STRONG ORBIT EQUIVALENCE OF LOCALLY COMPACT CANTOR MINIMAL SYSTEMS

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Abstract. We study minimal self-homeomorphisms of zero dimensional metrizable locally compact non-compact Hausdorff spaces. For this class of systems, we show that the ordered cohomology group is a complete invariant for strong orbit equivalence, i.e. topological orbit equivalence with continuous orbit cocycles. This is an "infinite" counterpart of a well known result of Giordano, Putnam and Skau about compact Cantor systems.

0. Introduction

The seminal paper [GPS1] demonstrates a comprehensive analysis of topological orbit equivalence of minimal Cantor systems. It appears that K_0-theory yields complete information about the orbit structure of such systems. However in that and the subsequent papers on topological orbit equivalence ([GW], [BH], [GPS2]) only homeomorphisms of compact spaces were considered. In contrast to that, the main purpose of the present work is to study minimal systems on *locally compact non-compact* totally disconnected metrizable spaces. It turns out that the dynamical properties of these systems are rather different from their compact counterparts. For instance, the set of points with dense semi-orbits is not the entire space but only a dense G_δ with empty(!) interior. It is well known that a topological version of the Rohlin lemma ("Kakutani-Rohlin" tower analysis) is crucial in studying Cantor systems on compact sets ([Ve], [Pu], [HPS]). However it fails in the locally compact case because of a "bad recurrence" property: every open compact set has a nonempty wandering subset. (See Section 1 for details.) In Section 2 we provide a family of examples of minimal systems on locally compact Cantor spaces. They are uniquely ergodic. This means that they admit a unique up to scaling σ-finite invariant Radon measure, i.e. a Borel measure which is finite on the compact subsets. (Remark that for some subclass of this systems, the invariant measure is finite.) We hope that they will work as a good source for modeling topological and measurable transformations with various dynamical properties like power minimality, power (weak) mixing, rank-one, etc. (see [Da]).

In Section 3 we show that the one-point compactification of a locally compact minimal system yields an *almost minimal* compact Cantor system. This means that there is a fixed point and the orbit of any other point is dense. We study these systems applying the machinery from [HPS]. Specifically, we describe the "almost minimality" in terms of the underlying Bratteli diagrams and the ordered

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cohomology groups. This analysis works in the final Section 4 to study the strong orbit equivalence. Two locally compact non-compact Cantor systems (X, ϕ) and (Y, ψ) are *strongly orbit equivalent* if there is a homeomorphism $F : X \to Y$ and two continuous maps $n, m : X \to \mathbb{Z}$ such that

$$F(\phi x) = \psi^{n(x)} F(x), \quad F(\phi^{m(x)} x) = \psi F(x).$$

The main result here is as follows: ϕ and ψ are strongly orbit equivalent if and only if $K^0(X, \phi)$ and $K^0(Y, \psi)$ are isomorphic as ordered groups. Remark that in the locally compact setting we define K^0 as a quotient of the continuous functions that vanish at the infinity.

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1. “Locally compact” systems

Let X be a locally compact totally disconnected metrizable space. It follows that X is zero-dimensional, i.e. there is a (countable) basis of closed and open sets (see, for instance [HR]). Clearly, we may assume even that these sets are compact. If, in addition, X has no isolated points then we shall call it a *locally compact Cantor* (“LCC”) set. A well known theorem of Cantor says that all compact LCC sets are homeomorphic. We record a “locally compact counterpart” of it as follows.

Proposition 1.1. *Every two non-compact LCC sets X and Y are homeomorphic.*

Proof. Take a countable family of compact open subsets $O_n \subset X$ such that $X = \bigcup_{n=1}^{\infty} O_n$ and put $X_1 := O_1$, $X_2 := O_2 \setminus O_1$, $X_3 := O_3 \setminus (O_1 \cup O_2), \dots$. The subsets X_n are compact, open, pairwise disjoint and $\bigcup_{n=1}^{\infty} X_n = X$. Since X is non-compact, we may assume without loss in generality that all X_n are nonempty. Every X_n has no isolated points, since X so is. In a similar way, we represent Y as a disjoint union of a countable family of compact Cantor sets Y_n. By the Cantor theorem, there is a homeomorphism $\phi_n : X_n \to Y_n$. “Glue” these homeomorphisms altogether to obtain a homeomorphism $\phi : X \to Y$ as desired. $\square$

Let ϕ denote a homeomorphism of the LCC set X. We say that ϕ is *minimal* if every ϕ-orbit is dense in X. Clearly, minimality is equivalent to ϕ having no non-trivial closed invariant subset Y, i.e. $\phi(Y) = Y$ implies that $Y = X$ or $Y = \emptyset$.

Given $x \in X$, let $w_+(x)$ denote the set of accumulation points of the forward semi-orbit $\{\phi^m x \mid m \geq 0\}$ of x. In a similar way, $w_-(x)$ denotes the set of accumulation points of the backward semi-orbit of x. Clearly, $w_+(x)$ and $w_-(x)$ are closed and invariant. Since $X = w_+(x) \cup w_-(x)$ and X is uncountable, at least one of the sets $w_+(x)$ or $w_-(x)$ is nonempty. We set

$$X_+ := \{x \in X \mid w_+(x) = X\}, \ X_- := \{x \in X \mid w_-(x) = X\}.$$

By the above observation $X_+ \cup X_- = X$. Clearly, X_+ and X_- are both invariant. It is well known that $X_+ = X_- = X$ if X is compact. However, in the locally compact case the situation is quite different.

Theorem 1.2. *If X is non-compact, then X_+ and X_- are invariant dense G_δ in X and their interiors are empty.*

Proof. Let $\{O_n\}_{n=1}^\infty$ be a base of the topology on X. One can verify that

$$X_+ = \bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{m>k}\phi^{-m}O_n.$$

Hence it is a G_δ in X. Suppose that the interior of X_+ is not empty. Take a compact open subset O in it. For each $x \in O$, denote by $n(x)$ the least positive integer n such that $\phi^n x \in O$. Since $x \in X_+$, a map $O \ni x \mapsto n(x) \in \mathbb{Z}_+$ is well defined. It is easy to verify that this map is continuous and hence bounded. We put $O' := \bigcup_{n=0}^N \phi^n O$, where $N := \max_{x\in O} n(x)$. Then O' is a compact subset of X and $\phi O' \subset O'$. Hence the subset $O'' := \bigcap_{n=1}^\infty \phi^n O'$ is nonempty, compact and ϕ-invariant. This contradicts to the minimality of ϕ (recall that $O'' \neq X$ since X is non-compact). Arguing in a similar way we deduce that X_- is also a G_δ in X and its interior is empty. To complete the proof it is enough to show that X_+ is dense in X. The closure $\overline{X_+}$ of X_+ is closed and invariant. If $\overline{X_+} = \emptyset$ then $X_- = X$, which contradicts to the fact that the interior of X_- is empty. Thus $\overline{X_+} = X$, i.e. X_+ is dense in X, as desired. □

Corollary 1.3. *If X is non-compact then $X_+ \cap X_-$ is an invariant dense G_δ with empty interior.*

Definition 1.4. A subset $Y \subset X$ is called wandering if the sets $\phi^n Y$, $n \in \mathbb{Z}$, are pairwise disjoint.

Given $Y \subset X$, we put $Y_w := \{y \in Y \mid \phi y, \phi^2 y, \dots \notin Y\}$. It is easy to verify that Y_w is wandering. Moreover, it is the largest wandering subset of Y. If X is compact then $Y_w = \emptyset$ for every clopen subset Y of X [Pu]. This property can be considered as a "good recurrence". Unlike this we have a "bad recurrence" in the non-compact case.

Proposition 1.5. *If X is non-compact and Y compact and open then Y_w is a nonempty closed subset with empty interior.*

Proof. Clearly, $Y_w = Y \cap (X \setminus \phi^{-1}Y) \cap (X \setminus \phi^{-2}Y) \cap \dots$. Hence it is closed. If $Y_w = \emptyset$ then "the first return map" $n : Y \ni y \mapsto n(y) \in \mathbb{Z}_+$ is well defined (see the proof of Theorem 1.2). Arguing in a similar way as in that proof one obtains a contradiction. Hence $Y_w \neq \emptyset$. If Y_w contains an open subset O then the subset $\bigcup_{n\in\mathbb{Z}} \phi^n O$ is invariant and open. Without loss in generality we may assume that $O \neq Y$. It follows that $\bigcup_{n\in\mathbb{Z}} \phi^n O \neq X$ and hence $O = \emptyset$. □

2. A family of examples

We provide here a class of examples of minimal dynamical systems on non-compact LCC sets. They are infinite counterparts of uniquely ergodic compact systems because they have a unique (up to scaling) σ-finite invariant measure. We record a necessary and sufficient condition under which this measure is finite.

Let C_n, F_n, $n = 1, 2, \dots$, be two sequences of finite $\mathbb{Z}$-subsets such that the following properties are satisfied for each n:

(i) $F_n + C_{n+1} + \{-1, 0, 1\} \subset F_{n+1}$,
(ii) $(F_n - F_n) \cap (C_{n+1} - C_{n+1}) = \{0\}$,
(iii) $|C_n| > 1$.

For each $n = 1, 2, \dots$, we put $X_n := F_n \times \prod_{k>n} C_k$ and define a map $i_n : X_n \to X_{n+1}$ be setting

$$i_n(f_n, c_{n+1}, c_{n+2}, \dots) = (f_n + c_{n+1}, c_{n+2}, \dots).$$

In view of (iii), X_n equipped with the product topology is a compact Cantor set. We deduce from (i) that i_n is well defined and continuous. It follows from (ii) that i_n is one-to-one. Moreover, $i_n(X_n)$ is a clopen subset of X_{n+1}. Denote by X the topological inductive limit of the sequence (X_n, i_n). Then X is a non-compact LCC set. We let

$$D_n := \{x = (f_n, c_{n+1}, \dots) \in X_n \mid f_n + 1 \in F_n\},$$
$$R_n := \{x = (f_n, c_{n+1}, \dots) \in X_n \mid f_n - 1 \in F_n\}.$$

It is clear that D_n and R_n are clopen subsets of X_n and $i_n(D_n) \subset D_{n+1}$, $i_n(R_n) \subset R_{n+1}$. From this and (i) we deduce that

$$\operatorname*{inj\,lim}_n (D_n, i_n) = \operatorname*{inj\,lim}_n (R_n, i_n) = X. \tag{1}$$

Now we define a map $\phi_n : D_n \to R_n$ by setting $\phi_n(f_n, c_{n+1}, \dots) = (f_n+1, c_{n+1}, \dots)$. It is a homeomorphism and the diagram

$$\begin{array}{ccc} D_n & \xrightarrow{i_n} & D_{n+1} \\ {\scriptstyle\phi_n}\downarrow & & \downarrow{\scriptstyle\phi_{n+1}} \\ R_n & \xrightarrow[i_n]{} & R_{n+1} \end{array}$$

commutes. We deduce from this and (1) that a homeomorphism $\phi : X \to X$ is well defined as the inductive limit of the sequence of "partial" transformations ϕ_n.

Theorem 2.1.

(i) *ϕ is minimal,*

(ii) *there exists a unique up to scaling ergodic σ-finite ϕ-invariant measure μ on X which is finite on the compact subsets,*

(iii) *μ is finite if and only if* $\lim_{n\to\infty} \frac{|F_n|}{|C_1|\cdots|C_n|} < \infty$.

Proof. Let us regard X as the union of X_n, $n \in \mathbb{N}$. It is clear that the ϕ-orbital equivalence relation on X_n is just the tail equivalence relation. Moreover, if $x = (f_n, c_{n+1}, \dots)$ and $x' = (f'_n, c'_{n+1}, \dots)$ are two points of X_n with $x = \phi^k x'$ then $k = f_n - f'_n + c_{n+1} - c'_{n+1} + \cdots$. (Remark that only finitely many terms here are non-zero.) (i) and (ii) follows directly from this. Notice that the restriction of μ onto X_n is just the infinite product of equidistributed measures on F_n and C_m, $m > n$. Hence

$$\frac{\mu(X_{n+1})}{\mu(X_n)} = \frac{|F_{n+1}|}{|F_n||C_{n+1}|}, \qquad n \in \mathbb{N}.$$

(iii) follows from this since $\mu(X) = \lim_{n\to\infty} \mu(X_n)$. $\square$

3. One-point compactification

Denote by $X_* = X \cup \{*\}$ the one-point compactification of X. If X is not compact then the point $*$ of X_* is not isolated. Hence X_* is a compact Cantor set. It is easy to see that there exists a unique homeomorphism ϕ_* of X_* whose restriction to X is ϕ: one should put $\phi_*(*) = *$. Although ϕ_* is not minimal any more, it is *almost minimal*. This means that

(a) there is a fixed point and
(b) the orbit of any other point is dense.

We record without proof the following obvious assertion

Proposition 3.1. *The one-point compactification provides a one-to-one correspondence between the minimal non-compact LCC-systems and the almost minimal compact Cantor systems. Furthermore, two LCC-systems are topologically conjugate if and only if their compactifications so are.*

Notice that an almost minimal Cantor system is essentially minimal in the sense of [HPS, Definition 1.2]: the fixed point is the unique minimal set. Hence we may apply certain structural theorems from that paper to almost minimal systems. To this end we introduce concepts of an *almost simple* ordered Bratteli diagram and an *almost simple* ordered group. (Below we freely use definitions and notations for ordered Bratteli diagram, Vershik transformation, ordered group etc. from [GPS1].)

Definition 3.2. An ordered Bratteli diagram $(V, E, \geq)$ is called *almost simple* if:

(a) There is a unique infinite path in $E_{\max}$ and it is the only infinite path in $E_{\min}$ as well. (It follows that each incidence matrix has a row consisting of "0"-s and one "1". Without loss of generality me may—and shall—assume that this is the last row and "1" stands at the right corner.)
(b) There is a telescoping (V', E') of (V, E) so that for each (V', E')-incidence matrix, the entries outside the last row are greater than 1.

Example 3.3. We define two sequence of finite $\mathbb{Z}$-subsets as follows:

$$F_n := [-3^n + 1, 3^n - 1] \cap \mathbb{Z}, \quad C_{n+1} := \{0, 2 \cdot 3^n - 1\},$$

$n = 0, 1, \ldots$. It is easy to verify that the conditions (i)–(iii) from the beginning of § 2 are satisfied. Denote by $\phi : X \to X$ the corresponding minimal transformation. Like in Theorem 2.1 let us regard X as a union of X_n. Now we set

$$\begin{aligned} &J(n,1) := 4 \cdot 3^n - 1, \quad J(n,2) := 1, \\ &Z(n,1,j) := \{-3^n + j\} \times \prod_{m > n+1} C_m \subset X_{n+1}, \quad j = 1, \ldots, J(n,1), \\ &Z(n,2,1) := X_* \setminus (X_n \sqcup Z(n,1,J(n,1))). \end{aligned}$$

Then we have (see [HPS, Theorem 4.2]):

(a) $\mathcal{P}_n := \{Z(n,k,j) \mid k = 1, 2, \ j = 1, \ldots, J(n,k)\}$ is a partition of X_* into clopen subsets,
(b) $Z(n,1,J(n,1)) \cup Z(n,2,J(n,2)) = X_* \setminus X_n$ and $\bigcap_n (X_* \setminus X_n) = \{*\}$,
(c) $Z(n,1,1) \cup Z(n,2,1) = X_* \setminus \phi(X_n)$,
(d) $\phi(Z(n,1,j)) = Z(n,1,j+1), \ 1 \leq j < J(n,1)$,
(e) $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$,
(f) $\bigcup_n \mathcal{P}_n$ generates the topology of X_*.

Now we construct the ordered Bratelli diagram associated to (a)–(f) via the procedure described on p. 841–842 of [HPS] (a routine calculation is omitted):

The only infinite path in $E_{\min}$ (and in $E_{\max}$) is

Each incidence matrix (except the first one) is $\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$. Hence the Bratelli diagram associated to ϕ_* is almost simple.

Definition 3.4. An ordered group (G, G^+) is called *almost simple* if G has a unique proper order ideal J and G/J is $\mathbb{Z}$.

Denote by $C_0(X, \mathbb{Z})$ the group of continuous functions from X to $\mathbb{Z}$ which vanish outside compact subsets. Clearly, the map $\phi_* : f \to f \circ \phi^{-1}$ is an automorphism of $C_0(X, \mathbb{Z})$. Let $K^0(X, \phi)$ stand for the quotient group $C_0(X, \mathbb{Z})/\mathrm{Im}(id - \phi^*)$. Denote by $K^0(X, \phi)^+$ the image of $C_0(X, \phi)^+$ in $K^0(X, \phi)$. For X compact, we obtain just the usual definition of K^0-invariants ([HPS], [GPS1]).

It follows from [HPS] that $(K^0(X_*, \phi_*), K^0(X_*, \phi_*)^+, \mathbf{1})$ is a dimension group with an order unit. It has only one proper order ideal which is the image in $K^0(X_*, \phi_*)$ of the elements of $C(X_*, \mathbb{Z})$ which vanish at $*$. Thus this ideal is isomorphic to $K^0(X, \phi)$. Since every order ideal of $K^0(X, \phi)$ is an order ideal of $K^0(X_*, \phi_*)$, we conclude that $K_0(X, \phi)$ is simple ordered. The quotient group $K^0(X_*, \phi_*)/K^0(X, \phi)$ is $\mathbb{Z}$. Thus $(K^0(X_*, \phi_*), K^0(X_*, \phi_*)^+)$ is almost simple. It is interesting to notice that the ordered group $(C_0(X, \mathbb{Z}), C_0(X, \mathbb{Z})^+)$ has no order units while its quotient $(K^0(X, \phi), K^0(X, \phi)^+)$ does have (the minimality of ϕ plays the role here).

Slightly modifying the argument of [HPS] we obtain

Theorem 3.5. *Let ϕ be a minimal homeomorphism of a non-compact LCC set X. Then (X_*, ϕ_*) is conjugate to the Vershik transformation constructed from an almost simple ordered Bratteli diagram $(V, E, \geq)$. Moreover, $K^0(X_*, \phi_*)$ is isomorphic to $K_0(V, E)$ as ordered groups with distinguished order units. The map $(X, \phi) \mapsto (V, E, \geq)$ is induced to a bijective correspondence between the conjugacy classes of minimal systems on non-compact LCC sets and the equivalence classes of almost simple ordered Bratteli diagrams. For each almost simple ordered dimension group (G, G^+), there is an almost simple ordered Bratteli diagram $(V, E, \geq)$ such that $(G, G^+, 1) = (K_0(V, E), K_0(V, E)^+, 1)$.*

It would be interesting to describe the unique order ideal in $K_0(V, E)$ in terms of the diagram (V, E). To this end we consider the partition of V into levels $V = V_0 \cup V_1 \cup \dots$ and denote by M_n the incidence matrix between levels n and

$n+1$. Since (V,E) is almost simple, M_n is of the form

$$M_n = \begin{pmatrix} & & & * \\ & \widetilde{M_n} & & \vdots \\ & & & * \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

Recall that $K_0(V,E) = \operatorname{inj}\lim\limits_{n\to\infty}(\mathbb{Z}^{|V_n|}, M_n)$ and put $J = \operatorname{inj}\lim\limits_{n\to\infty}(\mathbb{Z}^{|V_n|-1}, \widetilde{M_n})$. Then the following commutative diagram

$$\begin{array}{ccccccccc} \mathbb{Z}^{|V_0|} & \xrightarrow{M_0} & \mathbb{Z}^{|V_1|} & \xrightarrow{M_1} & \mathbb{Z}^{|V_2|} & \xrightarrow{M_2} & \cdots & \longrightarrow & K_0(V,E) \\ & & \uparrow & & \uparrow & & & & \\ & & \mathbb{Z}^{|V_1|-1} & \xrightarrow{\widetilde{M_1}} & \mathbb{Z}^{|V_2|-1} & \xrightarrow{\widetilde{M_2}} & \cdots & \longrightarrow & J \end{array}$$

determines an embedding $J \to K_0(V,E)$. The vertical arrows here are of the form:

$$\mathbb{Z}^{|V_n|-1} \ni (a_1, \dots, a_{|V_n|-1}) \mapsto (a_1, \dots, a_{|V_n|-1}, 0) \in \mathbb{Z}^{|V_n|}.$$

Thus if $K_0(V,E)$ corresponds to $K_0(X_*, \phi_*)$, then J corresponds to $K_0(X,\phi)$.

Proposition 3.6. *Two almost simple ordered dimension groups are isomorphic as ordered groups with distinguished order units if and only if their proper ideals are isomorphic as ordered groups.*

Proof. is routine if one represent the dimension group as an inductive limit of Cartesian powers of $\mathbb{Z}$ with positive homomorphisms. □

Remark that this claim is not true for arbitrary almost simple ordered groups.

4. Strong orbit equivalence

Recall that two minimal dynamical systems (X,ϕ) and (Y,ψ) are *topologically orbit equivalent* if there is a homeomorphism $F: X \to Y$ so that $F(\mathrm{Orbit}_\phi(x)) = \mathrm{Orbit}_\psi(Fx)$ for all $x \in X$. Two maps $n: X \to \mathbb{Z}$ and $m: Y \to \mathbb{Z}$ are well defined by

$$F(\phi x) = \psi^{n(F(x))} F(x), \quad F(\phi^{m(x)} x) = \psi F(x). \tag{2}$$

They are called the *orbit cocycles* associated to F.

Definition 4.1. Let ϕ and ψ be topologically orbit equivalent minimal homeomorphisms on non-compact *LCC*-sets X and Y respectively. They are *strongly orbit equivalent* if there is a topological orbit equivalence $F: X \to Y$ such that the associated orbit cocycles n and m are both continuous.

Remark that this definition is different from that used in the compact case (cf. [GPS1], [GW].) Actually, the continuity of the orbit cocycles (even one of them) implies that the compact systems are flip conjugate.

Extend F to a homeomorphism $F_*: X_* \to Y_*$ (this extension is unique). Clearly, F_* is an orbit equivalence of (X_*, ϕ_*) and (Y_*, ψ_*). However the associated orbit cocycles are no longer well defined at $* \in X_*$. Defining them at this point in an arbitrary way, we obtain some extensions n^* and m^* of n and m respectively. These orbit cocycles can have only one point of discontinuity, namely $*$. Thus the strong orbit equivalence of non-compact minimal systems corresponds to the strong orbit equivalence (in the usual sense) of their one-point compactifications.

Theorem 4.2. *Let ϕ and ψ be two minimal homeomorphisms of non-compact LCC sets X and Y respectively. The following are equivalent:*

(i) *ϕ and ψ are strongly orbit equivalent,*
(ii) *$K^0(X,\phi)$ and $K^0(Y,\psi)$ are isomorphic as ordered groups,*
(iii) *$K^0(X_*,\phi_*)$ and $K^0(Y_*,\psi_*)$ are isomorphic as ordered groups with distinguished order units.*

Proof. (ii)$\Longleftrightarrow$(iii) by Proposition 3.6.

(i)$\Longrightarrow$(ii) Without loss of generality we may assume that $X = Y$ (see Proposition 1.1). Slightly abusing notation (redenoting $F\circ\phi\circ F^{-1}$ by ϕ) we rewrite (2) as

$$\phi(x) = \psi^{n(x)}(x), \quad \phi^{m(x)} = \psi(x),$$

where $n, m : X \to \mathbb{Z}$ are continuous maps. Given an open compact subset $E \subset X$, we have a finite partition of E into clopen subsets $E = \bigcup_{k=1}^{N} E_k$, where

$$E_k = \{x \in E \mid \phi(x) = \psi^{n_k}(x)\}$$

(here we use the fact that n is continuous on E). It follows that

$$1_E - 1_E \circ \phi^{-1} = \sum_{k=1}^{N}(1_{E_k} - 1_{E_k} \circ \psi^{-n_k}) \in \mathrm{Im}(Id - \psi^*)$$

From this we deduce that $\mathrm{Im}(id - \phi^*) \subset \mathrm{Im}(id - \psi^*)$. The opposite inclusion is obtained in a similar way. Hence $K^0(X,\phi) = K^0(Y,\psi)$ as ordered groups.

(iii)$\Longrightarrow$(i) Let $\alpha : K^0(X_*,\phi_*) \to K^0(Y_*,\psi_*)$ stand for an order isomorphism preserving the distinguished order units. By Theorem 3.5, ϕ_* and ψ_* are the Vershik transformations associated to some Bratteli diagrams $(V^1, E^1, \geq^1)$ and $(V^2, E^2, \geq^2)$ respectively. Then we have

$$\mathbb{Z}^{|V_0^1|} \xrightarrow{M_0^1} \mathbb{Z}^{|V_1^1|} \xrightarrow{M_1^1} \cdots \longrightarrow K_0(V^1, E^1) = K^0(X_*,\phi_*)$$

and

$$\mathbb{Z}^{|V_0^2|} \xrightarrow{M_0^2} \mathbb{Z}^{|V_1^2|} \xrightarrow{M_1^2} \cdots \longrightarrow K_0(V^2, E^2) = K^0(Y_*,\psi_*).$$

Since the transformations are almost minimal, we may assume that the incidence matrices M_k^1, M_k^2, $k = 1, 2, \ldots$ of these diagrams satisfy (a) and (b) from Definition 3.2. Contracting, if necessary, these diagrams we obtain two families of one-to-one homomorphisms $A_k : \mathbb{Z}^{|V_k^1|} \to \mathbb{Z}^{|V_k^2|}$, $B_k : \mathbb{Z}^{|V_k^2|} \to \mathbb{Z}^{|V_{k+1}^1|}$, $k = 0, 1, \ldots,$ which "realize" α and α^{-1} respectively. This means that the following two diagrams commute:

$$\begin{array}{ccccccc}
\mathbb{Z}^{|V_0^1|} & \xrightarrow{M_0^1} & \mathbb{Z}^{|V_1^1|} & \xrightarrow{M_1^1} & \cdots & \longrightarrow & K_0(V^1,E^1) \\
{\scriptstyle A_0}\downarrow & & {\scriptstyle A_1}\downarrow & & & & \downarrow{\scriptstyle \alpha} \\
\mathbb{Z}^{|V_0^2|} & \xrightarrow{M_0^2} & \mathbb{Z}^{|V_1^2|} & \xrightarrow{M_1^2} & \cdots & \longrightarrow & K_0(V^2,E^2)
\end{array} \qquad (3)$$

$$
(4)\qquad
\begin{array}{ccccccccc}
\mathbb{Z}^{|V_0^1|} & \xrightarrow{M_0^1} & \mathbb{Z}^{|V_1^1|} & \xrightarrow{M_1^1} & \mathbb{Z}^{|V_2^1|} & \xrightarrow{M_2^1} & \cdots & \longrightarrow & K_0(V^1,E^1) \\
 & & {\scriptstyle B_0}\Big\uparrow & & {\scriptstyle B_1}\Big\uparrow & & & & \Big\uparrow{\scriptstyle \alpha^{-1}} \\
 & & \mathbb{Z}^{|V_0^2|} & \xrightarrow{M_0^2} & \mathbb{Z}^{|V_1^2|} & \xrightarrow{M_1^2} & \cdots & \longrightarrow & K_0(V^2,E^2)
\end{array}
.
$$

Notice that A_0 is well defined because of α preserves the distinguished order units. Since α is order preserving, it follows that A_k, B_k, viewed as matrices, have only non-negative entries. Moreover, since α preserves the distinguished order units, A_k and B_k are of the form

$$A_k = \begin{pmatrix} & * & & \\ 0 & \dots & 0 & 1 \end{pmatrix}, B_k = \begin{pmatrix} & * & & \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

Denote by (V_3, E_3) the (unordered) Bratteli diagram associated to the sequence

$$\mathbb{Z}^{|V_0^1|} \xrightarrow{A_0} \mathbb{Z}^{|V_0^2|} \xrightarrow{B_0} \mathbb{Z}^{|V_1^1|} \xrightarrow{A_1} \mathbb{Z}^{|V_1^2|} \xrightarrow{B_1} \dots$$

It follows from (3) and (4) that $B_k A_k = M_k^1$, $A_{k+1}B_k = M_{k+1}^2$ for all $k = 0, 1, \dots$, i.e. (V^1, E^1) and (V^2, E^2) are contractions of (V^3, E^3) on odd and even levels respectively. The special form of the incidence matrices from each of the three Bratteli diagrams implies that there is a distinguished vertex at each level. This is the vertex which is joined with only one vertex from the proceeding level. The infinite path passing through these vertices is a distinguished path in the Bratteli compactum associated to the diagram. As for X_* and Y_* are concerned this distinguished path is just $*$. Let Z stand for the Bratteli compactum associated to (V^3, E^3). Clearly the map which naturally identifies the corresponding infinite paths in the two Bratteli compactums associated to a diagram and to a contraction of it is a homeomorphism. Thus we obtain two homeomorphisms $F_1 : Z \to X_*$ and $F_2 : Z \to Y_*$. Notice that F_1 and F_2 preserve the tail equivalence relations and the distinguished paths. Hence $F_2 F_1^{-1}$ is a strong orbit equivalence of ϕ_* and ψ_*. □

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