is a compact metrizable space. Denote by $\mathcal{R}\subset V\times V$ the **tail** equivalence relation. Remind that two elements $v=(v_n)$ and $v'=(v_n')$ in V are $\mathcal{R}$-equivalent if $v_n=v_n'$ for all sufficiently large $n>0$. It is easy to verify the following properties of $\mathcal{R}$:

(i) $\mathcal{R}$ is a σ-compact subset of $V\times V$;

(ii) $\mathcal{R}$ is minimal, i.e. each $\mathcal{R}$-equivalence class is dense in V;
(iii) $\mathcal{R}$ is uniquely ergodic, i.e. there exists a unique probability $\mathcal{R}$-invariant measure on V (we call it Haar measure for $\mathcal{R}$).

AT-flows and fanny rank one. Let ν_n be a measure on V_n. Throughout this paper we assume that $\nu_n(v_n) > 0$ for all $v_n \in V_n$. If $\nu_n(V_n) = 1$ for all $n > 1$ then the product $\nu := \bigotimes_{n=1}^{\infty} \nu_n$ is a finite Borel measure on V. Clearly, it is non-atomic if and only if $\prod_{n=1}^{\infty} \max_{v_n \in V_n} \nu_n(v_n) = 0$. ¿From now on we shall assume that ν is non-atomic. It is well known that $\mathcal{R}$ is ν-nonsingular, ergodic and hyperfinite. Moreover,

$$\rho_\nu(v, v') = \sum_{n=1}^{\infty} (\log \nu_n(v'_n) - \log \nu_n(v_n)).$$

Notice that the sum contains only finitely many non-zero items. Clearly, ν is $\mathcal{R}$-invariant if and only if ν_n is equidistributed for every $n \in \mathbb{N}$. In this case $\mathcal{R}$ is of type II_1. In general $(\mathcal{R}, \nu)$ can be of an arbitrary Krieger's type.

Definition 1.2. A nonsingular flow $\{W_t\}_{t\in\mathbb{R}}$ on a standard measure space (X, μ) is **approximately transitive** (AT) if given $\epsilon > 0$ and finitely many non-negative functions $f_1, \dots, f_n \in L^1_+(X, \mu)$ there exists a function $f \in L^1_+(X, \mu)$ and reals $t_1, \dots, t_n$ such that

$$\left\| f_i - \sum_{k=1}^{m} a_{ik} f \circ W_{t_k} \frac{d\mu \circ W_{t_k}}{d\mu} \right\|_1 < \epsilon, \quad i = 1, \dots, n,$$

where a_{ik}, $i = 1, \dots, n$, $j = 1, \dots, m$, are some non-negative reals.

The following fundamental statement is due to A. Connes and J. Woods [CW] (see also [Haw] and [Ham] for a measure theoretical proof).

Theorem 1.3. *The associated flow of $(\mathcal{R}, \nu)$ is AT. Conversely, for every AT-flow $\{W_t\}_{t\in\mathbb{R}}$ there exists a sequence $(V_n, \nu_n)_n$ as above such that the associated flow of the tail equivalence relation on (V, ν) is conjugate to $\{W_t\}_{t\in\mathbb{R}}$.*

Definition 1.4. A nonsingular action S of G on a σ-finite Lebesgue space $(Y, \mathcal{A}, \nu)$ has **funny rank one** if there is a sequence $(Y_n)_{n=1}^{\infty}$ of measurable subsets of Y and a sequence $(G_n)_{n=1}^{\infty}$ of finite G-subsets such that

(i) the subsets $S_g Y_n$, $g \in G_n$, are pairwise disjoint for each $n > 0$,
(ii) given $A \in \mathcal{A}$ of finite measure, then $\inf_{P\subset G_n} \nu(A \triangle \bigcup_{g\in P} S_g Y_n) \to 0$ as $n \to \infty$,
(iii) $\sum_{g\in G_n} \inf\limits_{r\in\mathbb{R}} \int_{S_g Y_n} |\frac{d\nu\circ g}{d\nu} - r| d\nu \to 0$ as $n \to \infty$.

Remark that the fanny rank one was introduced by J.-P. Thouvenot for probability preserving $\mathbb{Z}$-actions (see also [Fe]) and extended to the general case by A. Sokhet [So]. This property does not depend on a particular choice of ν inside its equivalence class. Clearly, funny rank one implies ergodicity.

2. (C, F)-actions

Two finite subsets C_1 and C_2 of G are called **independent** if

$$(C_1 - C_1) \cap (C_2 - C_2) = \{0\}.$$

A sequence $(C_n)_{n=1}^{\infty}$ of finite G-subsets is **independent** if $C_1 + \cdots + C_n$ and C_{n+1} are independent for each n. This means that every element c of $C_1 + \cdots + C_n$ can be written uniquely as $c = c_1 + \cdots + c_n$ with $c_1 \in C_1, \dots, c_n \in C_n$.

G-actions associated to pairs of sequences of finite subsets. Let $(C_n)_{n=1}^{\infty}$ and $(F_n)_{n=0}^{\infty}$ be two sequences of finite G-subsets such that $F_0 = \{0\}$ and for each $n > 0$ the following are satisfied:

$$F_n + C_{n+1} \subset F_{n+1}, \ \#(C_n) > 1, \tag{2-1}$$

$$F_n, C_{n+1}, C_{n+2}, \dots \qquad \text{is independent.} \tag{2-2}$$

We put $X_n := F_n \times \prod_{k>n} C_k$ and define a map $i_n \colon X_n \to X_{n+1}$ by setting

$$i_n(f_n, c_{n+1}, c_{n+2}, \dots) := (f_n + c_{n+1}, c_{n+2}, \dots).$$

Clearly, i_n is a homeomorphism of X_n onto its image in X_{n+1}. Denote by X the topological inductive limit of the sequence (X_n, i_n) and by $\hat{i}_n \colon X_n \to X$ the canonical embeddings, $n > 0$. Clearly, X is a locally compact non-compact totally disconnected metrizable space without isolated points and $\hat{i}_n(X_n)$ is clopen in X.

Denote by $\mathcal{R}_n$ the tail equivalence relation on X_n. Clearly,

$$(i_n \times i_n)(\mathcal{R}_n) = \mathcal{R}_{n+1} \restriction i_n(X_n).$$

Hence an inductive limit $\mathcal{R}$ of $(\mathcal{R}_n, i_n \times i_n)$ is well defined. Clearly, $\mathcal{R}$ is a countable σ-compact minimal equivalence relation on X. Assume in addition that

(2-3) given $g \in G$, there is $m \in \mathbb{N}$ with $g + F_n + C_{n+1} \subset F_{n+1}$ for all $n > m$.

Given $g \in G$ and $n \in \mathbb{N}$, we set

$$D_g^{(n)} := (F_n \cap (F_n - g)) \times \prod_{k>n} C_k \qquad \text{and} \qquad R_g^{(n)} := D_{-g}^{(n)}.$$

Clearly, $D_g^{(n)}$ and $R_g^{(n)}$ are clopen subsets of X_n and the map $T_g^{(n)} \colon D_g^{(n)} \to R_g^{(n)}$ given by

$$T_g^{(n)}(f_n, c_{n+1}, \dots) := (f_n + g, c_{n+1}, \dots)$$

is a homeomorphism. Put

$$D_g := \bigcup_{n=1}^{\infty} \hat{i}_n(D_g^{(n)}),\ R_g := \bigcup_{n=1}^{\infty} \hat{i}_n(R_g^{(n)}).$$

Since the diagram

$$\begin{array}{ccc} D_g^{(n)} & \xrightarrow{T_g^{(n)}} & R_g^{(n)} \\ {\scriptstyle i_n}\Big\downarrow & & \Big\downarrow{\scriptstyle i_n} \\ D_g^{(n+1)} & \xrightarrow{T_g^{(n+1)}} & R_g^{(n+1)} \end{array}$$

commutes, a homeomorphism T_g: $D_g \to R_g$ is well defined by $T_g\hat{i}_n = \hat{i}_n T_g^{(n)}$. It follows from (2-3) that for each $g \in G$ there is m such that $D_g^{(n+1)} \supset i_n(X_n)$ for all $n > m$. Hence $D_g = X$. Since $R_g = D_{-g}$, we conclude that $R_g = X$. Moreover it is easy to verify that $T_{g_2}T_{g_1} = T_{g_2+g_1}$. Thus $T = \{T_g\}_{g\in G}$ is a topological action of G on X.

Theorem 2.1.

(i) *T is a minimal free action of G on X,*
(ii) *$\mathcal{R}$ is the T-orbit equivalence relation.*
(iii) *there is a unique (ergodic) σ-finite T-invariant measure on X such that $\mu(\hat{i}_0(X_0)) = 1$,*
(iv) *μ is finite if and only if*

$$\lim_{n\to\infty} \frac{\#(F_n)}{\#(C_1)\cdots\#(C_n)} < \infty,$$

(v) *T has fanny rank one.*

Proof. (i)–(iv) is routine.

(v) We put

$$Y_n := \hat{i_n}\left(\{0\} \times \prod_{k>n} C_k\right) \text{ and } G_n := F_n.$$

It is easy to verify that $(Y_n)_n$ and $(G_n)_n$ satisfy Definition 1.4 and

$$\bigcup_{g\in G_n} T_g Y_n = \hat{i_n}(X_n). \quad \square$$

Definition 2.2. We call T the (C, F)-action of G associated to $(C_n)_n$ and $(F_n)_n$. μ is called **Haar** measure for $\mathcal{R}$.

Notice that in the case of finite Haar measure, T is an analogue of the Chacon transformation. The difference $F_n \setminus (C_n + F_{n-1})$ plays the role of "spacers" at the n-th step in the classical construction [Fr].

It is possible that X is compact. This happens if and only if $F_{n+1} = F_n + C_{n+1}$ for all sufficiently large n. Consider, for example, $G = \mathbb{Z}$, $C_n = \{0, (-2)^{n-1}\}$, $F_n = C_1 + \cdots + C_n$, $n > 0$.

We record without proof a standard

Lemma 2.3. *Let $\beta_i \geq \alpha_i > 0$ and $\sum_{i=1}^n \alpha_i \geq (1-\epsilon)\sum_{i=1}^n \beta_i$. Then $\sum_{i\in I} \beta_i > \frac{2}{3}\sum_{i=1}^n \beta_i$, where $I = \{i | \ \alpha_i > (1-3\epsilon)\beta_i\}$.*

Let $(V, \nu) = \prod_{i=1}^\infty (V_n, \nu_n)$ for an independent sequence $(V_n)_{n=1}^\infty$ of finite G-subsets and probability measures ν_n on them. Given $g_1 \in V_1, \dots, g_n \in V_n$, we set $I(g_1, \dots, g_n) = \{v = (v_n) \in V | \ v_1 = g_1, \dots, v_n = g_n\}$.

Lemma 2.4. *let $\mathcal{S}$ be a ν-nonsingular equivalence relation on V and $\delta, \beta\colon G \to \mathbb{R}_+$ two maps. If for every $n \in \mathbb{N}$ and $g_1, g_1' \in V_1, \dots, g_n, g_n' \in V_n'$, there is a partial transformation $\gamma \in [[\mathcal{S}]]$ such that the following properties are satisfied:*

$$\begin{aligned} D(\gamma) \subset I(g_1, \dots, g_n), \qquad & R(\gamma) \subset I(g_1', \dots, g_n'), \\ \nu(D(\gamma)) \geq \delta(g_1 + \cdots + g_n - g_1' - &\cdots - g_n')\nu(I(g_1, \dots, g_n)), \\ \frac{d\nu\circ\gamma}{d\nu}(v) \geq \beta(g_1 + \cdots + g_n - g_1' - \cdots - g_n') & \qquad \text{for all } v \in D(\gamma), \end{aligned}$$

then $\mathcal{S}$ is ergodic.

Proof. Let A and A' be two Borel subset of V of positive measure. We can find $n \in \mathbb{N}$ and $g_1, g_1' \in V_1, \dots, g_n, g_n' \in V_n'$ such that

$$\nu(A_1) > \frac{4}{5}\nu(I(g_1, \dots, g_n)) \text{ and } \nu(A_1') > \frac{4}{5}\nu(I(g_1', \dots, g_n')),$$

where $A_1 = A \cap I(g_1, \dots, g_n)$ and $A_1' = A' \cap I(g_1', \dots, g_n')$. Since $V_1, \dots, V_n$ are independent, the map

$$(v_1, \dots, v_n) \mapsto v_1 + \cdots + v_n$$

is a natural bijection of $V_1 \times \cdots \times V_n$ onto $V_1 + \cdots + V_n$. Without loss of generality we may assume that $n = 1$. (Actually, replace the sequence $V_1, V_2, \dots$ by the following one $V_1 + \cdots + V_n, V_{n+1}, \dots$.) Next, we set

$$\epsilon := \frac{1}{4}\delta(g_1 - g_1') \ \text{ and } \ \epsilon' := \min\left(\frac{1}{15}, \frac{\epsilon\beta(g_1 - g_1')\nu(I(g_1))}{4\nu(I(g_1'))}\right).$$

There are clopen subsets $I_0 \subset I(g_1)$ and $I_0' \subset I(g_1')$ with $\nu(I_0 \triangle A_1) < \epsilon\nu(I_0)$ and $\nu(I_0' \triangle A_1') < \epsilon'\nu(I_0')$. Again we may assume that there are subsets $C, C' \subset V_2$ with $I_0 = \bigcup_{c\in C} I(g_1, c)$ and $I_0' = \bigcup_{c'\in C'} I(g_1', c')$. Set

$$C_1 := \{c \in C \mid \nu(A_1 \cap I(g_1, c)) > (1 - 3\epsilon)\nu(I(g_1, c))\}$$
$$C_1' := \{c' \in C' \mid \nu(A_1' \cap I(g_1', c')) > (1 - 3\epsilon')\nu(I(g_1', c'))\}.$$

Since $\nu(I_0 \cap A_1) > (1 - \epsilon)\nu(I_0)$ and $\nu(I_0' \cap A_1') > (1 - \epsilon')\nu(I_0')$, we deduce from Lemma 2.3 that

$$\nu_2(C_1) = \frac{\nu(\bigcup_{c\in C_1} I(g_1, c))}{\nu(I(g_1))} > \frac{2}{3}\frac{\nu(I_0)}{\nu(I(g_1))} > \frac{2}{3}\frac{\mu(A_1)}{(1 + \epsilon)\nu(I(g_1))} > \frac{8}{15(1 + \epsilon)}.$$

Without loss of generality we may assume that $\delta(g) < \frac{1}{4}$ for all $g \in G$. Therefore $\epsilon < \frac{1}{15}$ and $\nu_2(C_1) > \frac{1}{2}$. In a similar way $\nu_2(C_1') > \frac{1}{2}$. Thus there exists $c \in C_1 \cap C_1'$. We apply the hypothesis of the lemma to $I(g_1, c)$ and $I(g_1', c)$: there exists a partial transformation $\gamma \in [[\mathcal{S}]]$ such that

$$D(\gamma) \subset I(g_1, c),\ \nu(D(\gamma)) > \delta(g_1 - g_1')\nu(I(g_1, c)),$$
$$R(\gamma) \subset I(g_1', c), \text{ and } \frac{d\nu \circ \gamma}{d\gamma}(v) = \beta(g_1 - g_2) \text{ for all } v \in D(\gamma).$$

Since $\nu(D(\gamma) \cap A_1) > \epsilon\nu(I(g_1, c))$, we deduce

$$\begin{aligned}\nu(\gamma(D(\gamma) \cap A_1)) &\geq \beta(g_1 - g_1')\nu(D(\gamma) \cap A_1)\\ &> \epsilon\beta(g_1 - g_1')\frac{\nu(I(g_1, c))}{\nu(I(g_1', c))}\nu(I(g_1', c))\\ &= \epsilon\beta(g_1 - g_1')\frac{\nu(I(g_1))}{\nu(I(g_1'))}\nu(I(g_1', c)) \geq 4\epsilon'\nu(I(g_1', c)).\end{aligned}$$

Recall that $\nu(A_1' \cap I(g', c)) > (1 - 3\epsilon')\nu(I(g', c))$. Hence $\nu(\gamma(D(\gamma) \cap A) \cap A') > 0$, as desired. $\square$

In the following 5 subsections we demonstrate the 5 claims of Theorem 0.1 respectively. Theorem 0.2 is proved simultaneously.

Infinite ergodic index for (C, F)-transformations. From now on we shall assume that G has elements of infinite order. Enumerate them as $a_1, a_2, \dots$.

Lemma 2.5. *Let $\delta\colon G \to \mathbb{R}_+$ be a map with $\sum_{g\in G} \delta(g) < 1/2$. Then there exist a sequence of positive integers $(N_n)_{n=1}^\infty$ and two sequences $(C_n)_{n=1}^\infty$ and $(F_n)_{n=0}^\infty$ of finite G-subsets satisfying* (2-1)–(2-3) *such that $0 \in \bigcap_{n=1}^\infty (C_n \cap F_n)$ and*

$$\#(C_n(f)) > \delta(f)\#(C_n) \qquad \textit{for each } f \in F_{n-1}, \tag{2-4}$$

where $C_n(f) = \{c \in C_n | \ c' - c = N_n a_n + f$ *for some* $c' \in C_n\}$.

Proof. Let $G = \{g_i | \ i \in \mathbb{N}\}$ and $g_1 = 0$. Suppose that we already have $N_1, \dots, N_{n-1}$, $C_1, \dots, C_{n-1}$, $F_0, \dots, F_{n-1}$. Our purpose is to construct N_n, C_n and F_n. Let $F_{n-1} = \{f_i | \ i = 1, \dots, k\}$. Select positive integers $d_1, \dots, d_k$ in such a way that $\delta(f_i) < (d_i - 1)/d$, $i = 1, \dots, k$, where $d := d_1 + \cdots + d_k$. Now choose an integer N_n large so that

$$\mathbb{Z}(N_n a_n) \cap (\underbrace{F_{n-1} + \cdots + F_{n-1}}_{d \text{ times}} - \underbrace{F_{n-1} - \cdots - F_{n-1}}_{d \text{ times}}) = \{0\}, \tag{2-5}$$

We define C_n by listing its elements as follows:

$$\begin{aligned} &0,\ N_n a_n + f_1,\ 2N_n a_n + 2f_1, \dots,\ (d_1 - 1)N_n a_n + (d_1 - 1)f_1, \\ &\quad d_1 N_n a_n + f_2,\ (d_1 + 1)N_n a_n + 2f_2, \dots,\ (d_1 + d_2)N_n a_n + d_2 f_2, \\ &\quad \dots, \\ &\quad (d + 1 - d_k)N_n a_n + f_k, \dots\ , dN_n a_n + d_k f_k. \end{aligned} \tag{2-6}$$

Clearly, $C_n(f_i)$ is just the i-th line in (2-6) without the first (left) element. Hence $\#(C_n(f_i)) = d_i - 1 > \delta(f_i)d = \delta(f_i)\#(C_n)$. It follows from (2-5) that C_n and F_{n-1} are independent. Now we define F_n by setting

$$F_n := \bigcup_{i=1}^{n} (g_i + F_{n-1} + C_n). \quad \square$$

Remark 2.6.

(i) It is worthwhile to observe that (2-1)–(2-4) imply $C_n - C_n \supset N_n a_n + F_{n-1}$ and $C_n(f) \cap C_n(f') = \emptyset$ if $f \neq f'$.

(ii) In our inductive construction the "upper size" of F_n is not bounded, i.e. every finite set containing our F_n could also work as F_n. Hence without loss of generality we may assume that $\frac{\#(F_n)}{\#(F_{n-1}+C_n)} > n$.

Let T be a (C, F)-action of G associated with (C_n) and (F_n) satisfying (2-1)–(2-4). Without loss of generality we may assume that T is infinite measure preserving (see Remark 2.6(ii) and Theorem 2.1(iv)). We define a cocycle $\alpha_n \colon \mathcal{R}_n \to G$ by setting

$$\alpha_n(x, x') = \sum_{i=1}^{\infty} (x_i - x'_i), \qquad x = (x_i)_{i=1}^{\infty},\ x' = (x'_i)_{i=1}^{\infty} \in X_n = F_n \times \prod_{k>n} C_k.$$

It is easy to deduce from (2-2) that the subrelation $\alpha_n^{-1}(0)$ is trivial (diagonal), i.e.

$$\{(x, x') \in \mathcal{R}_n | \ \alpha_n(x, x') = 0\} = \{(x, x) | \ x \in X_n\}.$$

Given $a \in G$, we put $\mathcal{R}_n(a) := \{(x, y) \in \mathcal{R}_n | \ \alpha_n(x, y) \in \mathbb{Z}a\}$.

Lemma 2.7. *Let μ_n be Haar measure for $\mathcal{R}_n$. Given $a \in G$ of infinite order, the equivalence relation $\mathcal{R}_n(a)$ is ergodic with respect to μ_n.*

Proof. Let $k \in \mathbb{N}$ and $g_1, g_1' \in F_n$, $g_2, g_2' \in C_{n+1}, \dots, g_k, g_k' \in C_{k+n-1}$. Since $0 \in \bigcap_{n=1}^{\infty}(C_n \cap F_n)$, it follows from (2-3) that $F_1 \subset F_2 \subset \dots$ and $\bigcup_n F_n = G$. Take $l > k$ such that $g := g_1 + \dots + g_k - g_1' - \dots - g_k' \in F_{l-1}$ and $a_l \in \mathbb{Z}a$. Put

$$D(\gamma) := \bigcup_{c_{k+1} \in C_{k+1}, \dots, c_{l-1} \in C_{l-1}, c \in C_l(g)} I(g_1, \dots, g_k, c_{k+1}, \dots, c_{l-1}, c)$$

$$\gamma := T^{(n)}_{N_l a_l} \restriction D(\gamma).$$

Clearly, $D(\gamma) \subset I(g_1, \dots, g_k)$,

$$\gamma I(g_1, \dots, g_k, c_{k+1}, \dots, c_{l-1}, c) = I(g_1', \dots, g_k', c_{k+1}, \dots, c_{l-1}, c + N_l a_l + g)$$

and hence $R(\gamma) \subset I(g_1', \dots, g_k')$. From (2-4) we deduce that

$$\frac{\mu_n(D(\gamma))}{\mu_n(I(g_1, \dots, g_k))} = \frac{\#(C_l(g))}{\#(C_l)} > \delta(g).$$

Since γ is a partial transformation from $[[\mathcal{R}_n(a)]]$, we apply Lemma 2.4 to complete the proof. $\square$

For each $m > 1$, we let $(X_{n,m}, \mu_{n,m}) := \bigotimes_1^m (X_n, \mu_n)$, $\mathcal{R}_{n,m} := \bigotimes_1^m \mathcal{R}_n$ and $\alpha_{n,m} := \bigotimes_1^m \alpha_n$. Remark that $X_{n,m}$ can be considered as an infinite product space $X_{n,m} = F_n^m \times \prod_{k>n} C_k^m$, where the upper index m means the m-fold Cartesian product. Thus $\mathcal{R}_{n,m}$ is just the tail equivalence relation on $X_{n,m}$ and $\mu_{n,m}$ its Haar measure.

Corollary 2.8. *Let a be an element of G of infinite order and*

$$\mathcal{R}_{n,m}(a) := \alpha_{n,m}^{-1}(\mathbb{Z}(a, \dots, a)).$$

Then $\mathcal{R}_{n,m}(a)$ is an ergodic subrelation of $\mathcal{R}_{n,m}$.

Proof. Define a map $\delta_m \colon G^m \to \mathbb{R}_+$ by setting $\delta_m(g_1, \dots, g_m) = \delta(g_1) \cdots \delta(g_m)$. Replace $(C_k)_k$ and $(F_k)_k$ by $(C_k^m)_k$ and $(F_k^m)_k$ respectively, where the upper indices mean the m-fold Cartesian products. Then the later pair of sequences satisfies (2-1)–(2-4) with δ_m instead of δ. It remains to apply Lemma 2.7. $\square$

Remark 2.9. Since the set of cylinders is a base for the topology on X_n, it follows that $\mathcal{R}_n(a)$ and $\mathcal{R}_{n,m}(a)$ are topologically transitive.

It is easy to verify that $\alpha_{n+1} \circ (i_n \times i_n) = \alpha_n$ for each $n \in \mathbb{N}$. Hence an inductive limit α of $(\alpha_n, i_n \times i_n)$ is well defined. Clearly, α is a cocycle of $\mathcal{R}$ with values in G. It is straightforward that $\alpha(T_g x, x) = g$ for all $x \in X$, $g \in G$, i.e. α is a "return time" cocycle for T. Hence α is transient. Put $\mathcal{R}^m := \underbrace{\mathcal{R} \otimes \dots \otimes \mathcal{R}}_{m \text{ times}}$ and $\mathcal{R}^m(a) := (\alpha^m)^{-1}(\mathbb{Z}(a, \dots, a))$ for an element $a \in G$. Recall that μ is Haar measure for $\mathcal{R}$.

Theorem 2.10. *Let a be a G-element of infinite order.*

(i) *T_a is a μ-preserving transformation on X of infinite ergodic index,*
(ii) *$\mathcal{R}^m(a)$ is an ergodic subrelation of $\mathcal{R}^m$ for each $m > 0$.*

Proof. (ii) It is easy to verify that

$$\mathcal{R}^m(a) = \operatorname*{inj\,lim}_{n\to\infty}(\mathcal{R}_{n,m}(a), i_n^m \times i_n^m).$$

By Corollary 2.8, $\mathcal{R}_{n,m}(a)$ is ergodic. From this we deduce that so is $\mathcal{R}^m(a)$.

(i) follows from (ii), since $\mathcal{R}^m(a)$ is the $\underbrace{T_a \times \cdots \times T_a}_{m \text{ times}}$-orbit equivalence relation. $\square$

Remark 2.11. We observe that Theorem 0.2(1) was proved simultaneously (cf. Remark 2.9):

(i) $\underbrace{T_a \times \cdots \times T_a}_{m \text{ times}}$ is a topologically transitive transformation of X^m,
(ii) $\mathcal{R}^m(a)$ is a topologically transitive subrelation of $\mathcal{R}^m$.

Power weakly mixing actions. The above ideas can be adapted to construct G-actions with more stronger ergodic properties. Let $\mathfrak{S} = \{S_n |\ n \in \mathbb{N}\}$ be the set of finite sequences of G-elements (possibly equal) of infinite order. We record an analogue of Lemma 2.5.

Lemma 2.12. *Let $\delta\colon G \to \mathbb{R}_+$ be a map with $\sum_{g\in G}\delta(g) < 1/2$. Then there exist a sequence of positive integers $(N_n)_{n=1}^\infty$ and two sequences $(C_n)_{n=1}^\infty$ and $(F_n)_{n=0}^\infty$ of finite G-subsets satisfying* (2-1)–(2-3) *such that $0 \in \bigcap_n (C_n \cap F_n)$ and*

(2-7) $\#(C_n(f,b)) > \delta(f)\#(C_n)\#(S_n)^{-1}$ *for each $f \in F_{n-1}$ and every $b \in S_n$,*

where $C_n(f,b) = \{c \in C_n |\ c' - c = N_n b + f$ for some $c' \in C_n\}$.

Sketch of the proof. Let $S_n = (b_1, \dots, b_q)$. The argument is similar to what we used in Lemma 2.5. Notice that if A is a finite G-subset and b an element of infinite order then for each $a \in A$ and all sufficiently large n, the element $nb - a$ is of infinite order. Hence there is an increasing sequence $l_1 < \cdots < l_{q-1}$ such that $mb_{s+1} - ib_j$ is of infinite order for all $m \ge l_s, j \le s, i < l_{s-1} + d$, $s = 1, \dots, q-1$. Here d is just the same as in Lemma 2.5. Now we let

$$C_n := A_n(b_1) \cup (N_n l_1 b_2 + A_n(b_2)) \cup \cdots \cup (N_n l_{q-1} b_q + A_n(b_q)),$$

where $A_n(b_j)$ is the set "C_n" from Lemma 2.5 with b_j instead of "a_n". The integer N_n here is chosen so large to make C_n and F_{n-1} independent. We leave details to the reader. $\square$

Remark that the assertions 2.7—2.11 are corollaries from Lemma 2.5. In a similar way, one can deduce some analogues of them from Lemma 2.12 with an almost literal argument. We summarize them in the following

Theorem 2.13. *Let T be the (C,F)-action of G associated to $(C_n)_n$ and $(F_n)_n$ which satisfy (2-1)–(2-3), (2-7) and Remark 2.6(ii). Then T is infinite measure preserving. Moreover, the transformation $T_{a_1}\times\cdots\times T_{a_q}$ is topologically transitive and ergodic for each sequence $(a_1,\dots,a_q)\in\mathfrak{S}$.*

Remark 2.14.

(i) If $G=\mathbb{Z}$ we obtain an infinite measure preserving transformation T such that $T^{n_1}\times\cdots\times T^{n_k}$ is ergodic for each sequence of non-zero integers $n_1,\dots,n_k$. This property of T is called **power weakly mixing** in [DGMS].

(ii) The cocycle α has an interesting property. Suppose for simplicity that G is torsion-free. Then for each $m\in\mathbb{N}$ and a subgroup H of G^m with the nontrivial coordinate pullbacks, the quotient cocycle

$$\alpha^m+H\colon \bigotimes_1^m\mathcal{R}\to G^m/H$$

has dense range in G^m/H. Recall that α^m is transient.

Actions with nonconservative "square". Our purpose here is to demonstrate Theorem 0.1(4). To this end we replace (2-4) by some conditions of the "opposite" nature: there are two sequences $(C_n)_n$ and $(F_n)_n$ which satisfy (2-1)–(2-3) and

(a) the sequence $C_1-C_1, C_2-C_2,\dots$ is independent,

(b) for each $n>0$, the map

$$(C_n\times C_n)\setminus D\ni(c,c')\mapsto c-c'\in C_n-C_n'$$

is one-to-one, where $D:=\{(g,g)|\ g\in G\}$ is the diagonal in $G\times G$,

(c) $\sum_{n=1}^\infty\frac{1}{\#(C_n)}<\infty$.

These sequences can be constructed explicitly by an inductive process. This is routine.

Theorem 2.15. *Let T be the (C,F)-action of G associated to $(C_n)_n$ and $(F_n)_n$ as above. Then the Cartesian square $\{T_g\times T_g\}_{g\in G}$ of T is nonconservative.*

Proof. Recall that $X_0=\prod_{n=1}^\infty C_n$ and hence $X_0\times X_0=\prod_{n=1}^\infty(C_n\times C_n)$. We let $A:=\prod_{n=1}^\infty((C_n\times C_n)\setminus D)$. It follows from (c) that A has positive measure:

$$(\mu\times\mu)(A)=\prod_{n=1}^\infty\frac{\#(C_n)^2-\#(C_n)}{\#(C_n)^2}=\prod_{n=1}^\infty\left(1-\frac{1}{\#(C_n)}\right)>0.$$

If $(x,y)\in A$ and $(x',y'):=(T_gx,T_gy)\in A$ for some $g\in G$ then there is an integer $r>0$ such that

$$x_1+\cdots+x_r+g=x_1'+\cdots+x_r',$$
$$y_1+\cdots+y_r+g=y_1'+\cdots+y_r'.$$

Hence $x_1 - y_1 + \cdots + x_r - y_r = x_1' - y_1' + \cdots + x_r' - y_r'$. ¿From (a) we deduce that $x_1 - y_1 = x_1' - y_1', \dots, x_r - y_r = x_r' - y_r'$. It follows from (b) that $x_1 = x_1', \dots, x_r = x_r'$; $y_1 = y_1', \dots, y_r = y_r'$. Hence $g = 0$. □

Remark that the wandering subset A is compact without isolated points and its interior is empty.

Actions with continuous L^∞-spectrum, nonergodic Cartesian squares and all m-fold Cartesian products conservative. We first recall that given a nonsingular action S of G on (Y, ν), a measurable map $f\colon Y \to \mathbb{T}$ is called an eigenfunction of S if $f \circ S_g = \xi(g) f$ a.e. for a character $\xi \in \hat{G}$. S is said to have **trivial L^∞-spectrum** if every eigenfunction of S is constant.

The following lemma is standard and we state it without proof.

Lemma 2.16. *Let $\mathcal{S}$ be a nonsingular equivalence relation on a standard measure space $(Y, \mathfrak{A}, \nu)$, $\mathfrak{A}_0$ a dense subalgebra of $\mathfrak{A}$ and δ a positive real. If for every $A \in \mathfrak{A}_0$ there is a partial transformation $\gamma \in [[\mathcal{S}]]$ such that $D(\gamma) \cup R(\gamma) \subset A$, $\nu(D(\gamma) > \delta\nu(A)$, $\nu(R(\gamma)) > \delta\nu(A)$ and $\gamma x \neq x$ for each $x \in D(\gamma)$ then $\mathcal{S}$ is conservative.*

Let a be an element of infinite order in G and $\{g_n\}_{n=1}^\infty$ a sequence of G-elements in which every g occurs infinitely often. One can construct inductively two sequences $(C_n)_{n=1}^\infty$ and $(F_n)_{n=0}^\infty$ which satisfy (2-1)–(2-3) and such that

(a) $C_n = \{0, N_n a, 3N_n a + g_n\}$ for some integer N_n, $n = 1, 2 \dots,$
(b) the sequence $C_1 - C_1, C_2 - C_2, \dots$ is independent.

Notice that

$$C_n - C_n = \{-3N_n a - g_n, -2N_n a - g_n, -N_n a, 0, N_n a, 2N_n a + g_n, 3N_n a + g_n\}.$$

Theorem 2.17. *Let T be the (C,F)-action of G associated to $(C_n)_{n=1}^\infty$ and $(F_n)_{n=1}^\infty$ as above. Then T has trivial L^∞-spectrum, nonergodic Cartesian square but all k-fold Cartesian products conservative.*

Proof. Given $m > 1$, we denote by $\mathcal{S}$ the $\{\underbrace{T_g \times \cdots \times T_g}_{m \text{ times}}\}_{g \in G}$-orbit equivalence relation. First we prove that $\mathcal{S}$ is conservative. To this end it is enough to verify that the restriction of $\mathcal{S}$ to $X_{0,m}$ is conservative. Recall that $X_{0,m} = \prod_{k>0} C_k^m$. Given $c_1 \in C_1^m, \dots, c_l \in C_l^m$, we denote by $I(c_1, \dots, c_l)$ the corresponding cylinder in $X_{0,m}$. Put $v := (0, \dots, 0)$, $w := (N_{l+1}a, \dots, N_{l+1}a) \in C_{l+1}^m$ and define a partial transformation $\gamma \in [[\mathcal{S}]]$ by setting

$$D(\gamma) := I(c_1, \dots, c_l, v), \; R(\gamma) := I(c_1, \dots, c_l, w)$$
$$\gamma y := (T_{N_{l+1}a} \times \cdots \times T_{N_{l+1}a}) y$$

for all $y \in D(\gamma)$. Clearly, $D(\gamma) \cup R(\gamma) \subset I(c_1, \dots, c_l)$. We deduce from (a) that

$$\frac{\mu(D(\gamma))}{\mu(I(c_1, \dots, c_l))} = \frac{\mu(R(\gamma))}{\mu(I(c_1, \dots, c_l))} = \frac{1}{3^m}.$$

It follows from Lemma 2.16 that $\mathcal{S}$ is conservative. Actually, let $\mathfrak{A}_0$ be the algebra generated by the cylinders and $\delta := 1/(2\cdot 3^m)$. If A is a cylinder then—as we have just shown—the hypothesis of the lemma is satisfied. If A is the union of finitely many cylinders then define γ as the "concatenation" of the partial transformations acting within each of these cylinders. Clearly, γ is as desired.

We now show that $\{T_g\times T_g\}_{g\in G}$ is not ergodic. Let $A := \{0\}\times\prod_{k>1} C_k$ and $B := \{N_1 a\}\times\prod_{k>1} C_k$. Clearly, A and B are subsets of positive measure in $X_0\subset X$. If there is $g\in G$ with $(T_g\times T_g)(A\times B)\cap(A\times A)\neq\emptyset$, then

$$\begin{cases} g\in 0+\sum_{k>1}(C_k-C_k)\\ g\in N_1 a+\sum_{k>1}(C_k-C_k).\end{cases}$$

But this contradicts to (b).

It remains to show that the L^∞-spectrum of T is trivial. Let $f\colon X\to\mathbb{T}$ be a measurable map such that $f\circ T_g=\xi(g)f$ for all $g\in G$ and some character $\xi\in\widehat{G}$. Given $\epsilon>0$, there exists a subset $A\subset X_0$ of positive measure such that $|f(x)-f(y)|<\epsilon$ for all $x,y\in A$. Take a cylinder $I(b_1,\dots,b_q)\subset X_0$ such that

$$\mu(I(b_1,\dots,b_p)\cap A)>0.99\mu(I(b_1,\dots,b_p)). \tag{2-8}$$

Since

$$\begin{aligned}\mu(I(b_1,\dots,b_p,0))&=\mu(I(b_1,\dots,b_p,N_{p+1}a))\\ &=\mu(I(b_1,\dots,b_p,3N_{p+1}a+g_{p+1}))\end{aligned}$$

and T preserves μ, there is a subset $B\subset I(g_1,\dots,b_p,0)$ such that $\mu(B)>0$ and $B\cup T_{N_{p+1}a}B\cup T_{N_{p+1}a+g_{p+1}}B\subset A$. Hence

$$|1-\xi(N_{p+1}a)|\le\epsilon \text{ and } |1-\xi(2N_{p+1}a+g_{p+1})|\le\epsilon.$$

It follows that $|1-\xi(g_{p+1})|\le 3\epsilon$. For every $q>p$, there exists a cylinder $I(b_1,\dots,b_p,\dots,b_q)$ for which (2-8) holds. Repeating the argument we obtain that $|1-\xi(g_{q+1})|\le 3\epsilon$. Since every element of G occurs infinitely many times in $\{g_n\}_{n=1}^\infty$, it follows that $|1-\xi(g)|\le 3\epsilon$ for all $g\in G$. Thus ξ is trivial. $\square$

Infinite ergodic index without power weak mixing. It may seem that Theorem 0.1(1) implies Theorem 0.1(2). The purpose of this subsection is to disprove this conjecture: we demonstrate Theorem 0.1(3) here.

Lemma 2.18. *Let δ be as in Lemma 2.5. Then there are sequences (N_n), (C_n), (F_n) satisfying* (2-1)–(2-4) *such that $0\in\bigcap_n(C_n\cap F_n)$ and*

(i) *the sequence $2C_1-C_1, 2C_2-C_2,\dots$ is independent,*
(ii) *for each $n>0$, the map*

$$(C_n\times C_n)\setminus\Gamma\ni(c,c')\mapsto 2c-c'\in C_n-C_n$$

is one-to-one, where $\Gamma=\{(g,2g)\mid g\in G\}$.
(iii) $\sum_{n=1}^\infty\#(C_n)^{-1}<\infty$.

Sketch of the proof. One should repeat almost literally the proof of Lemma 2.5. The modification is as follows. Let a be an element of infinite order in G. Take some integer M_n (to be specified below) and set $C_n := (C'_n + M_n a) \cap \{0\}$, where C'_n is just the set C_n from Lemma 2.5. Clearly (2-4) remains true for this new C_n. It will be also true if we replace N_n by a larger integer. Now we select M_n in such a way that the "distance" between elements of $2C_n - C_n$ is greater than the "diameter" of $2C_{n-1} - C_{n-1}$. In order to achieve this we may need to enlarge N_n. □

Theorem 2.19. *Let T be the infinite measure preserving (C, F)-action of G associated to (C_n) and (F_n) from Lemma 2.18. Then for each $g \in G$ of infinite order the following is satisfied:*

(i) *T_g has infinite ergodic index,*
(ii) *$T_{2g} \times T_g$ is nonconservative.*

Proof. The first assertion follows from Theorem 2.10. To prove the second we let $A := \prod_{n=1}^{\infty}(C_n \times C_n) \setminus \Gamma \subset X_0 \times X_0$ and repeat the proof of Theorem 2.15 with an obvious modification. □

Thus Theorems 0.1 and 0.2 are proved completely (see Theorems 2.10, 2.13, 2.15, 2.17, 2.19; and Remark 2.11(i), Theorem 2.13 respectively).

3. Generalized Hajian-Kakutani actions

In this section we isolate a special class of (C, F)-actions which possess a number of interesting properties.

Exhausting weakly wandering subsets. Let $T = \{T_g\}_{g \in G}$ be a free Borel action of a countable Abelian group G on $(X, \mathfrak{B})$. A set $A \in \mathfrak{B}$ is called **exhausting weakly wandering** (e.w.w.) for T under a (countable) subset $S \subset G$ if the sets $T_g A$, $g \in S$, are disjoint and their union is X. The corresponding subset S is called **tiling** for T.

Proposition 3.1. *Let μ be a σ-finite T-invariant ergodic measure on $(X, \mathfrak{B})$.*

(i) *If $A, B \in \mathfrak{B}$ are e.w.w. under a very same tiling subset then $\mu(A) = \mu(B)$;*
(ii) *if there exists an e.w.w. subset $A \in \mathfrak{B}$ with $\mu(A) < \infty$ then every μ-nonsingular transformation commuting with T preserves μ.*

For the proof in the case $G = \mathbb{Z}$ we refer the reader to [EHI]. The general case is considered in a similar way.

We do not provide a proof of the following statement since it is routine.

Proposition 3.2. *There exists an independent sequence $(C'_n)_{n=1}^{\infty}$ of finite G-subsets such that $0 \in \bigcap_{n=1}^{\infty} C'_n$, $\#(C'_n) > 1$, and $C'_1 + C'_2 + \cdots = G$.*

It is clear that given $n_1 < n_2 < \dots$, the sequence

$$(C'_1 + \cdots + C'_{n_1}), (C'_{n_1+1} + \cdots + C'_{n_2}), \dots$$

is also independent. We call it a **telescoping** of $(C'_n)_n$.

We put $F_0 := \{0\}$, $F_n := C'_1 + \cdots + C'_{2n}$, and $C_n := C'_{2n}$ for $n \geq 1$. Clearly, (2-1) and (2-2) are satisfied.

Replacing $(C'_n)_n$ by an appropriate telescoping we may (and shall) assume that (2-3) holds.

Definition 3.3. The (C, F)-action T of G is called **a generalized Hajian-Kakutani action** or, more precisely, the **G-action associated to** $(C'_k)_{k=1}^\infty$.

It follows from Theorem 2.1 that T is free, minimal and the corresponding Haar measure is infinite. Moreover, T is ergodic and has funny rank one with respect to this measure.

Notice that T is an analogue of Hajian-Kakutani transformation—i.e. $\mathbb{Z}$-action—from [HK] (see also [EHI] and [M–Z]).

It follows straightforward from the definition of T that

$$\hat{i}_{n+1}(X_{n+1}) = \bigcup_{g \in C'_{2n+1}} T_g \hat{i}_n(X_n)$$

and $T_g(\hat{i}_n(X_n)) \cap T_h(\hat{i}_n(X_n)) = \emptyset$ for all $g, h \in C'_{2n+1}$ with $g \neq h$. ¿From this we deduce

Proposition 3.4.

(i) *Given $n > 0$, the subset $\hat{i}_n(X_n)$ is e.w.w. for T under $\sum_{k \geq n} C'_{2k+1}$.*
(ii) *If $T_g \hat{i}_0(X_0) \cap \hat{i}_0(X_0) \neq \emptyset$ for some $g \in G$ then $g \in \sum_{k=1}^\infty (C'_{2k} - C'_{2k})$.*

It follows that T is of **finite type** in the sense of [EHI], i.e. T admits e.w.w. sets of finite Haar measure. The following statement follows from this and Proposition 3.1.

Corollary 3.5.

(i) *For each $h \in G$, the set $A := T_h \hat{i}_0(X_0)$ is e.w.w. for T under $\sum_{k \geq 0} C'_{2k+1}$.*
(ii) *If $T_g A \cap A \neq \emptyset$ then $g \in \sum_{k=1}^\infty (C'_{2k} - C'_{2k})$.*
(iii) *Every μ-nonsingular transformation commuting with T preserves μ.*

Definition 3.6. We say that a tiling set for a G-action is **generating** if it is not contained in any proper subgroup of G.

Remark that in [M–Z], for $G = \mathbb{Z}^d$, the generating tiling sets are called **properly exhaustive**.

Proposition 3.7. *If G is not a torsion group then there exists a sequence of independent finite G-subsets (C'_n) satisfying* (2-3) *and such that $C'_1 + C'_2 + \cdots = G$, $0 \in \bigcap_n C'_n$ and the associated G-action has a generating tiling set. Moreover, the corresponding e.w.w. subset is of finite Haar measure.*

Proof. Let a be an element of G of an infinite order. Denote by π: $G \to G/\mathbb{Z}a$ the canonical projection map. One can choose a sequence of independent finite G-subsets $(C_n)_{n=1}^{\infty}$ in such a way that the following is satisfied:

(i) $a \in C_1$,
(ii) $\mathbb{Z}a = C_1 + C_2 + C_4 + C_6 \cdots$,
(iii) $\pi(C_3 + C_5 + \cdots) = G/\mathbb{Z}a$,
(iv) $0 \in \bigcap_n C_n$, $\#(C_n) > 1$.

Replace each of the two sequences $C_1, C_2, C_4, \dots$ and $C_3, C_5, \dots$ by some telescopings $C'_1, C'_2, C'_4, \dots$ and $C'_3, C'_5, \dots$ respectively in such a way that $(C'_n)_n$ satisfies (2-3). It remains to apply Proposition 3.4(i). □

Strong disjointness.

Definition 3.8. Let F and F' be two topological G-actions on Polish spaces Z and Z' respectively. We say that F and F' are **strongly disjoint** if there is no any non-atomic probability $\{F(g)\times F'(g)\}_{g\in G}$-quasi-invariant ergodic measure on $Z\times Z'$ whose Z-pullback is F-quasi-invariant or Z'-pullback is F'-quasi-invariant.

Let σ: $\mathbb{N} \to \mathbb{N}$ be a bijection such that $\{\sigma(2), \sigma(4), \dots\} \subset \{1, 3, 5 \dots\}$. Denote by T^σ the generalized Hajian-Kakutani G-action associated to the sequence $(C'_{\sigma(n)})_{n\in\mathbb{N}}$ and by X^σ the space of this action.

Theorem 3.9 (cf. [EHI, Theorem 2])**.** *T and T^σ are strongly disjoint.*

Proof. Let ν be a non-atomic probability $\{T_g \times T^\sigma_g\}_{g\in G}$-quasi-invariant ergodic measure on $X\times X^\sigma$ and μ its X-pullback. We assume that μ is T-quasi-invariant. Denote $\hat{i}_0(X_0)$ by W and the similar subset of X^σ by W^σ. If

$$(T_g \times T^\sigma_g)(W \times W^\sigma) \cap (W \times W^\sigma) \neq \emptyset$$

for some $g \in G$ then $T_g W \cap W \neq \emptyset$ and $T^\sigma_g W^\sigma \cap W^\sigma \neq \emptyset$. By Proposition 3.4(ii),

$$g \in \sum_{k=1}^{\infty}(C'_{2k} - C'_{2k}) \text{ and } g \in \sum_{k=1}^{\infty}(C'_{\sigma(2k)} - C'_{\sigma(2k)}).$$

Since the collection $(C_k)_{k=1}^{\infty}$ is independent, it follows that $g = 0$. Hence the sets $(T_g \times T^\sigma_g)(W \times W^\sigma)$, $g \in G$, are pairwise disjoint. Since ν is ergodic and non-atomic, we obtain $\nu(W \times W^\sigma) = 0$. In a similar way, $\nu(W \times T^\sigma_h W^\sigma) = 0$ for every $h \in G$ (see Corollary 3.5). Since W^σ is e.w.w. for T^σ, it follows that $\nu(W \times X^\sigma) = 0$ and hence $\mu(W) = 0$. In turn, W is e.w.w. for T^σ and this implies $\mu(X) = 0$ and hence $\nu(X \times X^\sigma) = 0$, a contradiction. □

Remark 3.10. Slightly modifying the above argument one can find countably many bijections σ_i: $\mathbb{N} \to \mathbb{N}$ such that the corresponding G-actions T^{σ_i}, $i \in \mathbb{N}$, are pairwise strongly disjoint. In particular, $(X^{\sigma_i}, \mu_i, T^{\sigma_i})$ is a countable family of pairwise disjoint (and hence non-isomorphic in the measure category sense) ergodic infinite measure preserving G-actions of finite type, where μ_i stands for Haar measure on X^{σ_i}.

4. Nonsingular (C,F)-actions and Theorem 0.3

To prove Theorem 0.3 we adapt the argument used in the proof of Theorem 0.1 to the nonsingular case. Trying to avoid repetitions we concentrate our attention on new phenomena only. We begin with an analogue of Lemma 2.5.

Recall that $(a_n)_{n=1}^\infty$ is a sequence consisting of all elements of G of infinite order.

Lemma 4.1. *Let $W = \{W_t\}_{t\in\mathbb{R}}$ be an AT-flow and $\delta\colon G \to \mathbb{R}_+$ a map with $\sum_{g\in G}\delta(g) < 1/2$. There exist a sequence of positive integers $(N_n)_{n=1}^\infty$, two sequences $(C_n)_{n=1}^\infty$ and $(F_n)_{n=0}^\infty$ of finite G-subsets and a sequence $(\kappa_n)_{n=1}^\infty$ of probability measures on $(C_n)_n$ such that:* (2-1)–(2-3) *are satisfied,* $0 \in \bigcap_{n=1}^\infty(C_n\cap F_n)$, *the associated flow of the (nonsingular) tail equivalence relation on the product measure space $\bigotimes_{n=1}^\infty(C_n,\kappa_n)$ is W and*

$$\kappa_n(C_n^0(f)) > \delta(f) \quad \text{for each } f\in F_{n-1}, \tag{4-1}$$

where

$$C_n^0(f) := \{c \in C_n \mid c' - c = N_n a_n + f \text{ for some } c' \in C_n \text{ with } \kappa_n(c') = \kappa_n(c)\}.$$

Proof. Let $G = \{g_i \mid i \in \mathbb{N}\}$ and $g_1 = 0$. By Theorem 1.3, W is the associated flow of the tail equivalence relation, say $\mathcal{S}_1$, on an infinite product space $\bigotimes_{n=1}^\infty(V_n,\nu_n)$, where each V_n is finite. Suppose that we already have $N_1,\dots,N_{n-1}$, $C_1,\dots,C_{n-1}$, $F_0,\dots,F_{n-1}$, $\kappa_1,\dots,\kappa_{n-1}$ and our purpose is to construct N_n, C_n, F_n, κ_n. Let $V_n = \{1,\dots,m\}$ and $F_{n-1} = \{f_j\}_{j=1}^k$. Select positive integers $d_1,\dots,d_k$ in such a way that $\frac{d_i}{d} > 2\delta(f_i)$, $d_i > 2m$ and d_i is divided by m for each $i = 1,\dots,k$. Now choose N_n large so that (2-5) holds and define C_n by (2-6). Consider a finite set D_n partitioned as $D_n = \bigcup_{i=1}^k D_n(i)$ with $\#(D_n(i)) = d_i/m$. Decompose the i-th line of (2-6) into m consecutive blocks of equal length as follows:

$$\overbrace{\underbrace{\bullet\cdots\bullet}_{d_i/m \text{ terms}}}^{1-\text{st block}}\cdots\overbrace{\underbrace{\bullet\cdots\bullet}_{d_i/m \text{ terms}}}^{m-\text{th block}}.$$

There is a bijection of $V_n \times D_n$ onto C_n which maps $\{j\}\times D_n(i)$ onto the j-th block of the i-th line. This bijection transfers the product measure $\nu_n \times$ (the equidistribution) to some probability measure on C_n. We call it κ_n. Clearly, $C_n^0(f_i)$ is just the i-th line of (2-6) without the "bad" elements—the first terms of the blocks. Hence

$$\kappa_n(C_n^0(i)) \ge \frac{d_i}{d}(1-\frac{m}{d_i}) > \frac{d_i}{2d} > \delta(f_i),$$

as desired. Now we define F_n just like in Lemma 2.5, i.e.

$$F_n := \bigcup_{i=1}^n (g_i + F_{n-1} + C_n).$$

It remains to find the associated flow of the tail equivalence relation $\mathcal{S}$ on the space $\bigotimes_{n=1}^{\infty}(C_n, \nu_n)$. To this end we remark that $\mathcal{S} = \mathcal{S}_1 \otimes \mathcal{S}_2$, where $\mathcal{S}_2$ is the tail equivalence relations on $\bigotimes_{n=1}^{\infty}(D_n,$ the equidistribution). Since $\mathcal{S}_2$ is of type II_1, the associated flows of $\mathcal{S}$ and $\mathcal{S}_1$ are conjugate. □

Suppose that $(C_n)_n$ and $(F_n)_n$ satisfy (2-1)–(2-3). Given a sequence κ_n of probability measures on C_n such that $\bigotimes_{n=1}^{\infty} \kappa_n$ is non-atomic, one can construct inductively a sequence (τ_n) of measures on (F_n) such that $\tau_0(0) = 1$ and $\tau_n(f_{n-1}+c_n) = \tau_{n-1}(f_{n-1})\kappa_n(c_n)$. We furnish $X_n = F_n \times \prod_{k>n} C_k$ with the product measure $\mu_n := \tau_n \otimes \bigotimes_{k>n} \kappa_k$. Clearly, $\mu_n \circ i_n^{-1} = \mu_{n+1} \restriction i_n(X_n)$. Hence an inductive limit μ of $(\mu_n)_{n=1}^{\infty}$ is well defined. Clearly, μ is a σ-finite measure on X.

Definition 4.2. We call μ a (C, F, κ)-measure.

Remark that the equivalence class of μ does not depend on a particular choice of (τ_n). It is determined uniquely by (κ_n).

Clearly, $\mathcal{R}$ (and the corresponding G-action T) is μ-nonsingular. Denote by ρ_μ its Radon-Nikodym cocycle. If $x = (f_n, c_{n+1}, \dots)$, $y = (f'_n, c_{n+1}, \dots) \in X_n$ then

$$\rho_\mu(\hat{i}_n x, \hat{i}_n y) = \log \tau_n(f'_n) - \log \tau_n(f_n) + \sum_{k>n} (\log \kappa_k(c'_k) - \log \kappa_k(c_k)).$$

It is easy to verify that T has funny rank one (cf. Theorem 2.1) with respect to μ. Since X_0 is a subset of positive measure in X, the associated flows of $\mathcal{R}$ and $\mathcal{R} \restriction X_0$ are conjugate.

The following statement is an analogue of Lemma 2.7.

Lemma 4.3. *$\mathcal{R}_n(a)$ is an ergodic equivalence relation on (X_n, μ_n) for every element $a \in G$ of infinite order.*

Sketch of the proof. The proof is similar to that of Lemma 2.7. The crucial point is to apply Lemma 2.4. Remark that $\mathcal{R}_n$ is no longer measure preserving. We define a map β: $G \to \mathbb{R}_+$ by setting

$$\beta(g) = \begin{cases} 1 & \text{if } g \notin \alpha_n(\mathcal{R}_n) \\ \exp(\rho_\mu(x, y)) & \text{if } g = \alpha_n(x, y) \end{cases}$$

With this β and δ from Lemma 4.1 we apply Lemma 2.4 in a way similar to that used in Lemma 2.7. Remark that we replaced C_n (used in Lemma 2.5) by C_n^0 (used in Lemma 4.1) just to obtain the required (in Lemma 2.4) inequality for the Radon-Nikodym derivative. □

Slightly modifying the proof of Theorem 2.10 we obtain

Theorem 4.4. *Let (C_n), (F_n), (κ_n) satisfy* (2-1)–(2-3) *and* (4-1). *Then the corresponding (C, F)-action T of G is has funny rank one with respect to a (C, F, τ)-measure. The associated flow of T is W. For every $a \in G$ of infinite order, the transformation T_a has infinite ergodic index.*

Thus Theorem 0.3(1) is done. Theorem 0.3(2) can be demonstrated in a similar way. As for Theorem 0.3(4,5) they follow from the following two statements (cf. Theorems 2.15, 2.17 respectively).

Theorem 4.5. *Given an AT-flow W, there exist (C_n), (F_n), (κ_n) satisfying* (2-1)–(2-3), *(a) and (b) before Theorem 2.15 and*

(c') $\sum_n \sum_{c\in C_n} \kappa_n(c)^2 < \infty$,

(d) *W is the associated flow of the tail equivalence relation on the product measure space $\prod_{n=1}^\infty (C_n, \tau_n)$.*

The corresponding (C,F)-action T of G has funny rank one with respect to a (C,F,κ)-measure. The associated flow of T is W. The action $T\times T$ is nonconservative.

Theorem 4.6. *Given an AT-flow W, there exist (C_n), (F_n), (κ_n) satisfying* (2-1)–(2-3), *(b) before Theorem 2.17 and*

(a') *$C_{2n} = \{0, N_n a, 3N_n a + g_n\}$ for some integer N_n, where a is a G-element of infinite order,*

(c) *κ_{2n} is equidistributed on C_{2n},*

(d) *W is the associated flow of the tail equivalence relation on the product measure space $\prod_{n=1}^\infty (C_n, \tau_n)$.*

The corresponding (C,F)-action T of G has funny rank one with respect to a (C,F,κ)-measure. The associated flow of T is W. T has trivial L^∞-spectrum, nonergodic Cartesian square but all k-fold Cartesian products conservative.

Combining the arguments of Theorems 4.5 and 4.6 one can deduce Theorem 0.3(3). We leave details to the reader.

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