Double Scaling Limit for Matrix Models with Non Analytic Potentials

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We study the double scaling limit for unitary invariant ensembles of random matrices with non analytic potentials and find the asymptotic expansion for the entries of the corresponding Jacobi matrix. Our approach is based on the perturbation expansion for the string equations. The first order perturbation terms of the Jacobi matrix coefficients are expressed through the Hastings-McLeod solution of the Painleve II equation. The limiting reproducing kernel is expressed in terms of solutions of the Dirac system of differential equations with a potential defined by the first order terms of the expansion.

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1. INTRODUCTION

We consider the unitary invariant matrix model, defined by the probability distribution

$$P_n(M)dM = Z_n^{-1}\exp\{-n\mathrm{Tr}V(M)\}dM, \qquad (1.1)$$

on the set of Hermitian $n \times n$ matrices. Here Z_n is a normalizing constant, $V : \mathbb{R} \to \mathbb{R}_+$ is a Hölder function satisfying the condition

$$V(\lambda) \geq (2+\epsilon)\log(1+|\lambda|). \qquad (1.2)$$

According to results of [5, 13], the Normalized Counting Measure (NCM) of eigenvalues $\{\lambda_j^{(n)}\}_{j=1}^n$ tends weakly in probability, as $n \to \infty$, to the non random limiting measure $\mathcal{N}$ known as the Integrated Density of States (IDS). The IDS is absolutely continuous if V' satisfies the Lipshitz condition. The IDS can be found as a unique solution of a certain variational problem [5, 13] which imply, in particularly, that if $V'(\lambda)$ satisfies the Lipshitz conditions on the support σ of limiting IDS, then the density of IDS $\rho(\lambda)$ is a solution of the following integral equation

$$V'(\lambda) = 2\int_\sigma \frac{\rho(\mu)d\mu}{\lambda-\mu}, \quad \sigma = \mathrm{supp}\mathcal{N}. \qquad (1.3)$$

While IDS depends strongly on the form of V, the local eigenvalue statistics is expected to be universal. Denote by $p_n(\lambda_1,...,\lambda_n)$ the joint eigenvalue probability density. It is known (see [15]) that

$$p_n(\lambda_1,...\lambda_n) = Q_n^{-1}\prod_{1\leq j<k\leq n}(\lambda_j-\lambda_k)^2\prod_{j=1}^n e^{-nV(\lambda_j)}, \qquad (1.4)$$

where Q_n is the respective normalization factor. Let

$$p_l^{(n)}(\lambda_1,...,\lambda_l) = \int p_n(\lambda_1,...,\lambda_l,\lambda_{l+1},...\lambda_n)d\lambda_{l+1}...d\lambda_n \qquad (1.5)$$

be the lth marginal distribution density of (1.4). Universality of local eigenvalue statistics means that if we consider some $\lambda_0 \in \sigma$, then all marginal distribution densities after a proper scaling (which depends on the behavior of the limiting DOS $\rho(\lambda)$ near the point $\lambda = \lambda_0$) tend to some universal limits.

The most known quantity probing universality is the gap probability

$$E_n(\Delta_n) = \mathbf{P}_n\{\lambda_l^{(n)} \notin \Delta_n,\ l = 1,...,n\}, \qquad (1.6)$$

where $\mathbf{P}_n\{...\}$ is defined by (1.1), and Δ_n is an interval of the spectral axis, whose order of magnitude is fixed by the condition $n\mathcal{N}(\Delta_n) \sim 1$. For unitary invariant matrix models $E_n(\Delta_n)$ can be obtained as the Fredholm determinant of a certain integral operator. This structure of the gap probability is a consequence of the structure of marginal densities, and the latter can be explained by the link of matrix models with orthogonal polynomials $p_l^{(n)}(\lambda)$, $(l = 1,...)$ on $\mathbb{R}$ associated with the weight $e^{-nV(\lambda)}$. The link is provided by the formula [15]

$$p_l^{(n)}(\lambda_1,...,\lambda_l) = \frac{(n-l)!}{n!}\det||K_n(\lambda_j,\lambda_k)||_{j,k=1}^l, \qquad (1.7)$$

where

$$K_n(\lambda,\mu) = \sum_{l=1}^n \psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu) \qquad (1.8)$$

is known as a reproducing kernel of an orthonormalized system

$$\psi_l^{(n)}(\lambda) = e^{-nV(\lambda)/2}P_{l-1}^{(n)}(\lambda),\ l \in \mathbb{N}, \qquad (1.9)$$

in which $P_l^{(n)}(\lambda)$ is a polynomial of l-th degree with a positive coefficient in front of λ^l. This polynomial is uniquely defined by the orthogonality conditions

$$\int P_l^{(n)}(\lambda)P_m^{(n)}(\lambda)e^{-nV(\lambda)}d\lambda = \delta_{l,m}. \qquad (1.10)$$

Formula (1.7) allows us to reduce the question on the behavior of the scaled ℓth marginal density to the

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question of the existence of the limit of $\mathcal{K}_n(s,t) = n^{-\gamma}K_n(s/n^\gamma, t/n^\gamma)$ for a proper chosen γ.

In the bulk case ($\rho(\lambda_0) \neq 0$) we choose $\gamma = 1$. Then the limiting hole probability is the Fredholm determinant of the integral operator, defined by the kernel $\sin\pi(t_1 - t_2)/\pi(t_1 - t_2)$ on the interval $(0,s)$. This fact for the GUE was established by M. Gaudin in the early 60s [15]. The same fact was proved recently in [8, 17] for certain classes of matrix models.

The edge case of local eigenvalue statistics was studied much later even for the GUE [11, 20]. It was found that if we choose $\gamma = 2/3$, then for the edge points $\lambda_0 = \pm a$ ($\sigma = [-a,a]$) the hole probability (1.6) of the GUE in the limit $n \to \infty$ is the Fredholm determinant of the integral operator, defined on the interval $(0,s)$ by the Airy kernel. This fact for real analytic potentials in (1.1) was obtained in [9]. In the paper [18] a more simple proof of the edge universality for the same class of potentials was given. An important advantage of the method of [18] is that it can be generalized to a class of non analytic potentials.

Universality near the critical point ($\gamma = 1/3$) was studied first for $V(\lambda) = \frac{1}{4}\lambda^4 - \lambda^2$ by using the Riemann-Hilbert approach in [4]. Here the asymptotic of the Jacobi matrix coefficients and the limiting reproducing kernel were found. The results of [4] after proper normalization (see Remark 3 after Theorem 1) coincide with results found below for more general V. The method of [4] was generalized on a class of real analytic symmetric potentials in [7] under additional assumptions that the limiting spectrum σ consists of one interval and the density $\rho(\lambda)$ behaves like a square root near the edge points and has only one critical point inside σ (cf. condition $C3$ below). But the asymptotic behavior of the Jacobi matrix coefficients was not studied.

In the present paper we find the asymptotic behavior of the Jacobi matrix coefficients and on the basis of this result prove universality near the critical point. We need not to assume that $V(\lambda)$ is a real analytic function. Our approach is based on the mathematical version of physical ideas proposed in [6]. The paper [6] was devoted to the case of V being a polynomial of minimal degree, which provides the condition $\rho(\lambda) \sim (\lambda - 2)^{m-1/2}$ near the edge point $\lambda = 2$. It was shown on the physical level of rigor that for these potentials $J^{(n)}_{n+k} = 1 + n^{-\nu} f_m(k/n^{1-\nu})$, where $\nu = 2m/(2m+1)$ and f_m is a solution of some non linear differential equation of order $2m - 2$. Moreover, then the resolvent $(J^{(n)} - z)^{-1}$ for $z = 2 + \zeta n^{-2/2m+1}$ can be written in terms of the resolvent of the second order differential operator $-\frac{d^2}{dx^2} + f_m(x)$ at the point ζ. In fact Theorems 1 and 2 of the present paper establish similar facts for the case of symmetrical potential with one critical point under proper smoothness conditions on the potential V.

Let us state our main conditions.

C1. *The support σ of the IDS of the ensemble consists of a single interval: $\sigma = [-2,2]$.*

C2. *$V(\lambda)$ is an even real locally Lipshitz function in $\mathbb{R}$.*

C3. *The DOS $\rho(\lambda)$ has the form*

$$\rho(\lambda) = \frac{1}{2\pi}\lambda^2 P_0(\lambda)\sqrt{4-\lambda^2}, \quad \lambda \in [-2,2], \tag{1.11}$$

where $P_0(\lambda) > \delta > 0$ for $\lambda \in [-2,2]$ and there exists $\varepsilon > 0$ such that $P^{(5)}(\lambda)_0 \in L_2[\sigma_\epsilon]$, where $\sigma_\varepsilon = [-2-\varepsilon, 2+\varepsilon]$.

C4. *The function*

$$u(\lambda) = 2\int \log|\mu - \lambda|\rho(\mu)d\mu - V(\lambda) \tag{1.12}$$

achieves its maximum if and only if $\lambda \in [-2,2]$.

Remark 1. *It follows from (1.3) that condition $C3$ imply that $V^{(6)}(\lambda) \in L_2[\sigma_\epsilon]$. In fact Theorems 1 and 2 below can be proved if $V^{(5)}(\lambda) \in L_2[\sigma_\epsilon]$, but the proof is more complicated. Since $V^{(4)}(0)$ is used in the limiting formulas for Theorem 1, it is natural to expect that the existence of continuous $V^{(4)}(\lambda)$ in some neighborhood of $\lambda = 0$ is a necessary condition for Theorem 1. Thus condition $C3$ does not look too restrictive.*

Remark 2. *It is well known DOS ρ of the ensemble (1.1) for $\sigma = [-2,2]$ has the form (1.11)*

$$\rho(\lambda) = \frac{1}{2\pi}\chi_\sigma(\lambda)P(\lambda)\sqrt{4-\lambda^2}, \tag{1.13}$$

where $\chi_\sigma(\lambda)$ is the indicator of σ and it follows from (1.3) that $P(\lambda)$ can be represented in the form

$$P(\lambda) = \frac{1}{\pi}\int_\sigma \frac{V'(\lambda) - V'(\mu)}{(\lambda-\mu)\sqrt{4-\mu^2}}d\mu, \tag{1.14}$$

So condition $C3$ means that $\rho(\lambda)$, behaves like square root near the edge points and has the second order zero at $\lambda = 0$.

Define a semi infinite Jacobi matrix $\mathcal{J}^{(n)}$, whose entries $J^{(n)}_{l-1,l} = J^{(n)}_{l,l-1} = J^{(n)}_l$ are defined by the recurrent relations

$$\lambda\psi^{(n)}_l(\lambda) = J^{(n)}_{l+1}\psi^{(n)}_{l+1}(\lambda) + J^{(n)}_l\psi^{(n)}_{l-1}(\lambda), \tag{1.15}$$

where $l = 0,1,\ldots,\quad J^{(n)}_0 = 0$ and $\psi^{(n)}_l$ is defined by (1.9). The main result of the paper is

Theorem 1. *Let conditions C1–C4 be fulfilled. Then for any $k : |k| \le n^{1/3}\log^2 n$*

$$J^{(n)}_{n+k} = 1 + \tilde{s}(-1)^k n^{-1/3} q(\frac{k}{n^{1/3}}) + \frac{k}{8nP_0(2)} + r_k, \tag{1.16}$$

where $q(x)$ is the Hastings-McLeod solution of the Painleve II equation

$$q''(x) = \frac{1}{2P_0(0)}xq(x) + 2q^3(x), \tag{1.17}$$

which is uniquely defined (see [12]) by the asymptotic conditions

$$\lim_{x\to+\infty} q(x)=0, \quad \lim_{x\to-\infty}\frac{q(x)}{(-x)^{1/2}}=\frac{1}{2P_0^{1/2}(0)}, \quad (1.18)$$

$P_0(\lambda)$ is defined by (1.11), $\tilde s=\mathrm{sign}(1-J_n^{(n)})$ and remainder terms r_k satisfy the bounds

$$|r_k|\le Cn^{-1}\left((k/n^{1/3})^2+1\right), \quad (1.19)$$

where C is some absolute constant.

Remark 3. *The result of Theorem 1 coincides with asymptotic of the Jacobi matrix coefficients obtained in [4] for the case $V(\lambda)=\lambda^4/4-\lambda^2$. Indeed, since coefficients R_k of [4] in our terms are J_k^2 (compare the recursion relations (1.15) of [4] with (1.15) of the present paper) taking into account that $P_0(x)\equiv 1$ for $V(\lambda)=\lambda^4/4-\lambda^2$, from formula (1.16) above we obtain:*

$$R_{n+k}=1+2\tilde s(-1)^k n^{-1/3}q(\frac{k}{n^{1/3}})$$
$$+n^{-2/3}q^2(\frac{k}{n^{1/3}})+\frac{k}{4n}+r_k',$$

Now, choosing as in [4] $y=2^{-1/3}k/n^{1/3}$ and denoting $u(y)=2^{1/3}q(2^{-1/3}y)$, we obtain (1.45)-(1.48) of [4] with $c_0=2^{1/3}$, $c_1=2^{2/3}$, $c_2=2^{-2/3}/2$ from (1.17)-(1.18). The only difference is that in [4] it is proved that $\tilde s=(-1)^{n+1}$, while the result of Theorem 1 does not justify the sign of $\tilde s$. But it is proved in Theorem 2 that the sign of $\tilde s$ has no influence on the behavior of the marginal densities in the double scaling limit.

To prove universality of local eigenvalue statistics we study

$$\mathcal{K}_n(t_1,t_2)=n^{-1/3}K_n(t_1n^{-1/3},t_2n^{-1/3}). \quad (1.20)$$

Theorem 2. *Under conditions C1–C4 for any $l\in\mathbb{N}$ there exists a weak limit of the marginal density (1.5)*

$$\lim_{n\to\infty}(2n^{2/3})^l p_l^{(n)}(2t_1/n^{1/3},\dots,2t_l/n^{1/3})$$
$$=\det\{\mathcal{K}(t_i,t_j)\}_{i,j=1}^l, \quad (1.21)$$

where

$$\mathcal{K}(t_1,t_2)=\frac{\Psi_1(0;t_1)\Psi_0(0;t_2)-\Psi_0(0;t_1)\Psi_1(0;t_2)}{\pi(t_1-t_2)}, \quad (1.22)$$

and $\mathbf{\Psi}(x,t)=(\Psi_0(x;t),\Psi_1(x;t))$ is a solution of the Dirac system of equations

$$\mathcal{A}\mathbf{\Psi}(x,t)=t\mathbf{\Psi}(x,t),$$
$$\mathcal{A}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\frac{d}{dx}+\begin{pmatrix}0&q(x)\\q(x)&0\end{pmatrix}, \quad (1.23)$$

with $q(x)$ defined by (1.17)-(1.18), and $\mathbf{\Psi}(x,t)$ chosen from the asymptotic conditions

$$\lim_{x\to-\infty}|\mathbf{\Psi}(x;t)|=0, \quad \lim_{x\to\infty}|\mathbf{\Psi}(x;t)|=1. \quad (1.24)$$

Corollary 1. *Under conditions C1–C4 the gap probability (1.6) for $\Delta_n=[n^{-1/3}a,n^{-1/3}b]$ converges, as $n\to\infty$, to the Fredholm determinant of the integral operator defined in $[a,b]$ by the kernel (1.21):*

$$\lim_{n\to\infty}E_n([an^{-1/3},bn^{-1/3}])=\det(I-\mathcal{K}([a,b])). \quad (1.25)$$

The paper is organized as follows. In Section 2 we prove Theorems 1 and 2. The proofs of the most of auxiliary results are given in Section 3. Some auxiliary results which have no direct links with matrix models (some properties of the Hastings-McLeod solution, bounds for smooth functions of Jacobi matrices etc.), are proven in Appendix.

2. PROOFS OF THEOREMS 1, 2

Proof of Theorem 1. The main idea of the proof is to use the perturbation expansion of the string equations:

$$J_k^{(n)}V'(\mathcal{J}^{(n)})_{k,k-1}=\frac{k}{n}, \quad (2.1)$$

which we consider as a system of nonlinear equations with respect to the coefficients $J_k^{(n)}$. Here and below we denote by $\mathcal{J}^{(n)}$ a semi-infinite Jacobi matrix, defined in (1.15). Relations (2.1) can be easily obtained from the identity

$$\int\left(e^{-nV(\lambda)}P_{k-1}^{(n)}(\lambda)P_k^{(n)}(\lambda)\right)'d\lambda=0.$$

To make the idea of the proof more understandable we first explain how does the method work in the simplest case $V(\lambda)=\lambda^4/4-\lambda^2$. In this case the sting equation (2.1) has the form

$$(J_{n+k}^{(n)})^2\left((J_{n+k-1}^{(n)})^2+(J_{n+k}^{(n)})^2+(J_{n+k+1}^{(n)})^2\right)-2(J_{n+k}^{(n)})^2=1+\frac{k}{n}. \quad (2.2)$$

Our first step is the following lemma, proven in Section 3:

Lemma 1. *Under conditions $C1-C4$ uniformly in k : $|k-n|\le n^{1/2}$*

$$|J_k^{(n)}-1|\le Cn^{-1/8}\log^{1/4}n,$$
$$|J_k^{(n)}+J_{k+1}^{(n)}-2|\le Cn^{-1/4}\log^{1/2}n.$$

Remark 4. *The convergence $J_k^{(n)}\to 1$, as $n\to\infty$ and $|k-n|=o(n)$ without uniform bounds for the remainders was proven in [2] under much more weak conditions ($V'(\lambda)$ is a Hölder function in some neighborhood of the limiting spectrum).*

The lemma allows us to write $J_{n+k}^{(n)}=1+\tilde J_k$, where $\tilde J_k$ is small for $|k|\le n^{1/2}$. Replacing $J_{n+k}^{(n)}$ by $1+\tilde J_k$ in (2.2) and keeping terms up to the order $\tilde J_k^3$ we get

$$\begin{aligned}&2\left(\tilde J_{k-1}+2\tilde J_k+\tilde J_{k+1}\right)+\tilde J_{k-1}^2+\tilde J_{k+1}^2-2\tilde J_k^2\\ &+4\tilde J_k\left(\tilde J_{k-1}+2\tilde J_k+\tilde J_{k+1}\right)+2\tilde J_k^2\left(\tilde J_{k-1}+2\tilde J_k+\tilde J_{k+1}\right)\\ &+2\tilde J_k\left(\tilde J_{k-1}^2+\tilde J_{k+1}^2\right)=\frac{k}{n}+O\left(\tilde J_{k-1}^4+\tilde J_k^4+\tilde J_{k+1}^4\right).\end{aligned}\tag{2.3}$$

To estimate the remainder terms we define

$$m_k:=\max\left\{\max_{|j|\le|k|+n^{1/3}/2}\left\{|\tilde J_j|,|\tilde J_j+\tilde J_{j+1}|^{1/2}\right\},(|k|/n)^{1/2}\right\}.\tag{2.4}$$

We will prove below that $|\tilde J_k|$, and $|\tilde J_k+\tilde J_{k+1}|^{1/2}$ are of the order $(|k|/n)^{1/2}$ ($|k|>n^{1/3}$), but we do not assume this from the very beginning. Besides, it is convenient to seek $\tilde J_k$ in the form

$$\tilde J_k=(-1)^k x_k+\frac{k}{8n}\tag{2.5}$$

and denote

$$\begin{aligned}d_k^{(1)}=x_{k+1}-x_k,\quad d_k^{(2)}&=d_k^{(1)}-d_{k-1}^{(1)},\\ d_k^{(3)}&=d_{k+1}^{(2)}-d_k^{(2)}.\end{aligned}\tag{2.6}$$

Then it follows from the definition (2.4) that $d_k^{(1)},d_{k-1}^{(1)},d_k^{(2)}=O(m_k^2)$. Using (2.5) in (2.3) and keeping only the terms up to the order $O(m_k^3)$, we get

$$\begin{aligned}&-2(-1)^k d_k^{(2)}+4(-1)^k x_k\left((-1)^k d_k^{(2)}+\frac{k}{2n}\right)\\ &\qquad+d_k^{(2)}\left(x_{k+1}+x_{k-1}\right)+2x_k^2\frac{k}{2n}\\ &-2(-1)^k\left(x_{k+1}\frac{k+1}{8n}+x_{k-1}\frac{k-1}{8n}+x_k\frac{k}{4n}\right)\\ &\qquad+2(-1)^k x_k\left(x_{k+1}^2+x_{k-1}^2\right)=O(m_k^4)\end{aligned}\tag{2.7}$$

Here we have used that

$$\begin{aligned}x_{k+1}^2+x_{k-1}^2-2x_k^2&=d_k^{(2)}\left(x_{k+1}+x_{k-1}\right)+2d_k^{(1)}d_{k-1}^{(1)}\\ &=d_k^{(2)}\left(x_{k+1}+x_{k-1}\right)+O(m_k^4)\end{aligned}$$

Equation (2.7) gives us immediately that $d_k^{(2)}=O(m_k^3)$. Hence, using that

$$\begin{aligned}x_{k+1}&=x_k+d_k^{(1)}=x_k+O(m_k^2),\\ x_{k-1}&=x_k-d_{k-1}^{(1)}=x_k+O(m_k^2)\end{aligned}$$

(2.7) can be rewritten in the form

$$d_k^{(2)}-2x_k^3-x_k\frac{k}{2n}=O(m_k^4).\tag{2.8}$$

This equation is a particular case of the equation

$$d_k^{(2)}-2x_k^3-\frac{k}{2P_0(0)n}x_k=r_k,\quad |r_k|\le\tilde C'\tilde m_k^4,\tag{2.9}$$

where

$$\tilde m_k:=m_{k+[n^{1/3}/2]}.$$

which we will obtain below for general V. Using (2.9) (or (2.8)) we can find first the order of m_k.

Lemma 2. *Let the sequence $\{x_k\}_{|k|\le n^{1/2}/2}$ satisfy equation (2.9) with m_k defined by (2.4) and $|x_k|\le Cn^{-1/8}\log^{1/4}n$. Then there exist $C^*,L^*>0$ such that for any $k:n^{1/2}/5>|k|>L^*n^{1/3}$*

$$\tilde m_k\le C^*\left(|k|/n\right)^{1/2}.\tag{2.10}$$

Besides, there exist $C_{1,2,3}$ such that for $n^{1/3}<k<k^*=[n^{1/3}\log^2 n]$

$$|x_k|\le C_1n^{-1/3}e^{-C_2(k/n^{1/3})^{3/2}}+C_3\tilde m_{2k^*}^4.\tag{2.11}$$

The proof is given in Section 3. It is based on the following proposition proven in Appendix.

Proposition 1. *Let $\{\tilde x_k\}_{|k|<M}$, satisfy the recursive relations:*

$$\begin{aligned}\tilde x_{k+1}-2\tilde x_k+\tilde x_{k-1}&=2\tilde x_k^3+\tilde r_k,\\ |\tilde r_k|\le\varepsilon^3,&\quad|\tilde x_k|\le\varepsilon_1.\end{aligned}\tag{2.12}$$

Then for any $|k|<M-2M_1$ with $M_1>2\varepsilon^{-1}/3$

$$\begin{aligned}|\tilde x_k|&\le\max\{\varepsilon,(2M_1^{-2}\varepsilon_1)^{1/3}\},\\ |\tilde x_{k+1}-\tilde x_k|&\le 4\max\{\varepsilon^2,(2M_1^{-2}\varepsilon_1)^{2/3}\}.\end{aligned}\tag{2.13}$$

Moreover, if for $|k| \le M$

$$\tilde{x}_{k+1} - 2\tilde{x}_k + \tilde{x}_{k-1} = f_k \tilde{x}_k + \tilde{r}_k, \tag{2.14}$$

with $f_k \ge d^2 > 0$, then for $|k| < M$

$$|\tilde{x}_k| \le C d^{-1} \Big(\sum_{|j|\le M} e^{-d|k-j|} \tilde{r}_j + |x_M| e^{-d|M-k|} + |x_{-M}| e^{-d|M+k|} \Big). \tag{2.15}$$

Starting from this point the proofs of Theorem 1) for the cases of $V(\lambda) = \lambda^4/4 - \lambda^2$ and general V coincide. That is why below we will consider equation (2.9) instead (2.8).

Define a continuous function $q_n(x)$, which for $x \in \mathbb{Z}/n^{1/3}$ coincides with x_k

$$q_n(\frac{k}{n^{1/3}}) = n^{1/3} x_k.$$

and is a linear function for $x \notin \mathbb{Z}/n^{1/3}$. For $x \in \mathbb{Z}/n^{1/3}$ Lemma 2 allows us to write (2.9) as

$$\frac{q_n(x+h) - 2q_n(x) + q_n(x-h)}{h^2} = 2q_n^3(x) + \frac{x}{2P_0(0)} q_n(x) + n^{-2/3} O(|x|^2 + 1), \tag{2.16}$$

where $h = n^{-1/3}$ and the bound for the remainder is uniform in $|x| \le \log^2 n$. We are interested in the behavior of the solution of this discrete equation which satisfies conditions (cf. (2.10) and (2.11)):

$$|q_n(x)| \le C|x|^{1/2},\ |q_n(x)| \le e^{-Cx^{3/2}/2},\ x \to +\infty \tag{2.17}$$

It follows from Lemma 2 that the functions $\{q_n(x)\}_{n=1}^{\infty}$ are uniformly bounded and equicontinuous for any bounded interval. Hence, this family is weakly compact in any compact set in $\mathbb{R}$ and any convergent subsequence converges uniformly to some solution of the Painleve equation (1.17), satisfying (2.17). Now we need to prove the asymptotic relations (1.18) for $x \to -\infty$. To this aim we use Lemma 3 below, which describes the behavior of the Stieltjes transform of the following densities

$$g_{k,n}(z) := \int \frac{\rho_{k,n}(\lambda) d\lambda}{\lambda - z},$$
$$\rho_{k,n}(\lambda) := \frac{1}{n} K_{n,k}(\lambda,\lambda),$$
$$K_{n,k}(\lambda,\mu) := \sum_{j=0}^{k} \psi_j^{(n)}(\lambda) \psi_j^{(n)}(\mu). \tag{2.18}$$

Lemma 3. *Under conditions $C1 - C4$ for any $k : |k| \le n^{1/3}\log^2 n$ $g_{n+k,n}(z)$ can be represented in the form*

$$g_{n+k,n}(z) = -\frac{1}{2}\Big(V''(0) z + \frac{V^{(4)}(0)}{6} z^3 \Big) + \frac{1}{2} X(z) \Big(P_0^2(0) z^4 + \frac{k}{n} P_0(0) z^2 + c_k - \delta_{n+k,n}(z) - \tilde{\delta}_{n+k,n}(z) \Big)^{1/2}, \tag{2.19}$$

where $X(z) = \sqrt{z^2-4}$ (here and below we choose the branch which behaves like z as $z \to +\infty$) and

$$c_k = \pm n^{-5/3} \sum_{j=0}^{|k|} \Big(2P_0(0) q_n^2(\frac{\pm j}{n^{1/3}}) \pm \frac{j}{2n^{1/3}} \Big). \tag{2.20}$$

($\pm$ corresponds to the sign of k). Moreover, the remainder terms $\delta_{n+k,n}(z)$ and $\tilde{\delta}_{n+k,n}(z)$ in (2.19) for $z : |z| < 1$ admit the bounds

$$|\delta_{n+k,n}(z)| = \Big| n^{-2} \int \frac{K_{n+k,n}^2(\lambda_1,\lambda_2)(\lambda_1-\lambda_2)^2}{(\lambda_1-z)^2(\lambda_2-z)^2} d\lambda_1 d\lambda_2 \Big| \le C \frac{(|k|/n)^{1/2} + n^{-1/3}}{n^2 |\Im z|^3}, \tag{2.21}$$

$$|\tilde{\delta}_{n+k,n}(z)| \le C\Big[n^{-4/3} + z^2 \Big(n^{-2/3} + (|k|/n)^{3/2} \Big) + |z|^5 + \frac{|z|^5 |\log^{1/2} n}{|\Im z|^2 n^{1/2}} \Big]. \tag{2.22}$$

The proof of the lemma is given in Section 3. Remark that, since $g_{n+k}(z)$ is the Stieltjes transform of some positive measure,

$$\Im z\, \Im g_{n+k}(z) > 0$$

Using the representation (2.19) we will show below that for any $k < -Ln^{1/3}$ with $L > 0$ big enough the above condition implies that c_k from (2.20) satisfies the bound

$$|c_k| \le C n^{-4/3} L^{1/3}$$

with some absolute C. And then from representation (2.20) we will derive (1.18). Having both asymptotic from (1.18), we can conclude that $q_n(x)$ converge uniformly on any compact in $\mathbb{R}$ to the Hastings-McLeod solution of (1.17), so that

$$\Delta_n(x) = q_n(x) - q(x) \to 0, \ \textit{as} \ \ n \to \infty.$$

But from (2.16) we derive that for any $x = k/n^{1/3}$ and $h = n^{-1/3}$ we have

$$\begin{aligned} h^{-2}\left(\Delta_n(x+h) + \Delta_n(x-h) - 2\Delta_n(x)\right) \\ = \left[2q_n^2(x) + 2q^2(x) + 2q_n(x)q(x) + \tfrac{x}{2P_0(0)}\right]\Delta_n(x) + r_n(x), \\ |r_n(x)| \le Cn^{-2/3}(|x|^2+1). \end{aligned} \qquad (2.23)$$

and uniformly in n

$$|\Delta_n(x)| \to 0, \ \textit{as} \ x \to \pm\infty.$$

Proposition 2. *For the Hastings - McLeod solution of (1.17) there exists $\delta > 0$ such that*

$$6q^2(x) + \frac{x}{2P_0(0)} \ge \delta^2. \qquad (2.24)$$

This proposition allows us to apply the assertion (2.15) of Proposition 1 to $\tilde{x}_k = \Delta(k/n^{1/3})$ with $d = n^{-1/3}\delta$ and $\tilde{r}_k = r_n(k/n^{1/3})$ with $r_n(x)$ from (2.23) The bound (1.19) follows.

As it was mentioned above, the main difference in the proof for the general case from the case $V(\lambda) = \lambda^4/4 - \lambda^2$ is in the derivation of the equation (2.9). For non polynomial V we cannot write $V'(\mathcal{J}^{(n)})$ directly like in (2.2) and therefore we need to use the Fourier expansion of V'. To construct this expansion it is convenient to consider $\mathcal{J}^{(0)}$ – an infinite Jacobi matrix with constant coefficients

$$\mathcal{J}^{(0)}_{k,k-1} = \mathcal{J}^{(0)}_{k-1,k} = 1 \qquad (2.25)$$

and to define for any positive $N < n$ an infinite Jacobi matrix $\tilde{\mathcal{J}}(N)$ with the entries

$$\tilde{J}_k = \begin{cases} J^{(n)}_{n+k} - 1, & |k| < N, \\ 0, & \text{otherwise.} \end{cases} \qquad (2.26)$$

Proposition 3. *For any function $v(\lambda)$, whose ℓth derivative belongs to $L_2[\sigma_\varepsilon]$ ($\sigma_\varepsilon = [-2-\varepsilon, 2+\varepsilon]$), consider a periodic function $\tilde{v}(\lambda) = \tilde{v}(\lambda + 4 + 2\varepsilon)$ with the same number of derivatives, and such that $\tilde{v}(\lambda) = v(\lambda)$ for $|\lambda| \le 2 + \varepsilon/2$. Let also $n^{1/2} \ge N, M > n^{1/3}$ and $\tilde{\mathcal{J}}(N+M)$ is defined by (2.26). Then uniformly in N, M and $|k| \le N$ for any fixed integer δ*

$$\begin{aligned} v(\mathcal{J}^{(n)})_{n+k,n+k+\delta} - \tilde{v}(\mathcal{J}^{(0)} \\ +\tilde{\mathcal{J}}(N+M))_{k,k+\delta} \ = \ O(M^{-\ell+1/2}). \end{aligned} \qquad (2.27)$$

The proof of the proposition is given in Appendix.

Lemma 4. *Let $v(\lambda)$ satisfy conditions of Proposition 3 with $\ell = 5$, δ be any fixed integer and $|k| \le 3n^{1/2}/4$. Then*

$$\begin{aligned} v(\mathcal{J}^{(n)})_{n+k,n+k-\delta} &= v(\mathcal{J}^{(0)})_{k,k-\delta} - c_1^{(\delta)}\tilde{J}_k + \sum\nolimits' \mathcal{P}^{(\delta)}_{k-l_1}\tilde{J}_{l_1} \\ &+ \sum\nolimits' \mathcal{P}^{(2,k,\delta)}_{l_1,l_2}\tilde{J}_{l_1}\tilde{J}_{l_2} + \sum\nolimits' \mathcal{P}^{(3,k,\delta)}_{l_1,l_2,l_3}\tilde{J}_{l_1}\tilde{J}_{l_2}\tilde{J}_{l_3} + r_k^{(\delta)} \\ &= v(\mathcal{J}^{(0)})_{k,k-\delta} - c_1^{(\delta)}\tilde{J}_k + \sum\nolimits^{(1)} + \sum\nolimits^{(2)} \\ &+ \sum\nolimits^{(3)} + r_k^{(\delta)}, \end{aligned} \qquad (2.28)$$

where $|r_k^{(\delta)}| \le Cm_k^4$ and $\mathcal{P}^{(2,k,\delta)}_{l_1,l_2}$ and $\mathcal{P}^{(3,k,\delta)}_{l_1,l_2,l_3}$ satisfy the bounds:

$$\begin{aligned} \left|\sum\nolimits' \mathcal{P}^{(2,k,\delta)}_{l_1,l_2}(l_1-k)(l_2-k)\tilde{x}_{l_1}\tilde{y}_{l_2}\right| &\le C\,||\tilde{x}||_0\,||\tilde{y}||_0, \\ \left|\sum\nolimits' \mathcal{P}^{(2,k,\delta)}_{l_1,l_2}(l_1-k)^2\tilde{x}_{l_1}\tilde{y}_{l_2}\right| &\le C\,||\tilde{x}||_0\,||\tilde{y}||_0, \\ \left|\sum\nolimits' \mathcal{P}^{(3,k,\delta)}_{l_1,l_2,l_3}(l_1-k)\tilde{x}_{l_1}\tilde{y}_{l_2}\tilde{z}_{l_3}\right| &\le C\,||\tilde{x}||_0\,||\tilde{y}||_0\,||\tilde{z}||_0 \end{aligned} \qquad (2.29)$$

for any bounded sequences $\{\tilde{x}_k\}$, $\{\tilde{y}_k\}$ and $\{\tilde{z}_k\}$. Here and below $||x||_0 = \max_k |x_k|$ and $\sum'$ means the summation over $|l_i| \le |k| + n^{1/3}/2$

Moreover,

$$\mathcal{P}^{(\delta)}_l = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^{(\delta)}(2\cos(x/2))e^{ilx}dx, \qquad (2.30)$$

with some smooth $F^{(\delta)}(\lambda)$ and for $\delta = 1$

$$\begin{aligned} c_1^{(1)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} v(2\cos x)\cos x\,dx, \\ F^{(1)}(\lambda) = 2P(\lambda) &= \frac{2}{\pi}\int_{-2}^{2}\frac{v(\lambda)-v(\mu)}{(\lambda-\mu)\sqrt{4-\mu^2}}d\mu. \end{aligned} \qquad (2.31)$$

For $\delta = 0$ $c_1^{(0)} = 0$

$$F^{(0)}(2\cos(x/2)) = \frac{\cos^2(x/2)}{2\pi}\int_{-\pi}^{\pi}\frac{v(2\cos x_1)dx_1}{\cos^2 x_1 - \cos^2(x/2)}. \qquad (2.32)$$

The proof of the lemma is given in Section 3.

Note, that if v coincides with V' for $\lambda \in \sigma_{\varepsilon/2}$, then

$$\begin{aligned} v(\mathcal{J}^{(0)})_{k,k-1} = c_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} v(2\cos x)\cos x\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}dx\int_{-2}^{2}\cos x\frac{\rho(\lambda)d\lambda}{2\cos x - \lambda} = 1. \end{aligned} \qquad (2.33)$$

It is easy to see also that, if in (2.31) $P(\lambda) = \lambda^2 P_0(\lambda)$, then for any $\tilde{x}_k$

$$\sum \mathcal{P}^{(1)}_{k-l}\tilde{x}_l = \sum \mathcal{P}_{0,k-l}(\tilde{x}_{l+1} + 2\tilde{x}_l + \tilde{x}_{l-1}), \qquad (2.34)$$

where

$$\mathcal{P}_{0,l} = \frac{1}{\pi}\int_{-\pi}^{\pi} P_0(2\cos(x/2))e^{ilx}dx.$$

Let us seek $\tilde{J}_k$ in the form (cf (2.5))

$$\tilde{J}_k = (-1)^k x_k + y_k, \qquad (2.35)$$

where in order to simplify notations we denote

$$y_k := k/(8P_0(2)n), \qquad (2.36)$$

with $P_0(\lambda)$ defined by (1.11).

Now, substituting (2.35) in (2.28) and keeping the terms up to the order m_k^3 (recall, that by definition (2.4) $y_k = O(m_k^2)$, $d_k^{(1)} = O(m_k^2)$), we get for $\delta = 1$

$$\begin{aligned}\sum{}^{(1)} &= -\sum{}' \mathcal{P}_{0,k-l}(-1)^l d_l^{(2)} + \sum{}' \mathcal{P}_{k-l}^{(1)} y_l \\ &= -\sum{}' \mathcal{P}_{0,k-l}(-1)^l d_l^{(2)} + y_k \sum{}' \mathcal{P}_{k-l}^{(1)} \\ &\qquad + O(n^{-13/6}), \quad (2.37)\end{aligned}$$

where we have used that $\mathcal{P}_{k-l}^{(1)} = \mathcal{P}_{l-k}^{(1)}$, so on the basis of(A.2) with $\ell = 4$ we have

$$\sum{}' \mathcal{P}_{k-l}^{(1)}(k-l)/n = \sum_{|k-l|>n^{1/3}} \mathcal{P}_{k-l}^{(1)}(k-l)/n = O(n^{-13/6}).$$

Similarly

$$\begin{aligned}\sum{}^{(2)} &= \sum{}' \mathcal{P}_{l_1,l_2}^{(2,k,1)}(-1)^{l_1+l_2} x_{l_1} x_{l_2} + \\ &\quad 2\sum{}' \mathcal{P}_{l_1,l_2}^{(2,k,1)}(-1)^{l_1} x_{l_1} y_{l_2} + O(m_k^4) \\ &= x_k^2 \sum{}' \mathcal{P}_{l_1,l_2}^{(2,k,1)}(-1)^{l_1+l_2} \\ &\quad + 2x_k d_k^{(1)} \sum{}' \mathcal{P}_{l_1,l_2}^{(2,k,1)}(-1)^{l_1+l_2}(l_1-k) \\ &\quad + 2x_k y_k \sum{}' \mathcal{P}_{l_1,l_2}^{(2,k,1)}(-1)^{l_1} \\ &\quad + 2x_k \sum{}_1^{(2)} + O(m_k^4), \qquad (2.38)\end{aligned}$$

where

$$\sum{}_1^{(2)} = \sum{}' \mathcal{P}_{l_1,l_2}^{(2,k,1)}(-1)^{l_1+l_2}\Big((x_{l_1}-x_k)-(l_1-k)d_k^{(1)}\Big). \qquad (2.39)$$

By the same way

$$\begin{aligned}\sum{}^{(3)} &= \sum{}' \mathcal{P}_{l_1,l_2,l_3}^{(3,k,1)}(-1)^{l_1+l_2+l_3} x_{l_1} x_{l_2} x_{l_3} + O(m_k^4) \\ &= x_k^3 \sum{}' \mathcal{P}_{l_1,l_2,l_3}^{(3,k,1)}(-1)^{l_1+l_2+l_3} + O(m_k^4). \quad (2.40)\end{aligned}$$

Proposition 4. *If $v(\lambda) = V'(\lambda)$, as $\lambda \in \sigma_{\varepsilon/2}$, then*

$$\begin{aligned}\sum{}' \mathcal{P}_{0,k-l}(-1)^l &= 2(-1)^k P_0(0) + O(n^{-3/2}), \\ \sum{}' \mathcal{P}_{k-l}^{(1)} &= 8P_0(2)k/n + O(n^{-13/6}), \\ \sum{}' \mathcal{P}_{l_1,l_2}^{(2,k,1)}(-1)^{l_1+l_2} &= 1 + O(n^{-5/6}), \\ \sum{}' \mathcal{P}_{l_1,l_2}^{(2,k,1)}(-1)^{l_1+l_2}(l_1-k) &= O(n^{-1/2}), \\ \sum{}' \mathcal{P}_{l_1,l_2}^{(2,k,1)}(-1)^{l_1} &= (-1)^k + O(n^{-5/6}), \\ \sum{}' \mathcal{P}_{l_1,l_2,l_3}^{(3,k,1)}(-1)^{l_1+l_2+l_3} &= (-1)^k(4P_0(0)-1) \\ &\quad + O(n^{-1/2}). \qquad (2.41)\end{aligned}$$

If $v^{(0)}(\lambda) = \lambda^{-1}V'(\lambda)$ for $\lambda \in \sigma_{\varepsilon/2}$, then

$$\begin{aligned}\sum{}' \mathcal{P}_{l-k}^{(0)} &= 4P_0(2) + O(n^{-7/6}), \\ \sum{}' \mathcal{P}_{l_1,l_2}^{(2,k,0)}(-1)^{l_1+l_2} &= 2P_0(0) + O(n^{-1/2}). \quad (2.42)\end{aligned}$$

Substituting (2.41) into (2.37)-(2.40) and using (2.41), we obtain

$$\begin{aligned}V'(\mathcal{J}^{(n)})_{n+k,n+k-1} &= 1 - (-1)^k x_k - y_k - \textstyle\sum' \mathcal{P}_{0,k-l}(-1)^l \\ & d_l^{(2)} + k/n + x_k^2 + 2x_k \textstyle\sum_1^{(2)} + 2x_k y_k(-1)^k \\ &+ (-1)^k(4P_0(0)-1)x_k^3 + O(m_k^4). \qquad (2.43)\end{aligned}$$

Using this expression in (2.1) and keeping the terms up to the order $O(m_k^3)$, we get

$$\begin{aligned}&\textstyle\sum' \mathcal{P}_{0,k-l}(-1)^{l-k} d_l^{(2)} = \\ &4P_0(0)x_k^3 + 8P_0(2)x_k y_k + 2x_k \textstyle\sum_1^{(2)} + O(m_k^4), (2.44)\end{aligned}$$

We consider this equations as a linear system of equations with respect to the variables $(-1)^l d_l^{(2)}$ for $l \le N_1 = |k| + 3n^{1/3}/4$.

Proposition 5. *Let the even function P satisfy the inequality $P(\lambda) \ge \delta > 0$ for $\lambda \in \sigma_\varepsilon = [-2-\varepsilon, 2+\varepsilon]$ and $P^{(\ell)} \in L_2[\sigma_\varepsilon]$. Let P_k and $(\mathcal{P}^{-1})_k$ be defined as*

$$\begin{aligned}P_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx} P(\cos(x/2))dx, \\ (\mathcal{P}^{-1})_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx} P^{-1}(\cos(x/2))dx\end{aligned}$$

Assume that for $|k| \le N_1$ ($N_1 > n^{1/3}$)

$$\sum_{|j|<N_1} P_{k-j}\tilde{x}_j = \tilde{z}_k + \tilde{\varepsilon}_k, \qquad (2.45)$$

and we have a priori bound $|\tilde{x}_k| \le \varepsilon_0$ ($|k| \le N_1$).

Then for any $|k| \le N_1/2$ and any $n^{1/3} \le N_2 < N_1/2$

$$\tilde{x}_k = \sum_{|j|\le N_2} (\mathcal{P}^{-1})_{k-j}\tilde{z}_j + \varepsilon_k, \qquad (2.46)$$

where

$$|\varepsilon_k| \le C\left(\max_{|j-k|\le N_2} |\tilde{\varepsilon}_k| + N_2^{-\ell+1/2}(\max_{|j|\le N_1}|z_j| + \varepsilon_0)\right), \qquad (2.47)$$

and C depends only on $||P^{(l)}||_2$ and δ.

Apply Proposition 5 to the system (2.44) with $\ell = 5$, $\tilde{x}_k = d_k^{(2)}$, $\tilde{z}_k = 0$. Then we get

$$|\tilde{\varepsilon}_k| = \left|4P_0(0)x_k^3 + 8P_0(2)x_k y_k + 2x_k \textstyle\sum_1^{(2)} + O(m_k^4)\right| \le Cm_k^3.$$

Moreover, since it follows from Lemma 1 that

$$|d_k^{(2)}| = |d_k^{(1)} - d_k^{(1)}| \le Cn^{-1/4}\log^{1/4} n$$

we can take $\varepsilon_0 = Cn^{-1/4}\log^{1/4} n$. Then, on the basis of Proposition 5, we obtain

$$|d_k^{(2)}| \le Cm^3_{k+[n^{1/3}/4]}, \quad k \le n^{1/2}/3. \tag{2.48}$$

Using this bound combined with (2.29), we get for $\sum_1^{(2)}$ from (2.39)

$$x_k \sum\nolimits_1^{(2)} = x_k \sum\nolimits' \mathcal{P}^{(2,k,1)}_{l_1,l_2} \sum_{k'=k}^{l_1} (l_1-k')d^{(2)}_{k'} = O(m^4_{k+[n^{1/3}/4]}).$$

Therefore (2.44) can be rewritten as

$$\sum\nolimits' \mathcal{P}_{0,k-l}(-1)^l d_l^{(2)} = 4P_0(0)x_k^3 + 8P_0(2)x_k y_k + O(m^4_{k+[n^{1/3}/4]}). \tag{2.49}$$

Now subtracting from (2.49) the same equation written for $k := k-1$, we get

$$\sum\nolimits' \mathcal{P}^{(0)}_{k-l}(-1)^l d_l^{(3)} = O(m^4_{k+[n^{1/3}/4]}).$$

Using Proposition 5 for the variables $(-1)^l d_l^{(3)}$, we obtain that $|d_k^{(3)}| \le Cm^4_{k+[n^{1/3}/2]}$ for $|k| \le n^{1/2}/4$. Hence, writing

$$\begin{aligned}\sum \mathcal{P}_{0,k-l}(-1)^l d_l^{(2)} &= d_k^{(2)} \sum\nolimits' \mathcal{P}_{0,k-l}(-1)^l + \\ &\sum\nolimits' \mathcal{P}_{0,k-l}(-1)^l (d_l^{(2)} - d_k^{(2)}) \\ &= d_k^{(2)} \sum\nolimits' \mathcal{P}_{0,k-l}(-1)^l + O(m^4_{k+[n^{1/3}/2]}),\end{aligned}$$

in view of the first relation in (2.41), we get (2.9) from (2.49). Now, using Lemma 2, we obtain the bound (2.10) for $\tilde{m}_k$ and (2.11). We are left to show that the second asymptotic of (1.18) can be obtained from Lemma 3.

Let us take $k = -[Ln^{1/3}]$ with L big enough. Since it is known (see [12]) that any solution of the Painleve II equations which satisfies (2.17) assumes also the bound

$$q^2(x) \le \frac{-x}{4P_0(0)}, \quad x \le -L_0, \tag{2.50}$$

we can conclude that

$$0 \le c_k \le \frac{k^2}{4n^2} + O\left((|k|/n)^{5/2}\right). \tag{2.51}$$

Now let us choose $\tilde{\varepsilon} = n^{-1/3}P_0^{-1/2}(0)$ and put in (2.19) $z = \tilde{\varepsilon}\zeta$. Then (2.19) takes the form

$$g_{n+k,n}(\tilde{\varepsilon}\zeta) = -\tilde{V}(\zeta) + \tfrac{1}{2}\tilde{\varepsilon}^2 P_0(0) X(\tilde{\varepsilon}\zeta)\sqrt{\zeta^4 - L\zeta^2 + \tilde{c}_k + \tilde{\phi}(\zeta)},$$

where $\tilde{V}$ is an analytic function,

$$0 \le \tilde{c}_k = P_0^{-2}(0)\tilde{\varepsilon}^{-4} c_k \le \frac{L^2}{4}, \tag{2.52}$$

(see (2.50)), and

$$\begin{aligned}|\tilde{\phi}(\zeta)| &= P_0^{-2}(0)\tilde{\varepsilon}^{-4}|\delta_{k,n}(\tilde{\varepsilon}\zeta) + \tilde{\delta}_{k,n}(\tilde{\varepsilon}\zeta) + O(kn^{-2})| \\ &\le C(1+|\zeta|^2),\end{aligned}$$

for $|\Im\zeta| \ge 1$ (see (2.22)). Let b be the smallest root of the quadratic equation

$$\zeta^2 - L\zeta + \tilde{c}_k = 0. \tag{2.53}$$

We note, that due to (2.52) b is real and positive. Consider

$$I(b,L) = \frac{\tilde{\varepsilon}^{-2}}{2\pi i}\oint_{\mathcal{L}} g_{n+k,n}(\tilde{\varepsilon}\zeta)e^{-\zeta^2/2}d\zeta \tag{2.54}$$
$$= \frac{P_0(0)}{2\pi i}\oint_{\mathcal{L}} X(\tilde{\varepsilon}\zeta)\sqrt{(\zeta^2-b)(\zeta^2-L+b)}e^{-\zeta^2/2}d\zeta + \tilde{r}_L,$$

where $\mathcal{L}$ consists of two lines $\Im\zeta = \pm 1$ and

$$|\tilde{r}_L| \le C\oint_{\mathcal{L}} \frac{|\phi(\zeta)|\cdot|X(\tilde{\varepsilon}\zeta)|e^{-|\zeta|^2/2}|d\zeta|}{\sqrt{(\zeta^2-b)(\zeta^2-L+b)}} \le CL^{-1/2}. \tag{2.55}$$

Then, using the Cauchy theorem, we get

$$\begin{aligned}I(b,L) &= \\ &\frac{P_0(0)}{2\pi}\Im\int\sqrt{((\varepsilon x)^2-4)(x^2-b)(x^2-L+b)}e^{-x^2/2}dx + \tilde{r}_L \\ &= -\frac{P_0(0)}{2\pi}\int_{|x|<b}\sqrt{(x^2-b)(x^2-L+b)}e^{-x^2/2}dx \\ &+\frac{P_0(0)}{2\pi}\int_{|x|\ge L-b}\sqrt{(x^2-b)(x^2-L+b)}e^{-x^2/2}dx \\ &+\tilde{r}_L + O(\tilde{\varepsilon}) = P_0(0)I_1(b,L) + \tilde{r}_L + O(\tilde{\varepsilon}).\end{aligned} \tag{2.56}$$

One can prove easily that for large L

$$I_1(b,L) \sim -C_0 L^{1/2} b^{3/2}, \quad (C_0 > 0).$$

On the other hand,

$$I(b,L) = \frac{\tilde{\varepsilon}^{-2}}{2\pi}\int e^{-x^2/2\sigma}\lim_{\varepsilon\to 0}\Im g_{n+k,n}(\tilde{\varepsilon}\zeta)dx > 0.$$

Thus, taking into account (2.55)

$$L^{1/2}b^{3/2} \le C'|\tilde{r}_L| \le C''L^{-1/2} \Rightarrow b \le CL^{-2/3}$$

Hance, since b is the root of quadratic equation (2.53), we have

$$|c_k| = P_1^2(0)\tilde{\varepsilon}^4(L-b)b/4 \le C_1' n^{-4/3}L^{1/3}. \tag{2.57}$$

The last inequality combined with (2.20), and the bound for the first differences $d_j^{(1)}$ imply for $k = [Ln^{1/3}]$, $l = [L^{-1/6}n^{1/3}]$

$$\begin{aligned}n^{-4/3}O(L^{1/3}) &= c_{-k} - c_{-k-l} \\ &= \frac{2P_0(0)}{n^{5/3}}\sum_{j=k}^{k+l} q_n^2(-\frac{j}{n^{1/3}}) - \frac{L+L^{-1/6}}{2n^{4/3}}L^{-1/6} \\ &= n^{-4/3}\left[L^{-1/6}\left((2P_0(0)q_n^2(-\tfrac{k}{n^{1/3}}) - \tfrac{L}{2} + O(L^{-1/6})\right)\right].\end{aligned}$$

Therefore,

$$|q_n(-L)| = (4P_0(0))^{-1/2}L^{1/2}(1+O(L^{-1/2})).$$

But it is known (see [12] that any bounded for positive x solution of (1.17), which possesses the above property satisfies also the asymptotic relations

$$q_n(-L) = \tilde{s}(4P_0(0))^{-1/2}L^{1/2}(1+O(L^{-2})),\ \tilde{s} = \operatorname{sign} q(0). \tag{2.58}$$

Hence, we have proved (1.18). Theorem 1 is proved. □

The proof of Theorem 2 is based on the following Proposition

Proposition 6. *Consider the sequence of functions $\mathcal{K}_n : \mathbb{R}^2 \to \mathbb{R}$ and for $\Im\zeta_{1,2} \neq 0$ define*

$$F_n(\zeta_1,\zeta_2) = \tag{2.59}$$
$$= \int \Im(t_1-\zeta_1)^{-1}\Im(t_2-\zeta_2)^{-1}(t_1-t_2)^2\mathcal{K}_n^2(t_1,t_2)dt_1dt_2,.$$

Assume that there exists $F(\zeta_1,\zeta_2)$ of the form

$$F(\zeta_1,\zeta_2) =$$
$$= \int\int \Im(t_1-\zeta_1)^{-1}\Im(t_2-\zeta_2)^{-1}(t_1-t_2)^2\Phi(t_1,t_2)dt_1dt_2$$

with $\Phi(t_1,t_2)$ bounded uniformly in each compact in $\mathbb{R}^2$ and such that for $\Im\zeta_{1,2} \geq 1$

$$|F_n(\zeta_1,\zeta_2)-F(\zeta_1,\zeta_2)| \leq C(1+|\zeta|^2)\varepsilon_n, \quad \varepsilon_n \to 0, \tag{2.60}$$

as $n \to \infty$. Assume also that for any fixed $\varepsilon \geq 0$ and uniformly in a varying in any compact in $\mathbb{R}$

$$I_\varepsilon(a) = \int_{|t-a|\leq\varepsilon} \mathcal{K}_n(t,t)dt \leq \varepsilon C + o(1). \tag{2.61}$$

Then for any intervals $I_1, I_2 \subset \mathbb{R}$

$$\lim_{n\to\infty}\int_{I_1} dt_1 \int_{I_2} dt_2\mathcal{K}_n^2(t_1,t_2) = \int_{I_1} dt_1 \int_{I_2} dt_2\Phi(t_1,t_2).$$

Proof of Proposition 6. Consider the integral

$$\int_{\Im\zeta_1=\pm1} d\zeta_1 \int_{\Im\zeta_2=\pm1} d\zeta_2(F_n(\zeta_1,\zeta_2)-F(\zeta_1,\zeta_2))e^{-(\zeta_1-a_1)^2/2\sigma_1}e^{-(\zeta_2-a_2)^2/2\sigma_2}. \tag{2.62}$$

Using the Cauchy theorem, we get that for any $\sigma_{1,2} > 0$, $a_{1,2} \in \mathbb{R}$

$$\left|\int\int (t_1-t_2)^2\left(\mathcal{K}_n^2(t_1,t_2)-\Phi(t_1,t_2)\right)e^{-(t_1-a_1)^2/2\sigma_1}e^{-(t_2-a_2)^2/2\sigma_2}dt_1dt_2\right| \leq C\varepsilon_n$$

with C, depending on $a_1,a_2,\sigma_1,\sigma_2$, but independent of n. This implies that for any Lipshitz f_1 and f_2 with a compact support

$$\int (t_1-t_2)^2\left(\mathcal{K}_n^2(t_1,t_2)-\Phi(t_1,t_2)\right)f_1(t_1)f_2(t_2)dt_1dt_2 \to 0. \tag{2.63}$$

For any small enough ϵ denote by $f_1^{(+\epsilon)}$ a Lipshitz function which coincides with the indicator χ_{I_1} of $I_1 = (a_1,b_1)$ inside this interval, equals to zero outside of $(a_1-\epsilon, b_1+\epsilon)$ and is linear in $(a_1-\epsilon,a_1), (b_1,b_1+\epsilon)$. Let $f_1^{(-\epsilon)}$ be a similar function for the interval $(a_1+\epsilon, b_1-\epsilon)$ and $f_2^{(\pm\epsilon)}$ be similar functions for I_2. Denote also

$$\phi_{\varepsilon_1}(t_1,t_2) = (t_1-t_2)^{-2}\mathbf{1}_{|t_1-t_2|>\varepsilon_1} + \varepsilon_1^{-2}\mathbf{1}_{|t_1-t_2|\leq\varepsilon_1}.$$

Then, evidently

$$\begin{aligned}\phi_{\varepsilon_1}(t_1,t_2)f_1^{(-\epsilon)}(t_1)f_2^{(-\epsilon)}(t_2) &\leq \phi_\varepsilon(t_1,t_2)\chi_{I_1}(t_1)\chi_{I_1}(t_2)\\ &\leq \phi_{\varepsilon_1}(t_1,t_2)f_1^{(+\epsilon)}(t_1)f_2^{(+\epsilon)}(t_2).\end{aligned}$$

Integrate this inequality with $(t_1-t_2)^2\mathcal{K}_n^2(t_1,t_2)$, and take the limits $n\to\infty$ and then $\epsilon\to 0$. We obtain

$$\int_{I_1\times I_2} dt_1dt_2\Phi(t_1,t_2)-O(\varepsilon_1)$$
$$\le \lim_{n\to\infty}\int_{I_1\times I_2} dt_1dt_2\mathcal{K}_n^2(t_1,t_2)$$
$$+\lim_{n\to\infty}\int_{I_1\times I_2} dt_1dt_2\mathcal{K}_n^2(t_1,t_2)\left(\frac{(t_1-t_2)^2}{\varepsilon_1^2}-1\right)\mathbf{1}_{|t_1-t_2|\le\varepsilon_1}$$
$$\le \int_{I_1\times I_2} dt_1dt_2\Phi(t_1,t_2)+O(\varepsilon_1).$$

But using the inequality $\mathcal{K}_n^2(t_1,t_2)\le\mathcal{K}_n(t_1,t_1)\mathcal{K}_n(t_2,t_2)$ and integrating first with respect to t_1 and then with respect to t_2, on the basis of (2.61) we obtain that

$$\int_{I_1\times I_2} dt_1dt_2\mathcal{K}_n^2(t_1,t_2)\left(\varepsilon_1^{-2}(t_1-t_2)^2-1\right)\mathbf{1}_{|t_1-t_2|\le\varepsilon_1}$$
$$\le 2\int_{I_1\times I_2} dt_1dt_2\mathcal{K}_n(t_1,t_1)\mathcal{K}_n(t_2,t_2)\mathbf{1}_{|t_1-t_2|\le\varepsilon_1}$$
$$\le 2C\varepsilon_1\int_{I_2} dt_2\mathcal{K}_n(t_2,t_2)\le C'\varepsilon_1.$$

Then, taking the limit $\varepsilon_1\to 0$ we get the assertion of Proposition 6.

Proof of Theorem 2. Take some fixed ζ_1, ζ_2 with $\Im\zeta_{1,2}\ne 0$, denote $z_{1,2}=\zeta_{1,2}n^{-1/3}$ and consider the function

$$F_n(\zeta_1,\zeta_2)=\Im z_1\Im z_2\int\frac{(\lambda_1-\lambda_2)^2K_n^2(\lambda_1,\lambda_2)d\lambda_1d\lambda_2}{|\lambda_1-z_1|^2|\lambda_2-z_2|^2},$$
$$F_n^{(1)}(\zeta_1)=n^{-2/3}\int\frac{(\lambda_1-\lambda_2)^2K_n^2(\lambda_1,\lambda_2)d\lambda_1d\lambda_2}{(\lambda_1-z_1)^2(\lambda_2-z_1)^2}. \tag{2.64}$$

Changing variables $\lambda_{1,2}=t_{1,2}n^{-1/3}$, and using (1.20), we get (cf (2.59))

$$F_n(\zeta_1,\zeta_2)=\Im\zeta_1\Im\zeta_2\int\frac{(t_1-t_2)^2\mathcal{K}_n^2(t_1,t_2)dt_1dt_2}{|t_1-\zeta_1|^2|t_2-\zeta_2|^2}. \tag{2.65}$$

Hence, according to Proposition 6, to prove the weak convergence of $\mathcal{K}_n^2(t_1,t_2)$ to $\mathcal{K}^2(t_1,t_2)$ of (1.21) we need to check (2.60) and (2.61). Observe now that to prove (2.61) it is enough to show that $g_{k,n}(z)$ defined in (2.19) for any $z=n^{-1/3}\zeta$ with $\Im\zeta\ge\varepsilon_n$ satisfy the bound

$$\left|g_{n,n}(\zeta n^{-1/3})+\tfrac{\zeta n^{-1/3}}{2}\left(V''(0)+(\zeta n^{-1/3})^2V^{(4)}(0)/6\right)\right|$$
$$\le Cn^{-2/3}(|\zeta|^2+1), \tag{2.66}$$

where C does not depend on n and ζ. Indeed, if we know (2.66), then using the Cauchy theorem with a the contour $\mathcal{L}$, consisting of two lines $\Im\zeta=\pm\varepsilon$, we obtain the bound

$$\int_{|t-a|\le\varepsilon}\mathcal{K}_n(t,t)dt\le e^{1/2}\int\mathcal{K}_n(t,t)e^{-(t-a)^2/2\varepsilon^2}dt$$
$$=\frac{e^{1/2}}{2\pi i}\oint_{\mathcal{L}}d\zeta\int dte^{-(\zeta-a)^2/2\varepsilon^2}\frac{\mathcal{K}_n(t,t)}{\zeta-t}$$
$$=\frac{e^{1/2}}{2\pi i}\oint_{\mathcal{L}}d\zeta e^{-(\zeta-a)^2/2\varepsilon^2}n^{2/3}\Big(g_{n,n}(\zeta n^{-1/3})$$
$$+\frac{\zeta n^{-1/3}}{2}\left(V''(0)+(\zeta n^{-1/3})^2V^{(4)}(0)/6\right)\Big)$$
$$\le C\oint_{\mathcal{L}}|d\zeta||e^{-(\zeta-a)^2/2\varepsilon^2}|(|\zeta|^2+1)\le C\varepsilon, \tag{2.67}$$

which proves (2.61). Consider $F_n^{(1)}(\zeta_1)$ defined in (2.64). It is easy to see that

$$F_n^{(1)}(\zeta)=n^{4/3}\delta_{n,n}(z),$$

where $\delta_{k,n}(z)$ is defined in (2.22). Therefore, if we prove that uniformly in $\Im\zeta\ge\varepsilon_n=(\log n)^{-1/2}$

$$|F_n^{(1)}(\zeta)|\le C(|\zeta|^2+1), \tag{2.68}$$

then, using this bound in (2.19), we get (2.66). Hence, our goal is to prove (2.60) and (2.68).

Using the Christoffel-Darboux formula, it is easy to derive from (2.64) that

$$\begin{aligned}F_n(\zeta_1,\zeta_2) &= (J_{n+1}^{(n)})^2[\Im R_{n,n}(z_1)\Im R_{n-1,n-1}(z_2)\\ &\quad+\Im R_{n,n}(z_2)\Im R_{n-1,n-1}(z_1)\\ &\quad-2\Im R_{n,n-1}(z_1)\Im R_{n-1,n}(z_2)],\\ F_n^{(1)}(\zeta_1) &= 2(J_{n+1}^{(n)})^2\Big[\frac{d}{d\zeta_1}R_{n,n}(\zeta_1/n^{1/3})\frac{d}{d\zeta_1}R_{n-1,n-1}(\zeta_1/n^{1/3})\\ &\quad-\Big(\frac{d}{d\zeta_1}R_{n,n-1}(\zeta_1/n^{1/3})\Big)^2\Big]\end{aligned} \tag{2.69}$$

where

$$R_{k,m}(z)=\int\frac{\psi_k^{(n)}(\lambda)\psi_m^{(n)}(\lambda)}{\lambda-z}d\lambda \tag{2.70}$$

is the resolvent of $\mathcal{J}^{(n)}$ ($R=(\mathcal{J}^{(n)}-z)^{-1}$).

Let us study first the case when in (1.16) $\tilde s=1$. Consider the Dirac operator $\mathcal{A}$ defined in $L_2(\mathbb{R})\times L_2(\mathbb{R})$ by the differential expression (1.23)-(1.18). Let $\mathcal{R}_{\alpha,\beta}(x,y;\zeta)$ ($\alpha,\beta=0,1$) be the kernel of the operator $\mathcal{R}(\zeta)=(2\mathcal{A}-\zeta)^{-1}$. It means that the coefficients $\mathcal{R}_{\alpha,\beta}(x,y;\zeta)$ satisfy the equations

$$\begin{aligned}2\frac{d}{dx}\mathcal{R}_{1,0}(x,y;\zeta)+2q(x)\mathcal{R}_{1,0}(x,y;\zeta)-\zeta\mathcal{R}_{0,0}(x,y;\zeta)&=\delta(x-y)\\ -2\frac{d}{dx}\mathcal{R}_{0,1}(x,y;\zeta)+2q(x)\mathcal{R}_{0,1}(x,y;\zeta)-\zeta\mathcal{R}_{1,1}(x,y;\zeta)&=\delta(x-y)\\ -2\frac{d}{dx}\mathcal{R}_{0,0}(x,y;\zeta)+2q(x)\mathcal{R}_{0,0}(x,y;\zeta)-\zeta\mathcal{R}_{1,0}(x,y;\zeta)&=0\\ 2\frac{d}{dx}\mathcal{R}_{1,1}(x,y;\zeta)+2q(x)\mathcal{R}_{1,1}(x,y;\zeta)-\zeta\mathcal{R}_{0,1}(x,y;\zeta)&=0\end{aligned} \tag{2.71}$$

Here $\delta(x)$ is the Dirac δ-function and, e.g., the first equation means that the l.h.s. is equal to zero, as $x \neq y$ and $2\mathcal{R}_{1,0}(x+0,x) - 2\mathcal{R}_{1,0}(x-0,x) = 1$.

Consider a semi infinite matrix with entries

$$R^*_{n+2k+\alpha,n+2m+\beta} = (-1)^{(k+m)}\mathcal{R}_{\alpha,\beta}\left(\frac{2k+\alpha}{n^{1/3}}, \frac{2m+\beta}{n^{1/3}};\zeta\right), \tag{2.72}$$

where $-n \leq k, m < \infty, \quad \alpha,\beta = 0,1$. Define

$$D := (\mathcal{J}^{(n)} - z)R^* - I. \tag{2.73}$$

Then

$$R = R^* - RD. \tag{2.74}$$

Lemma 5. *Set $M = [n^{1/3}\log^2 n]$. Then for $-M \leq k, m \leq M$,*

$$\begin{aligned}|D_{n+j,n+k}| &\leq Cn^{-2/3}\sum_{\alpha,\beta=0,1}\left(1+|\zeta|^2+q^2(x)\right)|\mathcal{R}_{\alpha,\beta}(x,y;\zeta)|\\ |D_{n+k,n+k}| &\leq Cn^{-1/3}\sum_{\alpha,\beta=0,1}(1+|\zeta|+q(x))|\mathcal{R}_{\alpha,\beta}(x,y;\zeta)|,\end{aligned} \tag{2.75}$$

with $x = \frac{k+\alpha}{n^{1/3}}, y = \frac{m+\theta(\alpha,\beta,k,m)}{n^{1/3}}$, and $|\theta_{\alpha,\beta,k,m}| \leq 2$.

Moreover, if $D^{(M)} = \{D_{n+j,n+k}\}_{j,k=-M}^{M}$, then

$$||D^{(M)}|| \leq \frac{Cn^{-1/3}}{|\Im\zeta|^{1/2}}\left(|\zeta|^2 + \left(M/n^{1/3}\right)^{3/2}\right), \tag{2.76}$$

The proof of (2.75 could be easily obtained from the definitions of R^*, representation (1.16) of $J^{(n)}_{n+k}$ and equations (2.71). The proof of (2.76) follows from the bound, valid for the norm of an arbitrary matrix $\mathcal{A}$

$$||\mathcal{A}||^2 \leq \max_i \sum_j |A_{i,j}| \cdot \max_j \sum_i |A_{i,j}|, \tag{2.77}$$

(2.75) and the bound for the resolvent of the Dirac operator (see [14])

$$\int |\mathcal{R}_{\alpha,\beta}(x,y;\zeta)|^2 dy \leq C|\Im\zeta|^{-1}\Im\mathcal{R}_{\alpha,\alpha}(x,x;\zeta) \leq C'|\Im\zeta|^{-1}, \tag{2.78}$$

To replace D by $D^{(M)}$ in (2.74) we use the following proposition

Proposition 7. *Let $\mathcal{J}$ be an arbitrary Jacobi matrix , with $|\mathcal{J}_{j,j+1}| \leq A$, for all j such that $|j-k| \leq M$. Consider $\mathcal{R}(z) = (z-\mathcal{J})^{-1}$ with $|\Im z| \leq A_1$. Then*

$$|\mathcal{R}_{k,j}(z)| \leq \frac{C_1'}{\Im z}e^{-C_2'|\Im z||k-j|} + \frac{C_1'}{|\Im z|^2}e^{-C_2'|\Im z|M}, \tag{2.79}$$

where $C_1', C_2' > 0$ depend only on A and A_1.

Using the bound (2.79) we derive from (2.73) that for $|j|, |k| \leq M/2$ and $\Im\zeta > \log^{1/2} n$

$$\begin{aligned}R_{n+j,n+k} &= R^*_{n+j,n+k} - (RD^{(M)})_{n+j,n+k} + O(e^{-c\log^{3/2} n})\\ &= R^*_{n+j,n+k} - (R^*D^{(M)})_{n+j,n+k} + (R(D^{(M)})^2)_{n+j,n+k}\\ &\quad + O(e^{-c\log^{3/2} n})\end{aligned} \tag{2.80}$$

Hence, using (2.75) and (2.78), we obtain that for $|j|, |k| \leq M/2$ and $\Im\zeta > \log^{1/2} n$

$$R_{n+j,n+k} = R^*_{n+j,n+k} + O(n^{-1/3}\log^p n) \tag{2.81}$$

with some positive n independent p. Thus, we have proved (2.60) with

$$\begin{aligned}F(\zeta_1,\zeta_2) = (&\mathcal{R}_{0,0}(0,0;\zeta_1)\mathcal{R}_{1,1}(0,0;\zeta_2)\\ &+\mathcal{R}_{0,0}(0,0;\zeta_2)\mathcal{R}_{1,1}(0,0,\zeta_1)\\ &-2\mathcal{R}_{0,1}(0,0+0;\zeta_1)\mathcal{R}_{1,0}(0+0,0;\zeta_2)),\end{aligned} \tag{2.82}$$

where we denote $\mathcal{R}_{0,1}(0,0+0,\zeta_1) = \lim_{x\to+0}\mathcal{R}_{0,1}(0,x,\zeta_1)$. But, according to the spectral theorem (see [14]),

$$\mathcal{R}_{\alpha,\beta}(x,y;\zeta) = \int \frac{\Psi_\alpha(x;t)\Psi_\beta(y;t)}{2t-\zeta}dt, \tag{2.83}$$

where $\mathbf{\Psi}(x;t) = (\Psi_0(x;t), \Psi_1(x;t))$ is the solution of the Dirac system (1.23), satisfying asymptotic conditions (1.24). The last two relations and the formula of the inverse Stieltjes transform yield

$$\begin{aligned}\Phi(t_1,t_2) = (2\pi)^{-2}\,(&\Psi_1(0;t_1/2)\Psi_0(0;t_2/2)\\ &-\Psi_0(0;t_2/2)\Psi_1(0;t/2))^2.\end{aligned} \tag{2.84}$$

Moreover, since $R_{k,j}(n^{-1/3}\zeta)$ and $R^*_{n+j,n+k}(\zeta/n^{1/3})$ are analytic functions for $\Im\zeta > 0$, taking the circle of the radius $\Im\zeta/2$ centered in ζ as a contour of integration and using the Cauchy representation for the derivative and (2.81), we obtain for $\Im\zeta > 2\log^{1/2} n$

$$\begin{aligned}\frac{d}{d\zeta}R_{n+j,n+k}(\zeta/n^{1/3}) = \frac{d}{d\zeta}&R^*_{n+j,n+k}(\zeta/n^{1/3})\\ &+O(n^{-1/3}\log^{p+1} n)\end{aligned} \tag{2.85}$$

Using the representation (2.83) and taking into account that $\Psi_\alpha(x;t)$ are smooth function with respect to t, according to the standard theory of the Cauchy type integrals (see [16]) we get that the $\frac{d}{d\zeta}R^*_{n+j,n+k}(\zeta/n^{1/3})$ is uniformly bounded up to the real line. Therefore we obtain (2.68) and prove the assertion of Theorem 2 for $l = 2$. For others l we study by the same way

$$F_n(\zeta_1,\dots,\zeta_l)=\int\prod_{i=1}^{l}\Im(t_i-\zeta_i)^{-1}(t_1-t_2)\dots(t_l-t_1)\mathcal{K}_n(t_1,t_2)\dots\mathcal{K}_n(t_l,t_1)dt_1\dots dt_l$$

Now, notice that the $(\Psi_0(x,t),\Psi_1(x,t))\to(-\Psi_1(x,t),\Psi_0(x,t))$ gives us the solution of (1.23) with potential $q_1(x)=-q(x)$ but does not change the expression (1.22). This completes the proof of Theorem 2.

To prove Corollary 1 we split the expansion for the Fredholm determinant in two parts: with $m<N$ and $m\ge N$ (m is the number of variables in the correspondent determinant). Using the Hadamard bound for determinants with $m>N$ and then (2.67) it is easy to see that the second sum possesses the bound $C^N/N!$. Hence using Theorem 2 we can take the limit $n\to\infty$ in the first sum and then take the limit $N\to\infty$. Relation (1.25) follows.

3. AUXILIARY RESULTS

Proof of Lemma 1. We introduce an eigenvalue distribution which is more general than (1.4), making different the number of variable and the large parameter in front of V in the exponent of the r.h.s of (1.4):

$$p_{k,n}(\lambda_1,...\lambda_k)=Z_{k,n}^{-1}\prod_{1\le j<m\le k}(\lambda_j-\lambda_m)^2\exp\prod_{j=1}^{k}e^{-nV(\lambda_j)},\tag{3.1}$$

where $Z_{k,n}$ is the normalizing factor. For $k=n$ this probability distribution density coincides with (1.4). Let

$$\tilde\rho_{k,n}(\lambda_1)=\int d\lambda_2...d\lambda_k p_{k,n}(\lambda_1,...\lambda_k),$$
$$\tilde\rho_{k,n}(\lambda_1,\lambda_2)=\int d\lambda_3...d\lambda_k p_{k,n}(\lambda_1,...\lambda_k)\tag{3.2}$$

be the first and the second marginal densities of (3.1). By the standard argument [15] we obtain

$$\tilde\rho_{k,n}(\lambda)=k^{-1}K_{k,n}(\lambda,\lambda),$$
$$\tilde\rho_{k,n}(\lambda,\mu)=\frac{K_{k,n}(\lambda,\lambda)K_{k,n}(\mu,\mu)-K_{k,n}^2(\lambda,\mu)}{k(k-1)},\tag{3.3}$$

where $K_{k,n}(\lambda,\mu)$ is defined in (2.18). Remark also that

$$\tilde\rho_{k,n}(\lambda)=\frac{n}{k}\rho_{k,n}(\lambda),$$

where $\rho_{k,n}$ is defined in (2.18). Taking any twice differentiable and vanishing outside $\sigma_{2\varepsilon}$ function $\phi(\lambda)$ and integrating by parts with respect to V, we come to the identity

$$\int V'(\lambda)\tilde\rho_{k,n}(\lambda)\phi(\lambda)d\lambda=\frac{1}{n}\int\tilde\rho_{k,n}(\lambda)\phi'(\lambda)d\lambda$$
$$+2\frac{k-1}{n}\int\tilde\rho_{k,n}(\lambda,\mu)\frac{\phi(\lambda)}{\lambda-\mu}d\lambda d\mu.\tag{3.4}$$

The symmetry property $\tilde\rho_{k,n}(\lambda,\mu)=\tilde\rho_{k,n}(\mu,\lambda)$ of (3.2) implies

$$\int\tilde\rho_{k,n}(\lambda,\mu)\frac{\phi(\lambda)}{\lambda-\mu}d\lambda d\mu=-\int\tilde\rho_{k,n}(\lambda,\mu)\frac{\phi(\mu)}{\lambda-\mu}d\lambda d\mu.$$

This allows us to rewrite (3.4) in the form

$$\int V'(\lambda)\tilde\rho_{k,n}(\lambda)\phi(\lambda)d\lambda=\frac{1}{n}\int\tilde\rho_{k,n}(\lambda)\phi'(\lambda)d\lambda$$
$$+\frac{k-1}{n}\int\tilde\rho_{k,n}(\lambda,\mu)\frac{\phi(\lambda)-\phi(\mu)}{\lambda-\mu}d\lambda d\mu.$$

Now, using (3.3) and the fact that

$$\int K_{k,n}^2(\lambda,\mu)d\mu=K_{k,n}(\lambda,\lambda),$$

we can rewrite the last equation as

$$\int\frac{\phi(\lambda)-\phi(\mu)}{\lambda-\mu}\rho_{k,n}(\lambda)\rho_{k,n}(\mu)d\lambda d\mu$$
$$-\int V'(\lambda)\rho_{k,n}(\lambda)\phi(\lambda)d\lambda+\delta_{k,n}(\phi)=0,\tag{3.5}$$

where we denote

$$\delta_{k,n}(\phi)=\frac{1}{2n^2}\int\Big(\phi'(\lambda)+\phi'(\mu)-2\tfrac{\phi(\lambda)-\phi(\mu)}{\lambda-\mu}\Big)K_{k,n}^2(\lambda,\mu)d\lambda d\mu.$$

Subtracting from (3.5) the relation obtained from (3.5) by the replacement $k\to(k-1)$ and multiplying the difference by n, we obtain:

$$2\int\frac{\phi(\lambda)-\phi(\mu)}{\lambda-\mu}\rho(\mu)[\psi_k^{(n)}(\lambda)]^2d\lambda d\mu$$
$$-\int V'(\lambda)\phi(\lambda)[\psi_k^{(n)}(\lambda)]^2d\lambda+\delta_{k,n}^{(R)}(\phi)+\tilde\delta_{k,n}^{(R)}(\phi)=0,\tag{3.6}$$

where

$$\delta_{k,n}^{(R)}(\phi)=\frac{1}{n}\int K_{k,n}(\lambda,\mu)\psi_k^{(n)}(\lambda)\psi_k^{(n)}(\mu)\Big(\phi'(\lambda)+\phi'(\mu)-2\frac{\phi(\lambda)-\phi(\mu)}{\lambda-\mu}\Big)d\lambda d\mu$$

$$\tilde\delta_{k,n}^{(R)}(\phi)=\int\frac{\phi(\lambda)-\phi(\mu)}{\lambda-\mu}(\rho_{k,n}(\mu)-\rho(\mu))[\psi_k^{(n)}(\lambda)]^2d\lambda d\mu-\tfrac{1}{n}\int\phi'(\lambda)[\psi_k^{(n)}(\lambda)]^2d\lambda.$$

By Schwartz inequality

$$|\delta_{k,n}^{(R)}(\phi)| \le \frac{2}{n}||\phi''||_0\Big(\int K_{k,n}^2(\lambda,\mu)(\lambda-\mu)^2 d\lambda d\mu\Big)^{1/2}$$
$$\cdot\Big(\int(\psi_k^{(n)}(\lambda))^2(\psi_k^{(n)}(\mu))^2 d\lambda d\mu\Big)^{1/2} \le \frac{C}{n}||\phi''||_0$$
$$|\tilde\delta_{k,n}^{(R)}(\phi)| \le |\tilde\delta_{0,n}^{(R)}(\phi)| + \frac{|k-n|}{n}||\phi'||_0$$
$$\le C\Big(||\phi'||_2^{1/2}||\phi''||_2^{1/2}\frac{\log^{1/2} n}{n^{1/2}} + ||\phi'||_0\frac{|k-n|}{n}\Big),$$

where the symbols $||...||_0$ and $||...||_2$ denotes the supremum and the L_2-norm on σ_ε. Here we have used the result of [5], valid for any smooth function $\phi(\mu)$ defined on σ_ε

$$\Big|\int\phi(\mu)\rho_{n,n}(\mu)d\mu - \int\phi(\mu)\rho(\mu)d\mu\Big|$$
$$\le C||\phi'||_2^{1/2}||\phi||_2^{1/2}n^{-1/2}\log^{1/2} n, \qquad (3.7)$$

where the symbol $||...||_2$ denotes the L_2-norm on σ_ε.

Now we are going to use (1.3) in the second integral in the r.h.s. of (3.6). But since this representation is valid only for $\lambda\in[-2,2]$ we need to restrict the integrals in (3.6) by some $\sigma_{\tilde\varepsilon} = [-2-\tilde\varepsilon, 2+\tilde\varepsilon]$ with some small $\tilde\varepsilon > 0$. To this aim we use

Proposition 8. *Consider any unitary invariant ensemble of the form (1.1) and assume that $V(\lambda)$ possess two bounded derivatives in some neighborhood of the support σ of the density of states ρ. Let also σ consist of a finite number of intervals, $\rho(\lambda)$ satisfy condition C4 and $\rho(\lambda)\sim C(a^*)|\lambda-a^*|^{1/2}$ near any edge point a^* of σ.*

Then there exist absolute constants $C, C_0, \varepsilon_0 > 0$ such that for any positive $C_0 n^{-1/2}\log n \le \varepsilon \le \varepsilon_0$ and for any integer $k : |k| \le n + n^{1/2}$ the bounds hold:

$$\int_{\mathbf{R}\setminus\sigma_\varepsilon}\rho_{k,n}(\lambda)d\lambda \le e^{-nC\varepsilon}, \int_{\mathbf{R}\setminus\sigma_\varepsilon}(\psi_k^{(n)}(\lambda))^2 d\lambda \le e^{-nC\varepsilon}. \qquad (3.8)$$

This proposition was proved in [3]. It allows us to restrict the integration in the first three integrals of (3.6) by $\sigma_{\tilde\varepsilon}$ with $\tilde\varepsilon = C_0 n^{-1/2}\log n$. Now we can use (1.3). The error, which appear because of this replacement is of the order $O(\tilde\varepsilon)$, because $V'(\lambda)$ is a smooth function in $\sigma_{\tilde\varepsilon}$. Hence, (3.6) can be rewritten in the form

$$2\int_{\sigma_{\tilde\varepsilon}}(\psi_k^{(n)}(\lambda))^2 d\lambda\int_{-2}^{2}\frac{\phi(\mu)}{\lambda-\mu}\rho(\mu)d\mu$$
$$= \delta_{k,n}^{(R)}(\phi) + \tilde\delta_{k,n}^{(R)}(\phi) + O(||\phi||_0 n^{-1/2}\log n). \qquad (3.9)$$

Take $\phi(\lambda) = P_0^{-1}(\lambda)(\lambda-z)^{-1}$ and substitute in (3.9). Then, according to (1.13), we get

$$2\int_{\sigma_{\tilde\varepsilon}}(\psi_k^{(n)}(\lambda))^2 d\lambda\int_{-2}^{2}\frac{\mu^2\sqrt{4-\mu^2}}{(\mu-z)(\lambda-\mu)}d\mu$$
$$= \delta_{k,n}^{(R)}(z) + \tilde\delta_{k,n}^{(R)}(z) + O(|\Im z|^{-1}n^{-1/2}\log n), \qquad (3.10)$$

where $\delta_{k,n}^{(R)}(z)$ and $\tilde\delta_{k,n}^{(R)}(z)$ have the form (3.7) and due to (3.7) satisfy the bound

$$|\delta_{k,n}^{(R)}(z)| \le \frac{C}{n|\Im z|^2}, \quad |\tilde\delta_{k,n}^{(R)}(z)| \le \frac{C|k-n|}{n|\Im z|^2} + \frac{C\log^{1/2} n}{n^{1/2}|\Im z|^2}. \qquad (3.11)$$

Thus, using the fact that

$$\frac{2}{\pi}\int\frac{\mu^2\sqrt{4-\mu^2}}{(\mu-z)(\lambda-\mu)}d\mu = \frac{z^2\sqrt{z^2-4}}{\lambda-z} + (z^2+z\lambda+\lambda^2) - 2,$$

we get from (3.10)

$$R_{k,k}(z) = -\Big(z^2 + a_k - \delta_{k,n}^{(R)}(z) - \tilde\delta_{k,n}^{(R)}(z)$$
$$+O(|\Im z|^{-1}n^{-1/2}\log n)\Big)\frac{1}{z^2\sqrt{z^2-4}}, \qquad (3.12)$$

where $R_{k,k}(z)$ is defined in (2.70) and we denote

$$a_k = \int\lambda^2[\psi_k^{(n)}(\lambda)]^2 d\lambda - 2. \qquad (3.13)$$

Let us assume that $a_k > Cn^{-1/2}\log^{1/2} n$ with C big enough. Then, using the bound (3.11) and the Rouchet theorem, we get that $R_{k,k}(z)$ has a root in the circle of radius $\frac{1}{2}a_k^{1/2}$ centered in the point $ia_k^{1/2}$. But, by definition (2.70),

$$\Im R_{k,k}(z)\Im z > 0, \qquad (3.14)$$

so $R_{k,k}(z)$ cannot have zeros, when $\Im z \ne 0$ and therefore we get that for $|k-n| \le n^{1/2}$ $a_k \le Cn^{-1/4}\log^{1/2} n$. Similarly, if we assume that $a_k \le -Cn^{-1/4}\log^{1/2} n$ we get that $\Im R_{k,k}(\frac{1}{2}|a_k|^{1/2}e^{i\pi/6}) > 0$, which also contradict to (3.14). Thus, we obtain that

$$|a_k| \le Cn^{-1/4}\log^{1/2} n, \quad |k-n| \le n^{1/2}. \qquad (3.15)$$

From (3.13) and (3.12) we find

$$(J_k^{(n)})^2 + (J_{k+1}^{(n)})^2 = \int_{\sigma_e}\lambda^2(\psi_k^{(n)}(\lambda))^2 d\lambda = 2 + a_k,$$
$$((J_k^{(n)})^2 + (J_{k+1}^{(n)})^2)^2 \qquad (3.16)$$
$$+(J_{k+1}^{(n)})^2(J_{k+2}^{(n)})^2 + (J_k^{(n)})^2(J_{k-1}^{(n)})^2 = \int_{\sigma_e}\lambda^4(\psi_k^{(n)}(\lambda))^2 d\lambda = \oint_L R_{k,k}(\zeta)\frac{\zeta^4 d\zeta}{2\pi i} = 6 + 2a_k + O(\frac{\log n}{n^{1/2}}).$$

Using here the first equation for $k := k\pm 1$ to express $(J_{k\pm1}^{(n)})^2$ and $(J_{k+2}^{(n)})^2$ through $(J_k^{(n)})^2$, we obtain

$$J_k^2 = 1 + \frac{a_k}{2} + \frac{a_{k+1}-a_{k-1}}{4}$$
$$\pm\Big[\frac{a_{k+1}+2a_k+a_{k-1}}{2} + O(\frac{\log n}{n^{1/2}})\Big]^{1/2}. \qquad (3.17)$$

Combining this relation with (3.15), we get the first statement of Lemma 1. The second statement follows from the first one and the first equation of (3.17).

□

Proof of Lemma 2. Relation (2.9) can be written as

$$d_k^{(2)} = 2x_k^3 + \tilde r_k, \quad \tilde r_k = x_k k(2P_0(0)n)^{-1} + r_k, \quad |\tilde r_k| \le C_*(\tilde m_k|k|/n + \tilde m_k^4), \quad (3.18)$$

where C_* is independent of N, n and we always can choose $C_* > 1$. If $\tilde m_k < k^{-1}$ for all $k > n^{1/3}$, then (2.10) is fulfilled. If $\tilde m_k > k^{-1}$ for some $k > 4n^{1/3}$, we can apply Proposition 1 to $\{x_j\}_{|j|\le M}$, with $M = k$, $M_1 = [n^{1/3}/2]$, $\varepsilon^3 = C_*\left(\tilde m_{k+2M_1}(k+2M_1)/n + m_{k+2M_1}^4\right)$, $\varepsilon_1 = m_{k+2M_1}$, because $M_1 > 2/3\varepsilon^{-1}$. Then, since

$$2\varepsilon_1 M_1^{-2} = 8\tilde m_{k+2M_1} n^{-2/3} < C_*\tilde m_{k+2M_1}(k+2M_1)/n < \varepsilon^3,$$

we obtain by (2.13) that

$$8\varepsilon^3 = 8C_*\left(\tilde m_{k+2M_1}(k+2M_1)/n + m_{k+2M_1}^4\right) \ge m_k^3.$$

Therefore at least one of the following inequalities holds

$$8C_*\tilde m_{k+2M_1}(k+2M_1)/n \ge m_k^3/2 \vee 8C_* m_{k+2M_1}^4 \ge m_k^3/2 \quad (3.19)$$

Since according to Lemma 1 $|m_{k+2M_1}| \le Cn^{-1/8}\log^{1/4} n$ the second inequality yields

$$\tilde m_{k+2M_1} > 2\tilde m_k \quad (3.20)$$

If the second inequality in (3.19) is false, then the first one holds. Write it as

$$\tilde m_{k+2M_1} \ge (16C_*)^{-1}\tilde m_k\left(\tilde m_k^2 n/k\right)\left(k/(k+n^{1/3})\right). \quad (3.21)$$

Assume that for some $k > 4n^{1/3}$

$$\tilde m_k^2 n/k \ge 40C_*. \quad (3.22)$$

Then (3.21)implies (3.20) and

$$\tilde m_{k+2M_1}^2 n/(k+2M_1) \ge 4\left(\tilde m_k^2 n/k\right)\cdot(k/(k+2M_1)) > 32C_*.$$

Hence, we can repeat this procedure l times with $l = [\log n]$. Then we obtain the inequality

$$\tilde m_{k+[\log n]M_1} > 2^{[\log n]}\tilde m_k,$$

which contradicts to Lemma 1. Thus, (3.22) is false and we have proved (2.10).

To prove (2.11) take any $k_0 > n^{1/3}$ denote $\tilde x_k = x_{k-2k_0}$ and, taking into account (2.9), apply (2.15) with $M = k_0$. Then since $f_k > (k_0/2P_0(0)n)$ we obtain (2.11).

□

Proof of Lemma 4 Choose $M = cn^{1/3}$, where the constant c is small enough to provide the condition

$$dC_2c < C_1/7, \quad (3.23)$$

where C_1 and C_2 are the constants from (A.5) and $d = \pi(2+\varepsilon)^{-1}$. This condition and (A.5) guarantee that for any $l, l' : |l-l'| > n^{1/3}/6$ and any $j : |j| < M$, $|t| \le 1$

$$|(e^{itdj\mathcal{J}^{(0)}})_{l,l'}| \le Ce^{dC_2M - C_1|l-l'|/4} \le Ce^{-C_1 n^{1/3}/42}. \quad (3.24)$$

Applying (A.3) three times we get (2.28) with

$$\mathcal{P}_{k-l}^{(\delta)} = c_1^{(\delta)}\delta_{k,l} + \int_{s_1+s_2=1} ds_1 ds_2 \sum_{j=\infty}^{\infty} v_j(ijd)\left(e^{ijds_1\mathcal{J}^{(0)}}E^{(l)}e^{ijds_2\mathcal{J}^{(0)}}\right)_{k,k+\delta},$$

$$\mathcal{P}_{l_1,l_2}^{(2,k,\delta)} = \int_{\sum s_i=1} ds_1 ds_2 ds_3 \sum_{j=-M}^{M} v_j(ijd)^2\left(e^{ijds_1\mathcal{J}^{(0)}}E^{(l_1)}e^{ijds_2\mathcal{J}^{(0)}}E^{(l_2)}e^{ijds_3\mathcal{J}^{(0)}}\right)_{k,k+\delta},$$

$$\mathcal{P}_{l_1,l_2,l_3}^{(3,k,\delta)} = \int_{\sum s_i=1} ds_1\dots ds_4 \sum_{j=-M}^{M} v_j(ijd)^3\left(e^{ijds_1\mathcal{J}^{(0)}}E^{(l_1)}e^{ijds_2\mathcal{J}^{(0)}}E^{(l_2)}e^{ijds_3\mathcal{J}^{(0)}}E^{(l_3)}e^{ijds_4\mathcal{J}^{(0)}}\right)_{k,k+\delta}, \quad (3.25)$$

$$r_k^{(\delta)} = \sum_{l_1,\dots,l_4}\int_{\sum s_i=1} ds_1\dots ds_5 \sum_{j=-M}^{M} v_j(ijd)^4\left(e^{ijds_1\mathcal{J}^{(0)}}\tilde{\mathcal{J}}e^{ijds_2\mathcal{J}^{(0)}}\tilde{\mathcal{J}}e^{ijds_3\mathcal{J}^{(0)}}\tilde{\mathcal{J}}e^{ijds_4\mathcal{J}^{(0)}}\tilde{\mathcal{J}}e^{ijds_5(\mathcal{J}^{(0)}+\tilde{\mathcal{J}})}\right)_{k,k+\delta}$$
$$+ \int_{s_1+s_2=1} ds_1 ds_2 \sum_{|j|>M} v_j(ijd)\left(e^{ijds_1\mathcal{J}^{(0)}}\tilde{\mathcal{J}}(e^{ijds_2\mathcal{J}^{(0)}} - e^{ijds_2(\mathcal{J}^{(0)}+\tilde{\mathcal{J}})})\right)_{k,k+\delta},$$

where we denote by $E^{(l)}$ a matrix with entries:

$$E_{k,m}^{(l)} = \delta_{k,l}\delta_{m,l+1} + \delta_{k,l+1}\delta_{m,l}.$$

Using the Schwartz inequality, we have

$$\sum_j |j|^4|v_j| \le \left(\sum_j |j|^{10}|v_j|^2\right)^{1/2}\left(\sum_{j\ne 0}|j|^{-2}\right)^{1/2} \le C,$$

Hence, using again the Schwartz inequality, we obtain

$$|r_k^{(\delta)}| \le m_k^4 d^4 \sum_{|j|<M} |j|^4|v_j| + m_k d \sum_{|j|>M} |j||v_j| \le Cm_k^4 + Cm_k M^{-7/2} \le Cm_k^4, \quad (3.26)$$

where the last inequality is valid because of the choice of M and (2.4).

To obtain (2.29) we use the representation (see [1]):

$$(e^{ijds\mathcal{J}^{(0)}})_{k,l} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ijds\cos x} e^{i(k-l)x} dx = \mathbf{J}_{k-l}(jds), \quad (3.27)$$

where $\mathbf{J}_k(s)$ is the Bessel function. But it is well known (see, e.g. [1]) that the Bessel functions satisfy the following recurrent relations:

$$k\mathbf{J}_k(s) = \frac{s}{2}\Big(\mathbf{J}_{k+1}(s) + \mathbf{J}_{k-1}(s)\Big).$$

Thus, e.g., the first sum in (2.29) can be expressed via the terms

$$\sum_{|j|<M} v_j(ijd)^2 \int_{\sum s_i=1} ds_1 ds_2 ds_3 (djs_1+\alpha_1)(djs_1+\alpha_2)$$
$$\cdot {\sum}' \mathbf{J}_{k-l_1+\alpha_3}(djs_1)\tilde{x}_{l_1}\mathbf{J}_{l_1-l_2+\alpha_4}(djs_2)\tilde{y}_{l_2}\mathbf{J}_{l_2-k+\alpha_5}(2js_3),$$

where $\alpha_1,\dots\alpha_5$ can take the values $0,\pm1,\pm(\delta+1)$. It is easy to see that any of these sums can be written in the form:

$$(e^{ijds_2\mathcal{J}^{(0)}} X^{(\alpha_3)} e^{ijds_2\mathcal{J}^{(0)}} Y^{(\alpha_4)} e^{ijds_2\mathcal{J}^{(0)}})_{k,k+\alpha_5},$$
$$X^{(\alpha)}_{k,l} = \delta_{k,\alpha+l}\tilde{x}_k, \quad Y^{(\alpha)}_{k,l} = \delta_{k,\alpha+l}\tilde{y}_k,$$

where evidently

$$||X^{(\alpha)}|| \le \max|\tilde{x}_k|, \quad ||Y^{(\alpha)}|| \le \max|\tilde{y}_k|.$$

Hence, similarly to (3.26) we obtain

$$\Big|{\sum}' \mathcal{P}^{(2,k,\delta)}_{l_1,l_2}(l_1-k)^2\tilde{x}_{l_1}\tilde{y}_{l_2}\Big| \le C||\tilde{x}||_0\,||\tilde{y}||_0 \sum_{|j|<M}|j|^4|v_j| \le C||\tilde{x}||_0\,||\tilde{y}||_0.$$

The other inequalities in (2.29) can be proved similarly.

We are left to prove (2.31). Due to representations (3.25) and (3.27), we derive that $\mathcal{P}^{(\delta)}$ can be represented in the form (2.30) with

$$F^{(\delta)}(x) = \sum \mathcal{P}^{(\delta)}_l e^{ilx}$$

Using (3.25) and (3.27) we get

$$\begin{aligned} F^{(1)}(x) &= c_1^{(1)} + \sum_j (ijd)v_j \int_0^1 ds_1 \sum_l \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{il(-x_1+x_2+x)}(1+e^{-i(x_1+x_2)}) \\ &\quad\cdot \exp\{2ijd[s_1\cos x_1 + (1-s_1)\cos x_2]\}dx_1dx_2 \\ &= c_1^{(1)} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{v(2\cos x_1) - v(2\cos(x_1-x))}{\cos x_1 - \cos(x_1-x)}(1+\cos(2x_1-x))dx_1 \\ &= c_1^{(1)} + \frac{1}{2\pi}\int_{-\pi}^{\pi} v(2\cos x_1)\Big(\frac{1+\cos(2x_1-x)}{\cos x_1-\cos(x_1-x)} + \frac{1+\cos(2x_1+x)}{\cos x_1 - \cos(x_1+x)}\Big)dx_1 \\ &= P(2\cos(x/2)) + P(-2\cos(x/2)). \end{aligned} \quad (3.28)$$

Representation (2.32) can be obtained similarly. Lemma 4 is proven.

□

Lemma 4 combined with Lemma 2 give us a useful corollary

Corollary 2. *For any even function $\phi(\lambda)$ which has three bounded derivatives on $[-2+\varepsilon, 2+\varepsilon]$*

$$\left|\phi(\mathcal{J}^{(n)})_{n+k,n+k} - \frac{1}{\pi}\int_{-2}^{2}\frac{\phi(\lambda)d\lambda}{\sqrt{4-\lambda^2}}\right| \le C\left((|k|/n) + n^{-2/3}\right). \quad (3.29)$$

□

Proof of Proposition 4. Let us remark first that all limiting expression in the r.h.s. of (2.41) and (2.42) correspond to infinite sums over j in the definitions (3.25) and infinite sums with respect to all l_i. The estimates for the remainder terms, which appears because of the restriction of summation in (3.25) over $|j| < M$, were obtained already in the proof of Lemma 4. And the remainders, which appear because of the replacement of infinite sums by sums over $|l_i| < N_k$, can be estimated by $O(e^{-C_1 n^{1/3}/12})$ due to (3.24). Thus we are left to compute infinite over l_i sums for $\mathcal{P}^{(2,k,\delta)}_{l_1,l_2}$ and $\mathcal{P}^{(3,k,\delta)}_{l_1,l_2,l_3}$

The first relation in (2.41) follows immediately from (2.30) and (3.28). To obtain the others let us consider an infinite Jacobi matrix $\mathcal{J}^{(\pi)}$ with $J^{(\pi)}_{k,k-1} = J^{(\pi)}_{k-1,k} = (-1)^k$

and define

$$V_k(a,b) = V'(a\mathcal{J}^0 + b\mathcal{J}^{(\pi)})_{k,k+1}$$
$$= \frac{a}{2\pi(a^2-b^2)}\int_\sigma V'(\lambda)\,\mathrm{sign}\lambda\frac{(\lambda^2-4b^2)^{1/2}}{(4a^2-\lambda^2)^{1/2}}d\lambda$$
$$+\frac{(-1)^k b}{2\pi(a^2-b^2)}\int_\sigma V'(\lambda)\,\mathrm{sign}\lambda\frac{(4a^2-\lambda^2)^{1/2}}{(\lambda^2-4b^2)^{1/2}}d\lambda. \quad (3.30)$$

It is easy to see, e.g., that

$$\sum_{l_1,l_2}\mathcal{P}^{(2,k,1)}_{l_1,l_2}(-1)^{l_1+l_2} = \frac{1}{2}\frac{\partial^2}{\partial b^2}V_k(a,b)\Big|_{a=1,b=0}$$
$$= \frac{1}{2\pi}\int_{-2}^{2}\frac{\lambda V'(\lambda)d\lambda}{\sqrt{4-\lambda^2}} - \frac{1}{\pi}\int_{-2}^{2}\frac{V'(\lambda)d\lambda}{\lambda\sqrt{4-\lambda^2}} = 1.$$

Here we have used (1.14) and (2.33).

Similarly

$$\sum_{l_1,l_2}\mathcal{P}^{(2,k,1)}_{l_1,l_2}(-1)^{l_1} = \frac{1}{2}\frac{\partial^2}{\partial a\partial b}V_k(a,b)\Big|_{a=1,b=0} = (-1)^k.$$

To compute the sum for $\mathcal{P}^{(3,k,1)}_{l_1,l_2,l_3}$ let us observe that

$$\sum_{l_1,l_2,l_3}\mathcal{P}^{(3,k,1)}_{l_1,l_2,l_3}(-1)^{l_1+l_2+l_3} = \frac{1}{6}\frac{\partial^3}{\partial b^3}V_k(1,b)\Big|_{b=0}$$
$$= \frac{(-1)^k}{2}\frac{\partial^2}{\partial b^2}\frac{1}{(1-b^2)}I(b)\Big|_{b=0},$$

where

$$I(b) = \frac{1}{2\pi}\int_\sigma (V'(\lambda)-\lambda V''(0))\,\mathrm{sign}\lambda\frac{(4-\lambda^2)^{1/2}}{(\lambda^2-4b^2)^{1/2}}d\lambda$$
$$= \frac{1}{2\pi}\int_\sigma V'(\lambda)\,\mathrm{sign}\lambda\frac{(4-\lambda^2)^{1/2}}{(\lambda^2-4b^2)^{1/2}}d\lambda - V''(0)(1-b^2).$$

Differentiating this expression, one can easily get the expression of (2.41).

To prove the last relation in (2.41) we use the symmetry arguments. Indeed, according to (3.25),

$$h(k,l_1) = \sum_{l_2}(\mathcal{P}^{(2,k,1)}_{l_1,l_2} + \mathcal{P}^{(2,k,1)}_{l_2,l_1})(-1)^{l_1+l_2} = \sum_j v_j(ijd)^2\int_{s_1+s_2+s_3=1} ds_1ds_2ds_3\Big(u_{s_1}(k-l_1)f_{s_2,s_3}(k-l_1)$$
$$+u_{s_1}(k-l_1-1)f_{s_2,s_3}(l_1-k_1-1) + u_{s_1}(k-l_1)f_{s_1,s_2}(k-l_1) + u_{s_1}(k-l_1+1)f_{s_1,s_2}(l_1-k_1+1)\Big),$$

where

$$u_{s_1}(k-l) = (e^{ijds_1\mathcal{J}^{(0)}})_{k-l},$$
$$f_{s_2,s_3}(l-k) = (-1)^l(e^{ijds_2\mathcal{J}^{(0)}}\mathcal{J}^{(\pi)}e^{ijds_3\mathcal{J}^{(0)}})_{l,k}.$$

Since both $u_{s_1}(k-l)$ and $f_{s_2,s_3}(l-k)$ are even functions with respect to $(l-k)$, after integration with respect to s_1,s_2,s_3 we get that

$$h(k,l_1) = h(k-l_1) = h(l_1-k) \Rightarrow \sum_{l_1}h(k,l_1)(k-l_1) = 0.$$

To prove (2.42) we define similarly to (3.30)

$$V^{(0)}_k(b) = v^{(0)}(\mathcal{J}^{(0)} + b\mathcal{J}^{(\pi)})_{k,k}$$
$$= \frac{1}{\pi}\int_\sigma\frac{V'(\lambda)\ \mathrm{sign}\lambda}{(4b^2-\lambda)^{1/2}X(\lambda)}d\lambda$$
$$= \frac{1}{\pi}\int_\sigma\frac{(V'(\lambda)-\lambda V''(0))\ \mathrm{sign}\lambda}{(4b^2-\lambda)^{1/2}X(\lambda)}d\lambda + V''(0).$$

Then

$$\sum_{l_1,l_2}\mathcal{P}^{(2,k,0)}_{l_1,l_2}(-1)^{l_1+l_2} = \frac{1}{2}\frac{\partial^2}{\partial b^2}V^0_k(1,b)\Big|_{b=0}$$
$$= \frac{2}{\pi}\int_{-2}^{2}\frac{(V'(\lambda)-\lambda V''(0))d\lambda}{\lambda^3X(\lambda)} = 2P_0(0).$$

□

Proof of Lemma 3.

Substituting in (3.5) $\phi(\lambda) = (\lambda-z)^{-1}$ we get easily the equation

$$g^2_{n+k,n}(z) + \int\frac{V'(\lambda)}{\lambda-z}\rho_{n+k,n}(\lambda)d\lambda = \delta_{n+k,n}(z) \quad (3.31)$$

with $\delta_{n+k,n}(z)$ of the form (cf. (3.6))

$$\delta_{n+k,n}(z) = \frac{(J^{(n)}_{n+k})^2}{n^2}\big((R^2)_{n+k,n+k}(z)(R^2)_{n+k-1,n+k-1}(z)$$
$$-(R^2)^2_{n+k,n+k-1}(z)\big), \quad (3.32)$$

where $(R^2)_{k,j} = (z-\mathcal{J}^{(n)})^{-2}_{k,j}$. Here we have used the Christoffel-Darboux formula in the numerator of the integrand in (2.22). Let us define

$$F(z) := \int \frac{V'(\lambda)}{\lambda - z}\rho_{n+k,n}(\lambda)d\lambda \\ -g_{n+k,n}(z)\left(zV''(0) + z^3\frac{V^{(4)}(0)}{6}\right) \\ = \int \frac{V'(\lambda) - \lambda V''(0) - \frac{1}{6}\lambda^3 V^{(4)}(0)}{\lambda^4(z-\lambda)}\rho_{n+k,n}(\lambda)d\lambda \\ + \left(V''(0) + z^2\frac{V^{(4)}(0)}{6}\right)\int \rho_{n+k,n}(\lambda)d\lambda \\ + \frac{1}{6}\lambda^3 V^{(4)}(0)\int \lambda^2\rho_{n+k,n}(\lambda)d\lambda.$$

Using in the first integral here the representation

$$\frac{1}{\lambda - z} = \frac{1}{\lambda} + \frac{z}{\lambda^2} + \frac{z^2}{\lambda^3} + \frac{z^3}{\lambda^4} + \frac{z^4}{\lambda^4(\lambda - z)}$$

and taking into account the evenness of the functions $\rho_{n+k,n}$ and V, we get

$$F(z) = \int \frac{V'(\lambda)}{\lambda}\rho_{n+k,n}(\lambda)d\lambda \\ + z^2\int \frac{V'(\lambda) - \lambda V''(0)}{\lambda^3}\rho_{n+k,n}(\lambda)d\lambda \\ + z^4\int \frac{V'(\lambda) - \lambda V''(0) - \frac{1}{6}\lambda^3 V^{(4)}(0)}{\lambda^4(z-\lambda)}\rho_{n+k,n}(\lambda)d\lambda \\ = c^{(0)}_{k,n} + z^2 c^{(2)}_{k,n} + z^4 c^{(4)}_{k,n}(z). \tag{3.33}$$

Denote

$$Q(\lambda) = \int \frac{V'(\lambda) - V'(\mu)}{\lambda - \mu}\rho(\mu)d\mu.$$

Taking the limit $n \to \infty$ in (3.31) and using (1.13), we get for any $\lambda \in [-2, 2]$

$$\lambda^4 P_0^2(\lambda)(\lambda^2 - 4) = [V'(\lambda)]^2 - 4Q(\lambda) \\ \Rightarrow Q(\lambda) = \frac{1}{4}\left([V'(\lambda)]^2 + \lambda^4 P_0^2(\lambda)(4 - \lambda^2)\right). \tag{3.34}$$

Therefore, denoting $v^{(0)}(\lambda) = V'(\lambda)\lambda^{-1}$, we get

$$c^{(0)}_{k,n} = Q(0) + \tilde{c}^{(0)}_n + c_k = \tilde{c}^{(0)}_n + c_k, \tag{3.35}$$

where

$$\tilde{c}^{(0)}_n = \int v^{(0)}(\lambda)(\rho_{n,n}(\lambda) - \rho(\lambda))d\lambda, \\ c_k = \pm n^{-1}\sum_{j=1}^{|k|} v^{(0)}(\mathcal{J}^{(n)})_{n\pm j, n\pm j}. \tag{3.36}$$

Here and below in the proof of Lemma 3 the sign $\pm$ corresponds to the sign of k. Repeating the argument of Lemma 4 for the function $v^{(0)}(\lambda)$, we obtain

$$v^{(0)}(\mathcal{J}^{(n)})_{n\pm j, n\pm j} = x_j^2 \sum_{|l_1|,|l_2|<|k|+n^{1/3}} \mathcal{P}^{(2,k,0)}_{l_1,l_2}(-1)^{l_1+l_2} \\ + y_j \sum_{|l_1|<|k|+n^{1/3}} \mathcal{P}^{(0)}_{l_1} + O\left((k/n)^{3/2}\right) \\ = 2P_0(0)x_j^2 + \frac{j}{2n} + O\left((k/n)^{3/2}\right).$$

Thus, using (2.42), we get (2.20). Now let us observe that

$$\int \frac{V'(\lambda) - \lambda V''(0)}{\lambda^3}\rho(\lambda)d\lambda = \frac{1}{2}\frac{d^2}{d\mu^2}Q(\mu)\Big|_{\mu=0} = \frac{1}{4}(V''(0))^2.$$

Hence,

$$c^{(2)}_{k,n} = \frac{1}{4}(V''(0))^2 + \tilde{c}^{(2)}_n \pm n^{-1}\sum_{j=1}^{|k|} v^{(2)}(\mathcal{J}^{(n)})_{n\pm j, n\pm j},$$

where

$$v^{(2)}(\lambda) = (V'(\lambda) - \lambda V''(0))\lambda^{-3}, \\ \tilde{c}^{(2)}_n = \int v^{(2)}(\lambda)(\rho_{n,n}(\lambda) - \rho(\lambda))d\lambda.$$

Using Corollary 2 from Lemma 2, we get

$$v^{(2)}(\mathcal{J}^{(n)})_{n+k,n+k} = \int \frac{v^{(2)}(\lambda)d\lambda}{\sqrt{4-\lambda^2}} + O(|k|/n) + O(n^{-2/3}) \\ = P_0(0) + O(|k|/n) + O(n^{-2/3}),$$

since

$$\int \frac{v^{(2)}(\lambda)d\lambda}{\sqrt{4-\lambda^2}} = \frac{1}{2}\frac{d^2}{d\mu^2}\int \frac{(V'(\lambda) - V'(\mu))d\lambda}{(\lambda-\mu)\sqrt{4-\lambda^2}}\Big|_{\mu=0} = P_0(0).$$

Therefore

$$c^{(2)}_{k,n} = \frac{1}{4}(V''(0))^2 + P_0(0)\frac{k}{n} + O\left(k^2/n^2\right) + \tilde{c}^{(2)}_n. \tag{3.37}$$

Now we apply (3.7) to

$$v^{(4)}(\lambda, z) = \frac{V'(\lambda) - \lambda V''(0) - \frac{1}{6}\lambda^3 V^{(4)}(0)}{\lambda^4(z-\lambda)}.$$

We get

$$c^{(4)}_{k,n}(z) = \int v^{(4)}(\lambda, z)\rho(\lambda)d\lambda \tag{3.38} \\ \pm n^{-1}\sum_{j=1}^{|k|} v^{(4)}(\mathcal{J}^{(n)}, z)_{n\pm j, n\pm j} + O\left(\frac{|z|\log^{1/2} n}{|\Im z|^2 n^{1/2}}\right) \\ = \frac{Q^{(4)}(0)}{4!} + O(z) + O\left(\frac{|z|\log^{1/2} n}{|\Im z|^2 n^{1/2}}\right) + O\left(\frac{|z|k}{|\Im z|n}\right) \\ = \frac{1}{4!}\left(24P_0^2(0) + 2V'(0)V''(0)\right) + O(z) + O\left(\frac{|z|\log^{1/2} n}{|\Im z|^2 n^{1/2}}\right),$$

where the last equality follows from (3.34). Collecting (3.35)-(3.38) we obtain from (3.31) that for $\Im z \ge n^{-1/3}\log^{-1/2} n$

$$g_{n+k,n}^2(z) + g_{n+k,n}(z)\left(zV''(0) + z^3\frac{V^{(4)}(0)}{6}\right) + c_k + z^2\left(\frac{1}{4}(V''(0))^2 + \frac{k}{n}P_0(0)\right) + z^4\left(P_0^2(0) + \frac{V''(0)V^{(4)}(0)}{12}\right) = \delta_{n+k,n}(z) - \tilde{c}_n^{(0)} - z^2\tilde{c}_n^{(2)} + z^2 O\left((k/n)^{3/2}\right) + |z|^5|\Im z|^{-2} O\left(n^{-1/2}\log^{1/2} n\right).$$

Solving this quadratic equation, we get

$$g_{n+k,n}(z) = -\frac{1}{2}\left(zV''(0) + z^3\frac{V^{(4)}(0)}{6}\right) + \Big(-P_0^2(0)z^4 - \frac{k}{n}P_0(0)z^2 - c_k + \delta_{n+k,n}(z) - \tilde{\delta}_{n+k,n}(z)\Big)^{1/2} \quad (3.39)$$

with c_k defined by (2.20), $\delta_{n+k,n}(z)$ defined by (3.32) and

$$\tilde{\delta}_{n+k,n}(z) = \tilde{c}_n^{(0)} + z^2\tilde{c}_n^{(2)} + z^2\left(O(n^{-1}) + O\left((k/n)^{3/2}\right)\right) + O(z^5) + |z|^5|\Im z|^{-2}O\left(n^{-1/2}\log^{1/2} n\right). \quad (3.40)$$

Since (2.19) follows from (3.39), we are left to estimate $\tilde{c}_n^{(0)}$, $\tilde{c}_n^{(2)}$ and $\delta_{n+k,n}(z)$.

Taking into account (3.32), to estimate $\delta_{n+k,n}(z)$ we need to estimate $(R^2)_{n+k,n+k}$ and $(R^2)_{n+k,n+k-1}$. Let us take $N' = k + \log^2 n n^{1/3}$, and consider $\tilde{\mathcal{J}}(N')$ defined by (2.26) and

$$R^{(1)}(z) = (z - \mathcal{J}^{(0)} - \tilde{\mathcal{J}}(N'))^{-1}.$$

Then, using the resolvent identity

$$H_1^{-1} - H_2^{-1} = H_1^{-1}(H_2 - H_1)H_2^{-1} \quad (3.41)$$

and (2.79), we get for any $z : \Im z > n^{-1/3}$

$$|(R^2)_{n+k,n+k}(z) - (R^{(1)}R^{(1)})_{k,k}(z)| \le Ce^{-C\log^2 n}. \quad (3.42)$$

Applying the resolvent identity (3.41) to $R^{(0)} = (z - \mathcal{J}^{(0)})^{-1}$ and $R^{(1)}(z)$ defined above, we get

$$|R^{(1)}_{k,k}(z) - R^{(0)}_{k,k}(z)| \le |R^{(0)}(z)\tilde{J}(N')R^{(1)}(\overline{z}))_{k,k}| \le \frac{(|k|/n)^{1/2} + n^{-1/3}}{|\Im z|^2}.$$

Now, using the Cauchy theorem and the above inequality, we obtain

$$|(R^{(1)}R^{(1)})_{k,k}(z) - (R^{(0)}R^{(0)})_{k,k}(z)| = \left|\frac{1}{2\pi i}\oint_{|z-\zeta|=|\Im z|/2}\frac{R^{(1)}_{k,k}(\zeta) - R^{(0)}_{k,k}(\zeta)}{(\zeta - z)^2}d\zeta\right| \le C\frac{(|k|/n)^{1/2} + n^{-1/3}}{|\Im z|^3}. \quad (3.43)$$

Moreover,

$$(R^{(0)}R^{(0)})_{k,k}(z) = -\frac{z}{(z^2-4)^{3/2}},$$

$$(R^{(0)}R^{(0)})_{k,k+1}(z) = -\frac{2}{(z^2-4)^{3/2}}$$

are bounded for $|z| \le 1$. Hence, substituting (3.43) in (3.32), we get the first estimate in (2.22).

To estimate $\tilde{c}_n^{(0)}$ and $\tilde{c}_n^{(2)}$ we subtract from (3.31) the same equation for $k := k-1$ and multiply the result by n (see the proof of Lemma 1 for the details). Then we get

$$2g_{n+k,n}(z)R_{n+k,n+k}(z) = \int\frac{V'(\lambda)}{z-\lambda}(\psi^{(n)}_{n+k}(\lambda))^2 d\lambda - \delta^{(R)}_{n+k,n}(z), \quad (3.44)$$

where

$$\delta^{(R)}_{n+k,n}(z) = \frac{1}{n}\sum_{j=1}^{\infty}R_{n+k,j}(z)R_{j,n+k}(z) - \frac{2}{n}\sum_{j=1}^{n+k}R_{n+k,j}(z)R_{j,n+k}(z).$$

Using the same trick as in (3.42), we get

$$\delta^{(R)}_{k,n}(z) = \frac{1}{n}\sum_{j=1}^{\infty}R^{(1)}_{k,j}(z)R^{(1)}_{j,k}(z) - \frac{2}{n}\sum_{j=1}^{k}R^{(1)}_{k,j}(z)R^{(1)}_{j,k}(z) + O(e^{-C\log^2 n}).$$

Besides, since $R^{(0)}_{k,j}(z)$ is an even function of $(j-k)$, we observe that

$$0 = \frac{1}{n}\sum_{j=-\infty}^{\infty}R^{(0)}_{k,j}(z)R^{(0)}_{j,k}(z) - \frac{2}{n}\sum_{j=-\infty}^{k}R^{(0)}_{k,j}(z)R^{(0)}_{j,k}(z) + \frac{1}{n}(R^{(0)}_{k,k}(z))^2,$$

Hence, to estimate $\delta^{(R)}_{n+k,n}(z)$ it is enough to estimate the difference between r.h.s. of the last two formulas. Using that for $|z| \le 1$

$$|R^{(0)}_{j,k}| \le Ce^{-|\Im z||j-k|},$$

we get

$$|\delta^{(R)}_{n+k,n}(z)| \le \frac{C}{n^{3/2}|\Im z|^{5/2}}. \quad (3.45)$$

Now performing transformations (3.33) for the integral in the r.h.s. of (3.44), we can rewrite it as

$$R_{n+k,n+k}(z)(2g_{n+k,n}(z) - V''(0)z - V^{(4)}(0)z^3/6) = a^{(2)}_{k,n}z^2 - a^{(0)}_{k,n} + \delta^{(R)}_{n+k,n}(z) + O(z^4), \quad (3.46)$$

where

$$a_{k,n}^{(0)} = v^{(0)}(\mathcal{J}^{(n)})_{n+k,n+k} = 2n^{-2/3}P_0(0)q_n^2(\frac{k}{n^{1/3}}) + \frac{k}{2n} + O(n^{-1}),$$
$$a_{k,n}^{(2)} = v^{(2)}(\mathcal{J}^{(n)})_{n+k,n+k} = P_0(0) + O(n^{-2/3}).$$

Let us take $k > 0$ and change the variable $z = \tilde{\varepsilon}\zeta$ with $\tilde{\varepsilon}^2 = k/P_0(0)n$ in (3.46). Then, using (3.39), we obtain from (3.46)

$$R_{n+k,n+k}(\tilde{\varepsilon}\zeta) = \frac{\zeta^2 + \frac{1}{2} + 2P_0(0)\frac{n^{1/3}}{k}q_n^2(\frac{k}{n^{1/3}}) + \tilde{\varepsilon}^{-2}P_0^{-1}(0)\delta_{n+k,n}^{(R)}(\tilde{\varepsilon}\zeta) + o(1)}{2i\Big(\zeta^4 + \zeta^2 + \tilde{\varepsilon}^{-4}P_0^{-2}(0)\left(c_k + \tilde{\delta}_{n+k,n}(\tilde{\varepsilon}\zeta) - \delta_{n+k,n}(\tilde{\varepsilon}\zeta)\right)\Big)^{1/2}} := \frac{R_1(\zeta)}{R_2^{1/2}(\zeta)}. \quad (3.47)$$

In view of (3.45)

$$\tilde{\varepsilon}^{-2}|\delta_{k,n}^{(R)}(\tilde{\varepsilon}\zeta)| \to 0, \quad as \quad \frac{k}{n^{1/3}} \to \infty, \quad \Im\zeta > d$$

with any fixed d (see (3.45)). Besides, $q_n^2(x) \to 0$, as $x \to \infty$, because of (2.11). Therefore there exists some fixed $l_0 > 0$, such that for $k > l_0 n^{1/3}$ and any $\zeta : \Im\zeta > 1/4$

$$\left|2P_0(0)\frac{n^{1/3}}{k}q_n^2(\frac{k}{n^{1/3}}) + \tilde{\varepsilon}^{-2}P_1^{-1}(0)\delta_{n+k,n}^{(R)}(\tilde{\varepsilon}\zeta)\right| < \frac{1}{4} < \min_{|\zeta - i/\sqrt{2}|=1/4}\left|\zeta^2 + \frac{1}{2}\right|.$$

Then according to the Rouchet theorem $R_1(\zeta)$ has a root inside the circle $\mathcal{B}$ of radius $1/4$ centered at $i/\sqrt{2}$. Thus, if $R_2(\zeta)$ has no roots of the second order inside $\mathcal{B}$, then similarly to the proof of Lemma 1 we obtain a contradiction with (3.14). Therefore, using the first inequality of (2.22), (3.40) and (3.45) we conclude that there exists an absolute constant C_0, such that

$$|\tilde{c}_n^{(0)}|, |\tilde{c}_n^{(2)}|^2 \le C_0 n^{-4/3}.$$

These bounds and (3.40) prove the second estimate of (2.22).

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APPENDIX A: SOME PROPERTIES OF JACOBI MATRICES

Proof of Proposition 3 Using the spectral theorem and Proposition 8, we get

$$\left|v(\mathcal{J}^{(n)})_{n+k,n+k-\delta} - \tilde{v}(\mathcal{J}^{(n)})_{n+k,n+k-\delta}\right| = \left|\int (v(\lambda) - \tilde{v}(\lambda))\psi_{n+k}^{(n)}(\lambda)\psi_{n+k-\delta}^{(n)}(\lambda)d\lambda\right| \le Ce^{-nC\varepsilon}.$$

Let us represent $\tilde{v}(\lambda)$ by its Fourier expansion

$$\tilde{v}(\lambda) = \sum_{j=-\infty}^{\infty} v_j e^{ijd\lambda}, \quad d = \frac{\pi}{2+\varepsilon}. \quad (A.1)$$

Then we have

$$\tilde{v}(\mathcal{J}^{(n)}) = \sum_{j=-\infty}^{\infty} v_j e^{ijd\mathcal{J}^{(n)}} = \sum_{|j|\le cM} v_j e^{ijd\mathcal{J}^{(n)}} + O(M^{-\ell+1/2}), \quad (A.2)$$

where c is some absolute constant which we will choose later. The bound for the remainder term in the last formula follows from the estimate

$$\left\|\sum_{|j|>cM} v_j e^{ijd\mathcal{J}^{(n)}}\right\| \le \sum_{|j|>cM}|v_j| \le \Big(\sum_{|j|>cM}|v_j|^2|j|^{2\ell}\Big)^{1/2}\Big(\sum_{|j|>cM}|j|^{-2\ell}\Big)^{1/2} \le ||v^{(\ell)}||_2(cM)^{-\ell+1/2}.$$

Consider now $N' = [N+M]+1$ and denote by $\mathcal{J}^{(n,N')}$ the matrix whose entries coincide with that of $\mathcal{J}^{(n)}$ with the only exception $\mathcal{J}^{(n,N')}_{n\pm N',n\pm N'+1} = 0$. We will use the Duhamel formula, valid for any matrices $\mathcal{J}_1, \mathcal{J}_2$

$$e^{it\mathcal{J}_2} - e^{it\mathcal{J}_1} = \int_0^t e^{i(t-s)\mathcal{J}_1}(\mathcal{J}_2 - \mathcal{J}_1)e^{is\mathcal{J}_2}ds \quad (A.3)$$

Let us take $|k| < N$ and apply (A.3) to $\mathcal{J}^{(n)}_{n+k,n+k-\delta}$ and $\mathcal{J}^{(n,N')}_{n+k,n+k-\delta}$. Then we get

$$v(\mathcal{J}^{(n)})_{n+k,n+k-\delta} - v(\mathcal{J}^{(n,N)})_{n+k,n+k-\delta}$$
$$= \int_0^t ds \sum_{|j|\le M} v_j \sum_{\pm} \Big((e^{ijd(t-s)\mathcal{J}^{(n,N')}})_{n+k,n\pm N'} \mathcal{J}^{(n)}_{n\pm N',n\pm N'-1} (e^{is\mathcal{J}^{(n)}})_{n\pm N'-1,n+k-\delta}$$
$$+ (e^{ijd(t-s)\mathcal{J}^{(n,N')}})_{n+k,n\pm N'-1} \mathcal{J}^{(n)}_{n\pm N'-1,n\pm N'} (e^{is\mathcal{J}^{(n)}})_{n\pm N',n+k-\delta} \Big) + O(M^{-\ell+1/2}). \qquad (A.4)$$

Now we use the bound, valid for any Jacobi matrix $\mathcal{J}$ with coefficients $J_{k,k+1} = J_{k+1,k} = a_k \in \mathbf{R}$, $|a_k| \le A$. Then there exist positive constants C_0, C_1, C_2, depending on A such that the matrix elements of $e^{it\mathcal{J}}$ satisfy the inequalities:

$$|(e^{it\mathcal{J}})_{k,j}| \le C_0 e^{-C_1|k-j|+C_2 t}. \qquad (A.5)$$

These bounds follow from the representation

$$(e^{it\mathcal{J}})_{k,j} = \frac{1}{2\pi i} \oint_l e^{itz} \tilde{R}_{k,j}(z) dz,$$

where $\tilde{R} = (zI - \mathcal{J})^{-1}$, and from (2.79).

Using (A.5) in (A.4), we get for any $c < C_0(C_1 d)^{-1}$

$$v(\mathcal{J}^{(n)})_{n+k,n+k-\delta} - v(\mathcal{J}^{(n,N')})_{n+k,n+k-\delta} = O((cM)^{-\ell+1/2}).$$

Similarly (see definitions (2.25) and (2.26)):

$$v(\mathcal{J}^{(0)} + \tilde{\mathcal{J}})_{k,k-\delta} - v(\mathcal{J}^{(n,N')})_{n+k,n+k-\delta} = O((cM)^{-\ell+1/2}).$$

These two bounds give us (2.27).

□

Proof of Proposition 1. Assume that $|\tilde{x}_k| > \varepsilon$ for some k. Without loss of generality we can assume that $\tilde{x}_k > 0$. Then due to (2.12)

$$\tilde{x}_{k+1} - 2\tilde{x}_k + \tilde{x}_{k-1} > \tilde{x}_k^3.$$

Consider first the case when also $\tilde{d}_k^{(1)} = \tilde{x}_{k+1} - \tilde{x}_k > 0$. Then by induction for any $M - k > i > 0$ we have $\tilde{d}_{k+i}^{(1)} > \tilde{d}_k^{(1)}$, $\tilde{x}_{k+i} > \tilde{x}_k$ and $\tilde{d}_{k+i}^{(2)} > \tilde{x}_k^3$. Hence

$$\varepsilon_1 > \tilde{x}_{k+M_1} > \tilde{x}_k + \tilde{x}_k^3 M_1^2/2.$$

If $\tilde{d}_k^{(1)} < 0$, then according to (2.12) we have $\tilde{x}_{-k-1} > \tilde{x}_k$ and we obtain (2.13) moving from k in the negative direction.

Similarly, assume that at some point $|k| \le M - 2M_1$

$$\tilde{d}_k^{(1)} > 4\max\{\varepsilon^2, (2\varepsilon_1 M_1^{-2})^{2/3}\} =: 4\mu^2. \qquad (A.6)$$

Since $|\tilde{d}_k^{(2)}| \le 3\mu^3$ because of (2.12) and the first inequality of (2.13), we have for any $|i| < i_0 := [2/(3\mu)] \le [2/(3\varepsilon)] \le M_1$

$$\tilde{d}_{k+i}^{(1)} > \tilde{d}_k^{(1)}/2 \Rightarrow \mu \ge |\tilde{x}_{k+si_0}| > i_0 \tilde{d}_k^{(1)}/2 = 3\mu^{-1}\tilde{d}_k^{(1)},$$

where $s = \text{sign}\tilde{x}_k$. The last inequality here contradicts to (A.6). Hence, (A.6) is false and we obtain the second inequality of (2.13).

To prove (2.15) observe that if we consider two $(2M+1)\times(2M+1)$ Jacobi matrices $J^{(f)}$ and $J^{(d)}$ with entries

$$J^{(f)} = D^{(f)} - J^{(0)}, \quad J^{(d)} = D^{(d)} - J^{(0)}$$
$$D^{(f)}_{k,j} = \delta_{k,j}(2 + f_k), \quad D^{(d)}_{k,j} = \delta_{k,j}(2 + d^2), \quad |k| \le 2M,$$

then, using the Neumann expansion for their inverse we get

$$0 \le (J^{(f)})^{-1}_{k,j} \le (J^{(f)})^{-1}_{k,j} \le (2d)^{-1} e^{-d|k-j|}. \qquad (A.7)$$

Hence, rewriting (2.14) as

$$(J^{(f)}\tilde{x})_k = -\tilde{r}_k + \delta_{k,2M}\tilde{x}_{2M+1} + \delta_{k,-2M}\tilde{x}_{-2M-1},$$

we get (2.15) from (A.7).

□

Proof of Proposition 7. We use the estimate for matrix elements of the resolvent of an arbitrary Jacobi matrix $\mathcal{J}$, with entries $|\mathcal{J}_{j,j+1}| \le A$:

$$|\mathcal{R}_{k,j}(z)| \le \frac{C_1'}{|\Im z|} e^{-C_2'|\Im z||k-j|}, \qquad (A.8)$$

This estimate is similar to well-known Combes- Thomas estimates for Schrödinger operator (see e.g.[19]).

Let $\mathcal{J}^{(k,M)}$ be the Jacobi matrix, whose entries coincide with that for $\mathcal{J}$ with the only exceptions $\mathcal{J}^{(k,M)}_{k\pm M,k\pm M+1} = \mathcal{J}^{(k,M)}_{k\pm M+1,k\pm M} = 0$, and $\mathcal{R}^{(k,M)} = (z - \mathcal{J}^{(k,M)})^{-1}$. Then, by the resolvent identity (3.41)

$$\mathcal{R}_{k,j} - \mathcal{R}^{(k,M)}_{k,j} = \sum_{\pm} \Big(\mathcal{R}^{(k,M)}_{k,k\pm M} \mathcal{J}_{k\pm M,k\pm M+1} \mathcal{R}_{k\pm M+1,j}$$
$$+ \mathcal{R}^{(k,M)}_{k,k\pm M+1} \mathcal{J}_{k\pm M+1,k\pm M} \mathcal{R}_{k\pm M,j} \Big).$$

Since $\mathcal{J}^{(k,M)}$ has a block structure, its resolvent $\mathcal{R}^{(k,M)}$ also has a block structure and its coefficients $\mathcal{R}^{(k,M)}_{k,j}$ do not depend on $\mathcal{J}_{j,j+1}$ with $|j-k| > M$. Hence, we can apply (A.8) to $\mathcal{R}^{(k,M)}_{k,j}$, $\mathcal{R}^{(k,M)}_{k,k\pm M}$ and $\mathcal{R}^{(k,M)}_{k,k\pm M+1}$. Then we get (2.79). Proposition 7 is proved.

□

Proof of Proposition 5 We would like to consider the system (of equations 2.45) like a linear equation in l_2. To this end we set

$$\tilde{z}_k = \sum_{|l|\le N_1} \mathcal{P}_{k-l}\tilde{x}_l, \quad \tilde{\varepsilon}_k = 0, \quad |k| > N_1.$$

Then it is easy to see that

$$|z_k| \le ||\mathcal{P}||\varepsilon_0,$$

and it follows from (2.45) extended for all k that with above z_k, $\tilde{\varepsilon}_k$ that

$$\tilde{x}_k = \sum_l \mathcal{P}^{-1}_{k-l}(\tilde{z}_l + \tilde{\varepsilon}_l). \tag{A.9}$$

Moreover, since by condition of the proposition P has ℓth derivative from $L_2[-2,2]$, P^{-1} also has ℓth derivative from $L_2[-2,2]$. Therefore, using (A.2) we have for any k

$$\sum_{|l-k|>N_2} |\mathcal{P}^{-1}_{k-l}| \le N_2^{-\ell+1/2}. \tag{A.10}$$

Using this bound in (A.9) we get (2.46).
□

APPENDIX B: PROOF OF PROPOSITION 2

. It is evident that it is enough to prove (2.24) for the case when $2P_0(0) = 1$ in (1.17) and (2.24). Hence, below we consider this case. For $x > 0$ the statement is evident.

To prove it for $x < 0$ remark first that if for some $x_1 > 0$

$$\frac{q'(x_1)}{q(x_1)} > \frac{Ai'(x_1)}{Ai(x_1)},$$

then since for $x > 0$ $q''(x) > xq(x)$ we have that for $x > x_1$

$$q(x) > y(x),$$

where $y(x)$ is the solution of the initial value problem

$$y''(x) = xy(x), \quad y(x_1) = q(x_1), \quad y'(x_1) = q'(x_1).$$

But according to the standard theory of the differential equations, the last problem has the solution

$$y(x) = c_1\, Ai(x) + c_2 Bi(x)$$

with

$$c_2 = W^{-1}\{Ai, Bi\} q(x_1) Ai(x_1) \left(\frac{q'(x_1)}{q(x_1)} - \frac{Ai'(x_1)}{Ai(x_1)} \right) > 0,$$

where $W\{Ai, Bi\} = Ai(x)Bi'(x) - Ai'(x)Bi(x) = \pi^{-1}$ (see [1]). But then $q(x) > y(x) \to \infty$ as $x \to \infty$, that contradicts to (1.18). Hence we conclude that for $x > 0$

$$(\log q(x))' \le (\log Ai(x))',$$

and since it is known (see [12]) that $q(x) = Ai(x)(1+o(1))$ as $x \to +\infty$, we obtain that

$$q(0) > Ai(0) = 0.355028... > 3^{2/3}/6,$$

(for $Ai(0)$ see [1])

For $x < 0$ set

$$f(x) := \sqrt{-x/6} - q(x),$$

and let x_0 be the first negative root of f. It is known that $q(x) > 0$, $q'(x) < 0$ (see [12]) and so

$$\sqrt{-x_0/6} = q(x_0) > q(0) > 3^{2/3}/6 \Rightarrow x_0 < -3^{1/3}/2.$$

But for any point $x \le x_0 < -3^{1/3}/2$ in which $q(x) \ge \sqrt{-x/6}$

$$\begin{aligned} q''(x) &\le \sqrt{-x/6}(x - x/3) \le -\frac{2}{3}\sqrt{-x_0^3/6} \\ &< -\left(4\sqrt{6(-x_0)^3}\right)^{-1} \le \left(\sqrt{-x/6}\right)''. \end{aligned}$$

Therefore $f''(x) > 0$ for $x \ge x_0$. Since by definition $f(x_0) = 0$, $f'(x_0) \le 0$ (because $f(0) < 0$ and x_0 is the first root of f) we conclude that $f(x) < 0$ for any $x < x_0$ that contradicts to (1.18). Thus we have proved that the left hand side of (2.24) is always positive. But since it tends to infinity as $x \to \pm\infty$, we conclude that there exists positive δ, satisfying (2.24).

□

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