EIGENVALUE DISTRIBUTION OF LARGE WEIGHTED RANDOM GRAPHS

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Abstract

We study eigenvalue distribution of the adjacency matrix $A^{(N,p)}$ of weighted random graphs $\Gamma = \Gamma_{N,p}$. We assume that the graphs have N vertices and the average number of edges attached to one vertex is p. To each edge of the graph e_{ij} we assign a weight given by a random variable a_{ij} with zero mathematical expectation and all moments finite.

In the first part of the paper, we consider the moments of normalized eigenvalue counting function $\sigma_{N,p}$ of $A^{(N,p)}$. Assuming all moments of a finite, we obtain recurrent relations that determine the moments of the limiting measure $\sigma_p = \lim_{N\to\infty} \sigma_{N,p}$. The method developed is applied to the Laplace operator Δ_Γ closely related with $A^{(N,p)}$. Using the recurrent relations, we analyze the form of σ_p for the both of random matrix families.

In the second part of the paper we consider the resolvents $G^{(A,\Delta)}(z)$ of $A^{(N,p)}$ and Δ_Γ of $\Gamma_{N,p}$ and study the functions $f_N^{(A,\Delta)}(z,u) = \frac{1}{N}\sum_{k=1}^N \exp\{-uG_{kk}^{(A,\Delta)}(z)\}$ in the limit $N \to \infty$. We derive closed equations that uniquely determine the limiting functions $f^{(A,\Delta)}(z,u)$. These equations allow us to prove the existence of the limiting σ_p for adjacency matrix and the Laplace operator under a rather weak condition that only the fourth moment of a_{ij} is finite. Besides, equations for $f^{(A,\Delta)}(z,u)$ give us the asymptotic expansions for the Stieltjes transform of the limiting σ_p with respect to z^{-k} and p^k.

1 Introduction

The spectral theory of graphs is an actively developing field of mathematics involving a variety of methods and deep results (see the monographs [5, 6, 11]). Given a graph with N vertices, one can associate with it many different matrices, but the most studied are the adjacency matrix and the Laplacian matrix of the graph. Commonly, the set of N eigenvalues of the adjacency matrix is referred to as the spectrum of the graph. In these studies, the dimension of the matrix N is usually regarded as a fixed parameter. The spectra of infinite graphs is considered in certain particular cases of graphs having a certain regular structure (see for example [13]).

Another large class of graphs, where the limiting transition $N \to \infty$ provides a natural approximation is represented by random graphs [3, 12]. In this branch, geometrical and topological properties of graphs are studied for a wide veriety of random graph ensembles. One of the classes of the prime reference is the *binomial random graph* originating by P. Erdős (see, e.g. [12]). Given a number $p_N \in (0,1)$, this family of graphs $\mathbf{G}(N,p_N)$ is defined by taking as Ω the set of all graphs on N vertices with the probability

$$P(G) = p_N^{e(G)}(1-p_N)^{\binom{N}{2}-e(G)}, \tag{1.1}$$

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where $e(G)$ is the number of edges of G. Most of the random graphs studies are devoted to the cases where $p_N \to 0$ as $N \to \infty$.

Intersection of these two branches of the theory of graphs contains the spectral theory of random graphs that is still poorly explored. However, a number of powerful tools can be employed here, because the ensemble of random symmetric $N \times N$ adjacency matrices A_N is a particular representative of the random matrix theory, where the limiting transition $N \to \infty$ is intensively studied during half of century since the pioneering works by E. Wigner [22]. Initiated by theoretical physics applications, the spectral theory of random matrices has revealed deep nontrivial links with many fields of mathematics.

Spectral properties of random matrices corresponding to (1.1) were examined in the limit $N \to \infty$ both in numerical and theoretical physics studies [7, 8, 9, 18, 20, 21]. There are two major asymptotic regimes: $p_N \gg 1/N$ and $p_N = O(1/N)$ and corresponding models can be called the *dilute random matrices* and *sparse random matrices*, respectively. The first studies of spectral properties of sparse and dilute random matrices in the physical literature are related with the works [20], [21], [18], where equations for the limiting density of states of sparse random matrices were derived. In papers [18] and [10] a number of important results on the universality of the correlation functions and the Anderson localization transition were obtained. Unfortunately all these results were obtained with non rigorous replica and supersymmetry methods.

On mathematical level of rigour, the eigenvalue distribution of dilute random matrices was studied in [16]. It was shown that the normalized eigenvalue counting function of

$$\frac{1}{\sqrt{Np_N}} A_{N,p_N} \tag{1.2}$$

converges in the limit $N, p_N \to \infty$ to the distribution of explicit form known as the semicircle, or Wigner law [22]. The moments of this distribution verify well-known recurrent relation for the Catalan numbers and can be found explicitly. Therefore one can say that the dilute random matrices represent explicitly solvable model (see also [20, 21]).

In the series of papers [2, 4, 3] and simultaneously in [15], the adjacency matrix and the Laplace matrix of random graphs (1.1) with $p_N = pN$ were studied. It was shown that this sparse random matrix ensemble can also be viewed as the explicitly solvable model. In particular, one can derive recurrent relations that determine the moments of the limiting eigenvalue distribution of $A_{N,pN}$, $N \to \infty$ depending on given value of p.

In the present paper we generalize the results of [4, 15] to the case of weighted random graphs. We study also the resolvent of the adjacency matrix and the Laplace operator of large weighted random graphs and derive rigorously equations for the Stieltjes transform $g(z)$ of the limiting eigenvalue distribution, obtained initially in [20], [21], [18] by using the replica and the supersymmetry approaches. We stress that our approach allows us to prove the existence of the limiting eigenvalue distribution under rather weak conditions, when only the fourth moment of a_{ij} is required to be bounded. Using our results it is not difficult to obtain the asymptotic expansions for $g(z)$ with respect to z^{-k}. Since it is well known that the coefficients of this expansion are the moments of the limiting IDS, we rediscover the recurrent formulas for the moments. Besides, constructing the asymptotic expansion of $g(z)$ with respect to p^k, it is easy to show that this expansion is convergent for $p < 1$. Since in the case $a_{ij} = 1$ the coefficients of this expansion are the rational functions on z, we can conclude that the limiting spectrum is pure point and consists of the spectra of finite blocks only.

2 Main results

Let $\mathcal{V}_N$ denotes the set of N vertices $v_1, v_2, ..., v_N$. We define the set $\mathcal{F}_{\mathcal{V}_N} = \mathcal{V}_N \to \mathbf{R}$ of all real functions $f = (f(v_1), f(v_2), \ldots, f(v_N))$. Let us assume that each pair of vertices of $\mathcal{V}_N$ is either connected by one no-oriented edge or not connected. Let us denote by E_N the set of the edges and by $\Gamma_N = (\mathcal{V}_N, E_N)$ the corresponding graph.

Assume that each edge $e=(v_i,v_j) \in E_N$ is assigned by the real weight $\xi(e)$. Then one can define a linear operator $\Delta_\Gamma^{(\xi)} : \mathcal{F}_{\mathcal{V}_N} \to \mathcal{F}_{\mathcal{V}_N}$ by relation

$$\Delta_\Gamma^{(\xi)}(f(v_i)) = \sum_{j:v_j\sim v_i} \xi(v_i,v_j)\cdot[f(v_i)-f(v_j)], \tag{2.1}$$

where the sum goes over all vertices v_j adjacent to given v_i. One can consider the operator $\Delta_\Gamma^{(\xi)}$ as a generalization of the discrete analogue of the Laplace operator on the graph Γ_N.

Clearly, $\Delta_\Gamma^{(\xi)}$ is a real symmetric $N\times N$ matrix that can be represented in the form

$$\Delta_\Gamma^{(\xi)} = B^{(N,\xi)} - A^{(N,\xi)},$$

where $A^{(N,\xi)}$ is a weighted adjacency matrix of Γ_N

$$A_{ij}^{(N,\xi)} = \begin{cases} \xi(v_i,v_j), & \text{if } v_i \sim v_j, \\ 0, & \text{if } v_i \not\sim v_j \end{cases}, \tag{2.2}$$

and $B^{(N,\xi)}$ is a diagonal matrix

$$B_{ii}^{(N,\xi)} = \sum_{j:v_j\sim v_i} \xi(v_i,v_j).$$

Note that $A_{ii}^{(N,\xi)} = 0$ and

$$\Delta_\Gamma^{(\xi)} = \mathrm{diag}[(MA^{(N,\xi)})] - A^{(N,\xi)}, \tag{2.3}$$

where

$$M_{ij} = 1-\delta_{ij} = \begin{cases} 0, & \text{if } i=j, \\ 1, & \text{if } i\neq j. \end{cases}$$

The set of eigenvalues $\lambda_1 \le \ldots \le \lambda_N$ of $A^{(N,\xi)}$ is referred to as the spectrum of the graph Γ.

With these definitions in hands, we can introduce the randomly weighted adjacency matrix of random binomial graphs. In this case the weights ξ are represented by the following family of random variables. Let $\Xi = \{a_{ij},\ i\le j,\ i,j\in\mathbf{N}\}$ be the set of jointly independent identically distributed (i.i.d.) random variables defined on the same probability space and possessing the moments

$$\mathbf{E}a_{ij}^k = X_k < \infty \qquad \forall\, i,j,k \in \mathbf{N}, \tag{2.4}$$

where $\mathbf{E}$ denotes the mathematical expectation corresponding to Ξ. We set $a_{ji}=a_{ij}$ for $i\le j$.

Given $0 < p \le N$, let us define the family $D_N^{(p)} = \{d_{ij}^{(N,p)},\ i\le j,\ i,j \in \overline{1,N}\}$ of jointly independent random variables

$$d_{ij}^{(N,p)} = \begin{cases} \frac{1}{\sqrt{p}}, & \text{with probability } p/N, \\ 0, & \text{with probability } 1-p/N, \end{cases} \tag{2.5}$$

We set $d_{ji} = d_{ij}$ and assume that $\Lambda_N^{(p)}$ is independent of Ξ.

Now one can consider the real symmetric matrix $A^{(N,p)}(\omega)$:

$$\left[A^{(N,p)}\right]_{ij} = a_{ij}d_{ij}^{(N,p)} \tag{2.6}$$

that has N real eigenvalues $\lambda_1^{(N,p)} \le \lambda_2^{(N,p)} \le \ldots \le \lambda_N^{(N,p)}$.

The normalized eigenvalue counting function (or integrated density of states (IDS)) of A is determined by the formula

$$\sigma\left(\lambda; A^{(N,p)}\right) = \frac{\sharp\{j : \lambda_j^{(N,p)} < \lambda\}}{N}.$$

Similarly we define $\sigma(\lambda; \Delta_\Gamma^{(\xi)})$.

In this paper we study the normalized eigenvalue counting functions by two complementary approaches: the moments and the resolvent techniques. Corresponding results are represented in the following two subsections.

2.1 Moment relations approach

The first group of results concerns the averaged moments

$$M_k^{(N,p)} = \mathbf{E}\left\{\int \lambda^k d\sigma\left(\lambda; A^{(N,p)}\right)\right\}.$$

Theorem 1. *Assuming conditions (2.4), there exist limits*

$$\lim_{N\to\infty} M_s^{(N,p)} = \begin{cases} m_k^{(p)} = \sum_{i=0}^k S(k,i), & \text{if } s = 2k, \\ 0, & \text{if } s = 2k-1, \end{cases} \tag{2.7}$$

where numbers $\mathrm{S}(k,i)$ *are determined by the system of recurrent relations*

$$\mathrm{S}(l,r) = \sum_{f=1}^{r} \binom{r-1}{f-1} \cdot \frac{X_{2f}}{p^{f-1}} \cdot \sum_{u=0}^{l-r} \mathrm{S}(l-u-f, r-f) \cdot \sum_{v=0}^{u} \binom{f+v-1}{f-1} \cdot \mathrm{S}(u,v) \tag{2.8}$$

with the initial condition $S(l,0) = \delta_{l,0}$.

The next theorem deals with the moments

$$L_s^{(N,p)} = \mathbf{E}\left\{\int \lambda^s d\sigma\left(\lambda; \Delta_\Gamma\right)\right\}.$$

Theorem 2. *Assume that (2.4) holds. Then given* $s \in \mathbf{N}$, *there exists the limit*

$$\lim_{N\to\infty} L_s^{(N,p)} = l_s^{(p)} = \sum_{i=0}^{s} \widehat{\mathrm{S}}(s,i). \tag{2.9}$$

where numbers $\widehat{\mathrm{S}}(s,i)$ *are determined by the system of recurrent relations*

$$\widehat{\mathrm{S}}(l,r_1) = \sum_{g_1=1}^{r_1} \binom{r_1-1}{g_1-1} \cdot \left(\widehat{\mathrm{S}}(l-g_1, r_1-g_1) \cdot \frac{X_{g_1}}{p^{g_1/2-1}} + \sum_{d=r_1-g_1}^{l-r_1} \widehat{\mathrm{S}}(d, r_1-g_1) \cdot \right.$$

$$\left. \cdot \sum_{g_2=1}^{l-d-g_1} \binom{g_1+g_2-1}{g_1-1} \cdot \frac{X_{g_1+g_2}}{p^{(g_1+g_2)/2-1}} \sum_{r_2=1}^{l-d-g_1-g_2} \binom{r_2+g_2-1}{g_2-1} \cdot \widehat{\mathrm{S}}(l-d-g_1-g_2, r_2) \right) \tag{2.10}$$

with the initial condition

$$\widehat{\mathrm{S}}(l,0) = \delta_{l,0}.$$

We discuss these results later. Let us only note that if $a_{ij} \equiv 1$ and $p = 1$, then $A^{(N,1)}$ becomes exactly the adjacency matrix of Γ and $\Delta_\Gamma^{(a)}$ takes the form of the Laplace operator on the graph. In this case formula (2.8) is reduced to

$$\mathrm{S}(l,r) = \sum_{f=1}^{r} \binom{r-1}{f-1} \cdot \sum_{u=0}^{l-r} \mathrm{S}(l-u-f, r-f) \cdot \sum_{v=0}^{u} \binom{f+v-1}{f-1} \cdot \mathrm{S}(u,v) \qquad (2.11)$$

This system of recurrent relations is obtained for the first time in [15]. It is simpler than that derived in [2] to determine $m_k^{(1)}$ (2.7). The difference is that our system (2.8) has one variable of summation less than the system of [2]. We explain this difference at the end of Section 4.

In the case $a_{ij} \equiv 1$ and $p = 1$ formulas (2.10) reduce to

$$\widehat{\mathrm{S}}(l,r_1) = \sum_{g_1=1}^{r_1} \binom{r_1-1}{g_1-1} \cdot \left(\widehat{\mathrm{S}}(l-g_1, r_1-g_1) + \sum_{d=r_1-g_1}^{l-r_1} \widehat{\mathrm{S}}(d, r_1-g_1) \cdot \right.$$

$$\left. \cdot \sum_{g_2=1}^{l-d-g_1} \binom{g_1+g_2-1}{g_1-1} \cdot \sum_{r_2=1}^{l-d-g_1-g_2} \binom{r_2+g_2-1}{g_2-1} \cdot \widehat{\mathrm{S}}(l-d-g_1-g_2, r_2) \right)$$

2.2 Resolvent approach

The resolvent approach is a powerful tool of the spectral theory in general and the spectral theory of random matrices also. In particular, it allows to simplify and generalize the pioneer method of Wigner [22] (based on the analysis of the moments of σ) used to study the IDS of the ensemble with independent gaussian entries. The resolvent approach produces also a lot of new results (see, e.g., the review papers [16], [19] and references therein). It is well known that the trace of the resolvent is the Stiltjes transform $g_N(z)$ of the normalized counting function of the matrix. Since the Stiltjes transform uniquely determines the measure, the proof of the existence of the limiting IDS is equivalent to the proof of the existence of the limit $\lim_{N\to\infty} g_N(z) = g(z)$. Besides, the equations for $g(z)$ give complete information about the limiting IDS.

For any z: $\Re z > 0$ consider the function $f_N(u,z) : \mathbf{R}_+ \to \mathbf{C}$:

$$f_N(u,z) = \frac{1}{N} \sum_{k=1}^{N} e^{-u a_k^2 G_{kk}^{(N,p)}(z)}, \quad G_{kk}^{(N,p)}(z) = (z - iA^{(N,p)})_{kk}^{-1}, \qquad (2.12)$$

where $\{a_i\}_{i=1}^{\infty}$ is a family of i.i.d. random variables which do not depend on $\{a_{i,j}\}_{i<j}^{\infty}$ and have the same probability distribution as $a_{1,2}$.

Theorem 3. *Assume that $\mu(a) = \mathbf{E}\{\theta(a - a_{i,j})\}$ the probability distribution of $a_{i,j}$ possesses the property*

$$\int a^4 d\mu(a) = X_4 < \infty. \qquad (2.13)$$

Then
(i) the variance of the function $f_N(u,z)$ defined by (2.12) vanishes in the limit $N \to \infty$:

$$\lim_{N\to\infty} \mathbf{E}\{|f_N(u,z) - \mathbf{E}\{f_N(u,z)\}|^2\} = 0, \qquad (2.14)$$

(ii) there exists the limit

$$\lim_{N\to\infty} \mathbf{E}\{f_N(u,z)\} = f(u,z), \qquad (2.15)$$

(iii) if we consider the class $\mathcal{C}$ of functions which are analytic with respect to $z:\ \Re z>0$ and for any fixed $z:\ \Re z>0$ possessing the norm

$$||f(u,z)|| = \max_{u>0} \frac{|f(u,z)|}{\sqrt{1+u}}, \tag{2.16}$$

then the limiting function is the unique solution in $\mathcal{C}$ of the functional equation

$$f(u,z) = 1 - u^{1/2}e^{-p}\int |a|d\mu(a)\int_0^\infty dv \frac{\mathcal{J}_1(2|a|\sqrt{uv})}{\sqrt{v}} \exp\{-zv + pf(v/p,z)\}, \tag{2.17}$$

where $\mathcal{J}_1(\zeta)$ is the Bessel function:

$$\mathcal{J}_1(\zeta) = \frac{\zeta}{2}\sum_{k=0}^{\infty} \frac{(-\zeta^2/4)^k}{k!(k+1)!}. \tag{2.18}$$

It is easy to see that equation (2.17) coincides with that obtained in [20], [18] by the replica trick and supersymmetry approach respectively by using the assumption that the solution of the problem is replica symmetric (or an equivalent assumption for the saddle point method). Our proof is rigorous and it needs not any additional assumption.

One can easily see that

$$-\frac{\partial}{\partial u} f_N(u,z)\Big|_{u=0} = \frac{X_2}{N}\sum_{k=1}^{N} \mathbf{E}\{G_{kk}^{(N,p)}(z)\} = X_2\mathbf{E}\{ig_{N,p}(-iz)\},$$

where $g_{N,p}(z)$ is the Stieltjes transform of the normalized counting function $\sigma(\lambda, A^{(N,p)})$.

$$g_{N,p}(z) = \int \frac{d\sigma(\lambda, A^{(N,p)})}{\lambda - z}$$

Hence, Theorem 3 implies that for any $z:\ \Im z \neq 0$

$$\lim_{N\to\infty} \mathbf{E}\{|g_{N,p}(z) - \mathbf{E}\{g_{N,p}(z)\}|^2\} = 0,$$

i.e., the fluctuations of $g_{N,p}(z)$ vanish in the limit $N\to\infty$. And (2.15) implies that

$$g(z) = \lim_{N\to\infty} \mathbf{E}\{g_{N,p}(z)\} = -X_2^{-1}\frac{\partial}{\partial u} f(u,z)\Big|_{u=0} \tag{2.19}$$

Thus, Theorem 3 states that under condition (2.13) there exists the weak limit $\sigma(\lambda, A)$ of the normalized counting measure $\sigma(\lambda, A^{(N,p)})$ and the Stieltjes transform $g_p(-iz)$ can be obtained as the first derivative of the solution of (2.17).

If the random variables $\{a_{i,j}\}$ possess the $2m-th$ moments, then on the basis of (2.17) it is easy to construct an asymptotic expansion of the function $f(u,z)$ in z^{-k} up to z^{-2m}:

$$f(u,z) = \sum_{k=0}^{2m} z^{-k}P_k(u) + o(z^{-2m}), \quad z\to\infty$$

where $P_k(u)$ are some polynomials. Since for any polynomial $P(u)$

$$u^{1/2}e^{-p}\int |a|d\mu(a)\int_0^\infty dv \frac{\mathcal{J}_1(2|a|\sqrt{uv})}{\sqrt{v}} \exp\{-zv\}P(v) = O(z^{-1}),$$

this expansion gives us recurrent formulas which express the coefficients of $P_k(u)$ via the coefficients of $P_{k-1}(u),\dots,P_1(u)$. By (2.19) it is evident, that the coefficient c_{k1} of $P_k(u)$ near u is the k-th coefficient of the expansion of $g_p(-iz)$, in z^{-k}. So, $c_{k1}=(-i)^kM_k$, where M_k- is the kth moment of the limiting measure $\sigma(\lambda)$.

Similarly one can construct an expansion of $g(z)$ with respect to p^k. To this end it is more convenient to study the case when $a_{ij}=1$ and $d_{ij}=0,1$ with probability $1-\frac{p}{N}$ and $\frac{p}{N}$ respectively. It is equivalent to the change of variables $z\to zp^{-1/2}$, $u\to up^{-1/2}$. Then we get the equation

$$\tilde f(u,z)=1-u^{1/2}e^{-p}\int_0^\infty dv\frac{\mathcal{J}_1(2\sqrt{uv})}{\sqrt{v}}.\exp\{-zv+p\tilde f(v,z)\}. \tag{2.20}$$

Let us seek the expansion of the form $\tilde f(u,z)=1+\sum p^kf_k(u,z)$. Since in the r.h.s. of (2.20) we have the exponent of $p(\tilde f(u,z)-1)$, it is evident that (2.20) gives us the recurrent formula for $f_k(u,z)$ and $f_k(u,z)$ is a linear combination of the functions $e^{-uR_{k,l}(z)}$ $(l=1,\dots k!)$ with $R_{k,l}(z)$ being rational functions of z. It is easy to prove that the expansion is convergent, if $p<1$. Therefore we can differentiate it with respect to u. Hence, for the function $\tilde g(z)$ defined as in Remark 1, we get the convergent expansion $\tilde g(z)=\sum p^k\tilde R_k(z)$, where $\tilde R_k(z)$ are the rational functions of z. Thus, we can state that for $p<1$ the spectrum of the adjacency matrix consists only from the spectrum of finite graphs.

Now let us study the IDS of the Laplace operator of the random graph. To this end for any z: $\Re z>0$ define the function $f_N^{(\Delta)}(u,z):\mathbf{R}_+\to\mathbf{C}$:

$$f_N^{(\Delta)}(u,z)=\frac{1}{N}\sum_{k=1}^N e^{-iua_k-ua_k^2G_{kk}^{(\Delta,N,p)}(z)},\quad G_{kk}^{(\Delta,N,p)}(z)=(z-iL^{(N,p)})^{-1}_{kk}, \tag{2.21}$$

where $\{a_i\}_{i=1}^\infty$ are defined by the same way as in (2.12).

Theorem 4. *Let the distribution of $a_{j,k}$ satisfy condition (2.13). Then*
(i) fluctuations of the function $f_N^{(\Delta)}(u,z)$ defined by (2.21) vanish in the limit $N\to\infty$:

$$\lim_{N\to\infty}\mathbf{E}\{|f_N^{(\Delta)}(u,z)-\mathbf{E}\{f_N^{(\Delta)}(u,z)\}|^2\}=0, \tag{2.22}$$

(ii) there exists the limit

$$\lim_{N\to\infty}\mathbf{E}\{f_N^{(\Delta)}(u,z)\}=f^{(\Delta)}(u,z), \tag{2.23}$$

(iii) the limiting function is the unique solution in the class $\mathcal{C}$ defined in Theorem 3 of the functional equation

$$f^{(\Delta)}(u,z)=\hat\mu(-u)-u^{1/2}e^{-p}\int|a|e^{-iau}d\mu(a)\int_0^\infty dv\frac{\mathcal{J}_1(2|a|\sqrt{uv})}{\sqrt{v}}\exp\{-zv+pf^{(\Delta)}(v/p,z)\}, \tag{2.24}$$

where $\hat\mu(u)=\int e^{iua}d\mu(a)$ is the Fourier transform of the measure $\mu(a)$, defined in Theorem 3.

3 Moments and trees

In this section we give an outlook of the method of computing moments. Rigorous description is given in the next section.

To study the mathematical expectation

$$\mathbf{E}\frac{1}{N}\mathrm{Tr}\left(\mathrm{A}^{(N,p)}\right)^k=\frac{1}{N}\sum_{i_l=1}^N\mathbf{E}\left\{A_{i_1i_2}^{(N,p)}A_{i_2i_3}^{(N,p)}\dots A_{i_ki_1}^{(N,p)}\right\}, \tag{3.1}$$

we give a further development to the method originated by E. Wigner (see e.g., [22]). In this approach the set of variables $I_k = \{i_1, i_2, \dots, i_k, i_1\}, i_l \in \overline{1,N}$ is regarded as a set of trajectories (walks) W_k of k steps. Each walk provides a contribution $\mathbf{E}\left\{A^{(N,p)}_{i_1 i_2} A^{(N,p)}_{i_2 i_3} \dots A^{(N,p)}_{i_k i_1}\right\}$. This mathematical expectation is non-zero only when each step (i_j, i_{j+1}) appears even number of times in this walk W_k. The order and the number of repetitions of the steps leads to partition of the set $\{W_k\}$ of the walks into the classes of equivalence.

In the case of the Wigner ensemble $A^{(N,N)}$, the classes of equivalence were labeled by the plane rooted trees τ_l of l edges for $k = 2l$. Such a tree can be run over by $2l$ steps starting and finishing at the root and passing each edge two times exactly (there and back). This path is made in the lexicographical order. This means that each time when there is a choice where to go, the most left edge is passed. The set T_l of all trees τ_l contains $C_l = \frac{(2l)!}{l!(l+1)!}$ elements.

The situation is more delicate in the case of dilute matrices (2.6), when p is fixed and $N \to \infty$. In the paper [14] it is shown that the leading contribution to (3.1) in this limit is provided by the walks $I_k, k = 2l$ that fall into the classes of equivalence described as follows.

We consider an element $\tau_m \in T_m, m \leq l$ and construct a path of $2l$ steps over this tree. Each edge is passed even number of times. If $m < l$, then there exist one or several edges passed even number of times, which is greater than 2. This path is made in the lexicographical order that chooses the most left edge among those that are yet not passed. The number of such paths and corresponding contribution were estimated in [14].

The case of non-weighted adjacency matrix and corresponding Laplace operator is considered in [2, 15], where these paths were computed exactly and recurrent relations for their number were obtained.

In the present paper we develop the method to compute these paths and corresponding contributions in the case of weighted matrices. It is similar to the method of decomposition of trees by one edge that is well-known combinatorial tool to obtain recurrent relations needed.

Let us briefly describe our method. Consider a tree τ_m with m edges and the root ρ and denote by (ρ, ν) the edge that is passed the first. If one removes this edge, one gets two subtrees G_2 and G_1 (see Figure 1).

Denote by f the number of passages $\rho \to \nu$. Then the path over the tree τ_m is described as follows: after the first passage $\rho \to \nu$ one enters the tree G_2 and goes over its edges. Each time when one gets into the vertex ν, there is a choice where to go: either to the leave G_u and enter the subtree G_1 by $\nu \to \rho$, or to continue the path over G_2. It is clear that the paths over the subtrees G_2 and G_1 are performed independently. More precisely, when leaving the subtree G_1, one keeps the information about its part already passed. Returning back to it by the passage $\nu \to \rho$, we continue this path with no regard what part of the path over G_1 is performed. The number of passages f over the edge (ρ, ν) in direction $\overrightarrow{(\rho, \nu)}$ determines the weight factor $\mathbf{E}a^{2f} = X_{2f}$.

This splitting of trees (and paths) in two parts leads to the recurrent relations for the number of the paths and corresponding contributions.

Certainly this brief presentation does not reflect all of the details of the procedure. Moreover, in the rigorous proof we study the classes of walks W_k directly and the trees arise as somehow supplementary objects. We used them here as more visual illustrations than the walks.

4 Proof of Theorem 1

4.1 Walks and contributions

Using independence of families Ξ and $\Lambda^{(p)}_N$, we have

$$M^{(N,p)}_k = \int \mathbf{E}\{\lambda^k d\sigma_{A^{(N,p)}}\} = \mathbf{E}\left(\frac{1}{N}\sum_{i=1}^{N}[\lambda^{(N,p)}_i]^k\right) = \frac{1}{N}\mathbf{E}\left(Tr[A^{(N,p)}]^k\right) =$$

$$=\frac{1}{N}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\cdots\sum_{j_k=1}^{N}\mathbf{E}\left(A^{(N,p)}_{j_1,j_2}A^{(N,p)}_{j_2,j_3}\cdots A^{(N,p)}_{j_k,j_1}\right)=$$

$$=\frac{1}{N}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\cdots\sum_{j_k=1}^{N}\mathbf{E}\left(a_{j_1,j_2}a_{j_2,j_3}\cdots a_{j_k,j_1}\right)\cdot\mathbf{E}\left(d^{(N,p)}_{j_1,j_2}d^{(N,p)}_{j_2,j_3}\cdots d^{(N,p)}_{j_k,j_1}\right). \quad (4.1)$$

Consider $W_k^{(N)}$ the set of closed walks of k steps over the set $\overline{1,N}$:

$$W_k^{(N)}=\{w=(w_1,w_2,\cdots,w_k,w_{k+1}=w_1):\forall i\in\overline{1,k+1}\ \ w_i\in\overline{1,N}\}.$$

For $w\in W_k^{(N)}$ let us denote $a(w)=\prod_{i=1}^k a_{w_i,w_{i+1}}$ and $d^{(N,p)}(w)=\prod_{i=1}^k d^{(N,p)}_{w_i,w_{i+1}}$. Then we have

$$M_k^{(N,p)}=\frac{1}{N}\sum_{w\in W_k^{(N)}}\mathbf{E}a(w)\cdot\mathbf{E}d^{(N,p)}(w). \quad (4.2)$$

Let $w\in W_k^{(N)}$ and $f,g\in\overline{1,N}$. Denote by $n_w(f,g)$ the number of steps $f\to g$ and $g\to f$;

$$n_w(f,g)=\$\{i\in\overline{1,k}:\ (w_i=f\ \wedge\ w_{i+1}=g)\vee(w_i=g\ \wedge\ w_{i+1}=f)\}.$$

Then

$$\mathbf{E}a(w)=\prod_{f=1}^{N}\prod_{g=f}^{N}X_{n_w(f,g)}.$$

Given $w\in W_k^{(N)}$, let us define the sets $V_w=\cup_{i=1}^k\{w_i\}$ and $E_w=\cup_{i=1}^k\{(w_i,w_{i+1})\}$, where (w_i,w_{i+1}) is a non-ordered pair. It is easy to see that $G_w=(V_w,E_w)$ is a simple non-oriented graph and the walk w covers the graph G_w. Let us call G_w the skeleton of walk w. We denote by $n_w(e)$ the number of passages of the edge e by the walk w in direct and inverse directions. For $(w_j,w_{j+1})=e_j\in E_w$ let us denote $a_{e_j}=a_{w_j,w_{j+1}}=a_{w_{j+1},w_j}$. Then we obtain

$$\mathbf{E}a(w)=\prod_{e\in E_w}\mathbf{E}a_e^{n_w(e)}=\prod_{e\in E_w}X_{n_w(e)}.$$

Similarly we can write

$$\mathbf{E}d^{(N,p)}(w)=\prod_{e\in E_w}\mathbf{E}\left([d_e^{(N,p)}]^{n_w(e)}\right)=\prod_{e\in E_w}\frac{1}{N\cdot p^{n_w(e)/2-1}}.$$

Then, we can rewrite (4.2) in the form

$$M_k^{(N,p)}=\frac{1}{N}\sum_{w\in W_k^{(N)}}\prod_{e\in E_w}\frac{X_{n_w(e)}}{N\cdot p^{n_w(e)/2-1}}=$$

$$=\sum_{w\in W_k^{(N)}}\left(\frac{1}{N^{|E_w|+1}\cdot p^{k/2-|E_w|}}\prod_{e\in E_w}X_{n_w(e)}\right)=\sum_{w\in W_k^{(N)}}\theta(w), \quad (4.3)$$

where $\theta(w)$ is the contribution of the walk w to the mathematical expectation of the corresponding moment. To perform the limiting transition $N\to\infty$ it is natural to separate W_k^N into classes of equivalence. Walks $w^{(1)}$ and $w^{(2)}$ are equivalent $w^{(1)}\sim w^{(2)}$, if and only if there exists a bijection f between the sets of vertices $V_w^{(1)}$ and $V_w^{(2)}$ such that for $i=1,2,\ldots,k$ $w_i^{(2)}=f(w_i^{(1)})$

$$w^{(1)}\sim w^{(2)}\iff\exists f:\ V_w^{(1)}\overset{bij}{\to}V_w^{(2)}:\ \forall\, i\in\overline{1,k+1}\ \ w_i^{(2)}=f(w_i^{(1)})$$

Let us denote by $[w]$ the class of equivalence of walk w and by $C_k^{(N)}$ the set of such classes. It is obvious that if two walks $w^{(1)}$ and $w^{(2)}$ are equivalent then their contributions are equal.

$$w^{(1)} \sim w^{(2)} \implies \theta(w^{(1)}) = \theta(w^{(2)})$$

Cardinality of the class of equivalence $[w]$ is equal the number of all mappings $f : V_w \to R_f \subset \overline{1,N}$ i.e. $N \cdot (N-1) \cdot \ldots \cdot (N - |V_w| + 1)$. Then we can rewrite (4.3) in the form

$$M_k^{(N,p)} = \sum_{w \in W_k^{(N)}} \left(\frac{1}{N^{|E_w|+1} \cdot p^{k/2-|E_w|}} \prod_{e \in E_w} X_{n_w(e)} \right) =$$

$$= \sum_{[w] \in CW_k^{(N)}} \left(\frac{N \cdot (N-1) \cdot \ldots \cdot (N - |V_w| + 1)}{N^{|E_w|+1} \cdot p^{k/2-|E_w|}} \prod_{e \in E_w} X_{n_w(e)} \right) = \sum_{[w] \in CW_k^{(N)}} \hat{\theta}([w]). \tag{4.4}$$

4.2 Minimal and essential walks

To consider the set $C_k^{(N)}$, it is convenient to relate a given class of equivalence $[w]$ with one particular walk from this class. More precisely, we give the rule to determine this walk that we will call the minimal walk.

Definition 1. *The walk w is a minimal walk, if w_1 (the root of walk) has the number 1 and the number of each new vertex is equal to the number of all already passed vertices plus 1.*

Example 1. The sequences (1,2,1,2,3,1,4,2,1,4,3,1) and (1,2,3,2,4,2,3,2,1,2,4,1,5,1) represent the minimal walks.

Let us denote the set of all minimal walks of $W_k^{(N)}$ by $MW_k^{(N)}$. It is clear that there is only one minimal walk at each class of equivalence and vice versa. Therefore we can rewrite (4.4) in the form

$$M_k^{(N,p)} = \sum_{w \in MW_k^{(N)}} \left(\frac{N \cdot (N-1) \cdot \ldots \cdot (N - |V_w| + 1)}{N^{|E_w|+1} \cdot p^{k/2-|E_w|}} \prod_{e \in E_w} X_{n_w(e)} \right) = \sum_{w \in MW_k^{(N)}} \hat{\theta}([w]). \tag{4.5}$$

Walk w of $W_k^{(N)}$ has at least k vertices. Hence, $MW_k^{(1)} \subset MW_k^{(2)} \subset \ldots \subset MW_k^{(i)} \subset \ldots MW_k^{(k)} = MW_k^{(k)} = \ldots$. It is natural to denote $MW_k = MW_k^{(k)}$. Then (4.5) can be written as

$$m_k^{(p)} = \lim_{N \to \infty} M_k^{(N,p)} = \lim_{N \to \infty} \sum_{w \in MW_k} \left(\frac{N \cdot (N-1) \cdot \ldots \cdot (N - |V_w| + 1)}{N^{|E_w|+1} \cdot p^{k/2-|E_w|}} \prod_{e \in E_w} X_{n_w(e)} \right) =$$

$$= \lim_{N \to \infty} \sum_{w \in MW_k} \left(N^{|V_w|-|E_w|-1} \prod_{e \in E_w} \frac{X_{n_w(e)}}{p^{k/2-1}} \right). \tag{4.6}$$

The set MW_k is finite. Regarding this and (4.6), we conclude that the minimal walk w has non-vanishing contribution, if $|V_w| - |E_w| - 1 \geq 0$. But for each simple connected graph $G = (V, E)$ $|V_w| \leq |E_w| + 1$, and the equality takes place if and only if the graph G is a tree.

Definition 2. *The minimal walk w is an essential walk, if its contribution in the limit $N \to \infty$ is not zero.*

Clearly, each essential walk is a minimal walk that has a tree as a skeleton and vice versa. Then the number of passages of each edge e belonging to the essential walk w is even. Hence, the limiting mathematical expectation $m_k^{(p)}$ depends only on the even moments of random variable of a. It is clear that the limiting mathematical expectation $\lim_{N \to \infty} M_{2s+1}^{(N,p)}$ is equal to zero.

4.3 First edge decomposition of essential walks

Let us start with necessary definitions. The first vertex $w_1 = 1$ of the essential walk w is called the root of the walk. We denote it by ρ. Let us denote the second vertex $w_2 = 2$ of the essential walk w by ν. We denote by l the half of walk's length and by r the number of steps of w starting from root ρ. In Section 3 we explained that we derive the recurrent relations by splitting of the walk (or of the tree) into two parts. To describe this procedure, it is convenient to consider the set of the essential walks of length $2l$ such that they have r steps starting from the root ρ. We denote this set by $\Lambda(l,r)$. One can see that this description is exact, in the sense that it is minimal and gives complete description of the walks we need. Denote by $S(l,r)$ the sum of contributions of the walk of $\Lambda(l,r)$. Let us remove the edge $(\rho,\nu) = (1,2)$ from G_w and denote by $\hat{G}_w$ the graph obtained . The graph $\hat{G}_w$ has two components. Denote the component that contains the vertex ν by G_2 and the component containing the root ρ by G_1. Add the edge (ρ,ν) to the edge set of the tree G_2. Denote the result of this operation by $\hat{G}_2$. On Figure 2 one can see examples of G_2, G_1, $\widehat{G}_2$. Denote by u the half of the walk's length over the tree G_2 and by f the number of steps (ρ,ν) in the walk w. It is clear that the following inequalities hold for all essential walks (excepting the walk of length zero) $1 \le f \le r$, $r + u \le l$. Let us denote by $\Lambda_1(l,r,u,f)$ the set of the essential walks with fixed parameters l, r, u, f and by $S_1(l,r,u,f)$ the sum of contributions of the walks of $\Lambda_1(l,r,u,f)$. Denote by $\Lambda_2(l,r)$ the set of the essential walks of $\Lambda(l,r)$ such that their skeleton has only one edge attached the root ρ. Also we denote by $S_2(l,r)$ the sum of contributions of the walk of $\Lambda_2(l,r)$. Now we can formulate the first lemma of decomposition. It allows express S as a function of the S, S_2.

Lemma 1 (First decomposition lemma). *The following relation holds*

$$\mathrm{S}(l,r) = \sum_{f=1}^{r} \sum_{u=0}^{l-r} \mathrm{S}_1(l,r,u,f), \tag{4.7}$$

where

$$\mathrm{S}_1(l,r,u,f) = \binom{r-1}{f-1} \cdot \mathrm{S}_2(f+u,f) \cdot \mathrm{S}(l-u-f, r-f) \tag{4.8}$$

Proof. The first equality is obvious. The second equality follows from the bijection

$$\begin{aligned} \Lambda_1(l,r,u,f) \overset{bij}{\to} \Lambda_2(f+u,f) \times \Lambda(l-u-f, r-f) \times \\ \times \Theta_1(r,f), \end{aligned} \tag{4.9}$$

where $\Theta_1(r,f)$ is the set of sequences of 0 and 1 of length r such that there are exactly f symbols 1 in the sequence and the first symbol is 1. Equality (4.8) is illustrated by Figure 3.

Let us construct this mapping F. Regarding one particular essential walk w of $\Lambda_1(l,r,u,f)$, we consider the first edge e_1 of the graph G_w and separate w in two parts, the left and the right ones with respect to this edge e_1. Then we add a special code that determines the transitions from the left part to the right one and back through the root ρ. Obviously these two parts are walks, but not necessary minimal walks. Then we minimize these walks. This decomposition is constructed by the following algorithm. We run over w and simultaneously draw the left part, the right part, and code. If the current step belongs to G_l, we add it to the first part, otherwise we add this step to the second part. The code is constructed as follows. Each time the walk leaves the root the sequence is enlarged by one symbol. If current step is (ρ,ν) and "0" otherwise, this symbol is "1". It is clear that the first element of the sequence is "1", the number of signs "1" is equal to f, and the full length of the sequence is r. Now we minimize the left and the right parts. Thus, we have constructed the decomposition of he essential walk w and the mapping F.

Example 2. For $w = (1,2,1,2,3,2,1,4,1,2,5,2,1,4,6,4,1,2,5,2,3,2,3,2,1,4,1)$ the left part, the right one, and the code are $(1,2,1,2,3,2,1,2,4,2,1,2,4,2,3,2,3,2,1)$, $(1,2,1,2,3,2,1,2,1)$, $(1,1,0,1,0,1,0)$, respectively.

Let us denote the left part by $(w^{(f)})$ and the right part by $(w^{(s)})$. These parts are really walks with the root ρ. For each edge e in the tree $\hat{G}_2$ the number of passages of e of the essential walk w is equal to the corresponding number of passages of e of the left part $(w^{(f)})$. Also for each edge e belonging to the tree G_1 the number of passages of e of essential walk w is equal to the corresponding number of passages of e of the right part $(w^{(s)})$. The weight of the essential walk is multiplicative with respect to edges. Then the weight of the essential walk w is equal to the product of weights of left and right parts. The walk of zero length has unit weight. Combining this with (4.9), we obtain

$$\mathrm{S}_1(l,r,u,f) = |\Theta_1(r,f)| \cdot \mathrm{S}_2(f+u,f) \cdot \mathrm{S}(l-u-f, r-f). \tag{4.10}$$

Taking into account that $|\Theta_1(r,f)| = \binom{r-1}{f-1}$, we derive from (4.10) (4.8).

Now let us prove that for any given elements $w^{(f)}$ of $\Lambda_2(f+u,f)$, $w^{(s)}$ of $\Lambda(l-u-f,r-f)$, and the sequence $\theta \in \Theta_1(r,f)$, one can construct one and only one element w of $\Lambda_1(l,r,u,f)$. We do this with the following gathering algorithm. We go along either $w^{(f)}$ or $w^{(s)}$ and simultaneously draw the walk w. The switch from $w^{(f)}$ to $w^{(s)}$ and back is governed by the code sequence θ. In fact, this procedure is inverse to the decomposition procedure described above up to the fact that $w^{(s)}$ is minimal. This difficulty can be easily resolved for example by coloring vertices of $w^{(f)}$ and $w^{(s)}$ in red and blue colors respectively. Certainly, the common root of $w^{(f)}$ and $w^{(s)}$has only one color. To illustrate the gathering procedures we give the following example.

Example 3. For $w^{(f)} = (1,2,1,2,3,2,1,2,4,2,1,2,4,2,3,2,3,2,1)$, $w^{(s)} = (1,2,1,2,3,2,1,2,1)$, $\theta = (1,1,0,1,0,1,0)$ the gathering procedure gives $w = (1,2,1,2,3,2,1,4,1,2,5,2,1,4,6,4,1,2,5,2,3,2,3,2,1,4,1)$.

It is clear that the decomposition and gathering are injective mappings. Their domains are finite sets, and therefore the corresponding mapping (4.9) is bijective. This completes the proof of Lemma 1.■

To formulate Lemma 2, let us give necessary definitions. We denote by v the number of steps starting from the root ρ except the step $\overrightarrow{(\rho,\nu)}$ and by $\Lambda_3(u+f,f,v)$ the set of essential walks of $\Lambda_2(u+f,f)$ with fixed parameter v. Also we denote by $S_3(u+f,f,v)$ the sum of weights of walks of $\Lambda_3(u+f,f,v)$. Let us denote by $G_{1,2}$ the graph consisting of only one edge (ρ,ν) and by $\Lambda_4(f)$ the set of essential walks of length $2f$ such that their skeleton coincides with the graph $G_{1,2}$. It is clear that $\Lambda_4(f)$ consists of the only one walk (1,2,1,2,...,2,1) of weight $\frac{X_{2f}}{p^{f-1}}$. The previous lemma allows us to express S_2 as a function of S. The next lemma allows to express S_2 as a function of S. Thus, two lemmas allow us to express S as a function of S.

Lemma 2 (Second decomposition lemma).

$$\mathrm{S}_2(f+u,f) = \sum_{v=0}^{u} \mathrm{S}_3(f+u,f,v) \tag{4.11}$$

$$\mathrm{S}_3(f+u,f,v) = \binom{f+v-1}{f-1} \cdot \frac{X_{2f}}{p^{f-1}} \cdot \mathrm{S}(u,v) \tag{4.12}$$

The first equality is trivial, the second one follows from the bijection

$$\Lambda_3(f+u,f,v) \overset{bij}{\to} \Lambda(u,v) \times \Lambda_4(f) \times \Theta_2(f+v,f), \tag{4.13}$$

where $\Theta_2(f+v,f)$ is the set of sequences of 0 and 1 of length $f+v$ such that there are exactly f symbols 1 in the sequence and last symbol of it is 1. The proof is analogous to the proof of the first decomposition lemma. Equality (4.12) is illustrated by Figure 4.

4.4 Recurrent relations for S

Combining these two decomposition lemmas and changing the order of summation, we get the recurrent relations

$$\mathrm{S}(l,r) = \sum_{f=1}^{r} \binom{r-1}{f-1} \cdot \frac{X_{2f}}{p^{f-1}} \cdot \sum_{u=0}^{l-r} \mathrm{S}(l-u-f, r-f) \cdot \sum_{v=0}^{u} \binom{f+v-1}{f-1} \cdot \mathrm{S}(u,v),$$

with the initial condition $\mathrm{S}(l,0) = \delta_l$. This gives (2.8).

Using this system of recurrent relations, one can obtain information about limiting σ. For example, one can observe that the support of the limiting measure σ is unbounded even when the support of the distribution of $\{a_{i,j}\}$ is finite. This fact follows from inequality

$$M_{4k} \geq (\mathrm{C} \cdot k)^k, \tag{4.14}$$

where C is a constant. To explain (4.14), let us denote by Ψ the set of essential walks of length $4k$ such that the root ρ belongs to each of the edges of the skeleton and each edge is passed 4 times. Weight of the essential walk of Ψ is equal to $\left(\frac{X_4}{p}\right)^k$. Cardinality of Ψ equals $(2k-1)!!$. This implies (4.14).

Finally, let us note that using the technique developed, one can derive recurrent relations that determine the coefficients of $\frac{1}{p}$-expansion of $m_l^{(p)}$

$$m_l^{(p)} = \sum_{i=0}^{l-1} \left(\sum_{r=0}^{l} \mathrm{S}(l,r,i) \right) \cdot \frac{1}{p^i}. \tag{4.15}$$

Then we get

$$\mathrm{S}(l,r,i) = \sum_{f=1}^{r} \binom{r-1}{f-1} \cdot X_{2f} \cdot \sum_{u=0}^{l-r} \sum_{j=0}^{(l-u-f-1)\cdot(1-\delta_{l-u-f})} \mathrm{S}(l-u-f, r-f, j) \cdot$$

$$\cdot \sum_{v=0}^{u} \binom{f+v-1}{f-1} \cdot \mathrm{S}(u,v,i-f-j+1) \tag{4.16}$$

with the initial condition $S(l,0,i) = \delta_l \cdot \delta_i$. We do not explain the detail of this derivation. Similar formulas are obtained in [2]. The difference is that in [2] matrices are not normalized by $\frac{1}{\sqrt{p}}$. This leads to expressions for $\frac{1}{p}$-terms different from our (4.16). Relations (4.12) provide more information about the properties of $m_l^{(p)}$ than relations (2.8). As the result, (4.13) are more combersome than (2.8).

5 Laplace operator

Regarding the Laplace operator, we have to modify our method. In this case the random variable Δ_{ii} is given by the sum of A_{ij} and therefore is dependent on random variables a_{ij}. Each of non-diagonal entries differs from the corresponding entry of the weighted adjacency matrix by the sign only. Each diagonal entry of $\Delta_\Gamma^{(\xi)}$ equals to the sum of all entries of the same line of the corresponding weighted adjacency matrix. Taking into account this observation one can write

$$L_s^{(N,p)} = \mathbf{E}\left\{\int \lambda^s d\sigma\left(\lambda; \Delta_\Gamma\right)\right\} = \mathbf{E}\frac{1}{N}\mathrm{Tr}\left[\Delta_\Gamma\right]^s = \\ = \mathbf{E}\frac{1}{N}\mathrm{Tr}\left[B - A\right]^s = \sum_{i\in\overline{1,N}^s}\sum_{b\in\{0,1\}^s}\mathbf{E}\left(K_{i_1 i_2}^{(b_1)}\cdot K_{i_2 i_3}^{(b_2)}\cdot\ \ldots\cdot K_{i_s i_1}^{(b_s)}\right), \tag{5.1}$$

where $K_{ij}^{(0)} = -A_{ij}$, $K_{ij}^{(1)} = B_{ij}$. Let us introduce the symbol $M_{ij} = 1 - \delta_{ij} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i \neq j \end{cases}$, then B can be rewritten in the form

$$B_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ \sum_{l=1}^{N} A_{il}\cdot M_{li}, & \text{if } i = j \end{cases}. \tag{5.2}$$

Given numbers $b_1, b_2, \ldots, b_s$, we substitute (5.2) into (5.1) and change the order of sums over l's and $\mathbf{E}\{\cdot\}$ and observe that the mathematical expectation depends the product of A's only. The difference between this representation and that of (4.1) is that the moment $L_s^{(N,p)}$ is expressed as the sum of weights of closed walks of s steps. A step can be usual or special (double). Let us explain the nature of the special step that corresponds to the factor $A_{ij}\cdot M_{ji}$. We denote it by an arrow from i to j. To turn back to the walk we add the step $\overrightarrow{(j,i)}$ which is represented by M_{ji}. This step can be regarded as the imaginary one because it does not contribute to the length of the walk and to the weight (mathematical expectation) of the walk. In the figures we denote the special step corresponding to the factor $A_{ij}\cdot M_{ji}$ by an arrow from i to j.

As before, we determine the classes of equivalence of the walks, the minimal walks, and the essential walks. In the case of Δ_Γ the essential walks are the minimal walks that have a tree as a skeleton. Each of usual steps $\overrightarrow{(j,k)}$ of the essential walk corresponds to one

usual step $\overrightarrow{(k,j)}$ only. Then if there are b usual steps $\overrightarrow{(j,k)}$, c special steps $\overrightarrow{(j,k)}$, and d special steps $\overrightarrow{(k,j)}$, then the edge (k,j) has the weight $(-1)^{2b}\cdot X_{2b+c+d} = X_{2b+c+d}$.

Let us give necessary definitions and formulate two analogs of the decomposition lemmas. Denote by l the number of usual and special steps of the essential walk w, by r_1 the number of steps starting from the root ρ and by f the number of usual and special steps $\overrightarrow{(\rho,\nu)}$. We denote by $\widehat{\Lambda}(l,r_1)$ the set of the essential walks of l steps such that they have r_1 steps starting from the root ρ and by $\widehat{\mathrm{S}}(l,r_1)$ the sum of contributions of the walk of $\widehat{\Lambda}(l,r_1)$. Let d be the length of the walk over the tree G_1. Denote by $\widehat{\Lambda}_1(l,r_1,d,f)$ the set of essential walks with fixed parameters l,r_1,d, and f and by $\widehat{\mathrm{S}}_1(l,r_1,d,f)$ the sum of weights of walks of $\widehat{\Lambda}_1(l,r_1,d,f)$. Let us denote by $\widehat{\Lambda}_2(l,r_1)$ the set of essential walks of $\widehat{\Lambda}(l,r_1)$ such that their skeleton has only one edge attached to the root ρ and by $\widehat{\mathrm{S}}_2(l,r_1)$ the sum of contributions of the walks of $\widehat{\Lambda}_2(l,r_1)$.

Lemma 3 (Third decomposition lemma).

$$\widehat{\mathrm{S}}(l,r_1) = \sum_{f=1}^{r_1}\sum_{d=0}^{l-f}\widehat{\mathrm{S}}_1(l,r_1,d,f) \tag{5.3}$$

$$\widehat{\mathrm{S}}_1(l,r_1,d,f) = \binom{r_1-1}{f-1}\cdot\widehat{\mathrm{S}}_2(l-d,f)\cdot\widehat{\mathrm{S}}(d,r_1-f) \tag{5.4}$$

The first equality is trivial, the second one follows from the bijection

$$\widehat{\Lambda}_1(l, r_1, d, f) \overset{bij}{\to} \widehat{\Lambda}_2(l-d, f) \times \widehat{\Lambda}(d, r_1 - f) \times \Theta_1(r_1, f). \tag{5.5}$$

The proof is analogous to the proof of the first decomposition lemma. Equality (5.4) is illustrated by Figure 5.

To formulate the fourth decomposition lemma, let us give necessary definitions. We denote by r_2 the number of steps starting from the root ρ (excepting the usual and special steps $\overrightarrow{(\rho,\nu)}$), by n the number of usual steps $\overrightarrow{(\rho,\nu)}$, by f_1 the number of special steps $\overrightarrow{(\rho,\nu)}$ and by f_2

the number of special steps $\overrightarrow{(\nu,\rho)}$. Denote by $\widehat{\Lambda}_3(l, n, f_1, f_2, r_2)$ the set of essential walks with fixed parameters l,n,f_1,f_2,r_2 such that their skeletons have only one edge attached to the root ρ. Let $\widehat{\mathrm{S}}_3(l, n, f_1, f_2, r_2)$ be the sum of weights of walks of $\widehat{\Lambda}_3(l, n, f_1, f_2, r_2)$. Denote by $\widehat{\Lambda}_4(n, f_1, f_2)$ the set of essential walks of length $2n + f_1 + f_2$ with fixed parameters n,f_1,f_2 such that their skeletons coincide with the graph $G_{1,2}$. It is not hard to see that for the case $n \geq 1$ one has the equality $|\widehat{\Lambda}_4(n, f_1, f_2)| = \binom{n+f_1}{n} \cdot \binom{n+f_2-1}{n-1}$. It is clear that each of the walks of $\widehat{\Lambda}_4(n, f_1, f_2)$ has the weight $\frac{X_{2n+f_1+f_2}}{p^{n+f_1/2+f_2/2-1}}$.

Lemma 4 (Fourth decomposition lemma).

$$\widehat{\mathrm{S}}_2(l, r_1, d, f) = \sum_{n=0}^{min\{[(l-d)/2)], f\}} \sum_{f_2=0}^{l-d-f-n} \sum_{r_2=0}^{l-d-f-n-f_2} \widehat{\mathrm{S}}_3(l-d, n, f-n, f_2, r_2) \tag{5.6}$$

$$\widehat{\mathrm{S}}_3(l-d, n, f_1, f_2, r_2) = \begin{cases} \binom{n+f_2+r_2-1}{r_2} \cdot \binom{n+f_1}{n} \cdot \binom{n+f_2-1}{n-1} \cdot \widehat{\mathrm{S}}(l-d-n-f-f_2, r_2), & \text{if } n \geq 1 \\ \delta_{l-d-f_1} \delta_{f_2} \delta_{r_2} \frac{X_{2n+f_1+f_2}}{p^{n+f_1/2+f_2/2-1}}, & \text{if } n=0 \end{cases} \tag{5.7}$$

The first equality is trivial, the second one follows from the bijection

$$\widehat{\Lambda}_2(l-d, n, f_1, f_2, r_2) \overset{bij}{\to} \widehat{\Lambda}(l-d-2n-f_1-f_2, r_2) \times \widehat{\Lambda}_3(n, f_1, f_2) \times$$
$$\times \Theta_2(n + f_2 + r_2, n + f_2), \tag{5.8}$$

The proof is analogous to the proof of the first decomposition lemma. Equation (5.7) is illustrated by Figure 6.

Combining this two lemmas, we get expression for $\widehat{\mathrm{S}}$. This expression is the sum over all admissible values of f, d, n, f_2, r_2. Let us change the order of summation. On the one hand, the number n of usual steps $\overrightarrow{(\rho,\nu)}$ is not greater than the number f of all steps $\overrightarrow{(\rho,\nu)}$, on the other hand the inequality $2n + f_1 \leq l$ holds because each of usual steps $\overrightarrow{(\rho,\nu)}$ corresponds to the step $\overrightarrow{(\nu,\rho)}$. Then $n \leq min(r_1, l - r_1)$. The number f_1 of special steps $\overrightarrow{(\rho,\nu)}$ is not greater than the number r_1 of all steps starting from the root ρ minus the number n usual steps $\overrightarrow{(\rho,\nu)}$. Then f_1 changes from 0 to $r_1 - n$. Now there are only $l - r_1 - n$ free steps. Then the number f_2 of special steps $\overrightarrow{(\nu,\rho)}$ can be changed from 0 to $l - r_1 - n$. Now it remains $l - r_1 - n - f_2$ free steps. The walk's length d over the tree G_1 is not less than the number r_1 of steps starting from the root ρ minus the number $n + f_1$ of steps $\overrightarrow{(\rho,\nu)}$. Then $r_1 - f_1 - n \leq d \leq l - r_1 - n - f_2$. Now there are only $l - d - 2n - f_1 - f_2$ free steps. In the case $n = 0$, the expression is simplified to $\sum_{f_1=1}^{r_1} C_{r_1-1}^{f_1-1} \cdot \widehat{\mathrm{S}}(l - f_1, r_1 - f_1)$. The relations described above are illustrated by Figure 7.

$$\widehat{\mathrm{S}}(l, r_1) = \sum_{n=1}^{min(r_1, l-r_1)} \sum_{f_1=0}^{r_1-n} \binom{n+f_1}{n} \binom{r_1-1}{n+f_1-1}.$$

$$\sum_{f_2=0}^{l-r_1-n} \binom{n+f_2-1}{n-1} \cdot \frac{X_{(2n+f_1+f_2)}}{p^{(2n+f_1+f_2)/2-1}} \sum_{d=r_1-f_1-n}^{l-r_1-n-f_2} \widehat{\mathrm{S}}(d, r_1-n-f_1)\cdot$$

$$\cdot \sum_{r_2=0}^{l-d-2n-f_1-f_2} \binom{r_2+f_2+n-1}{r_2} \cdot \widehat{\mathrm{S}}(l-d-2n-f_1-f_2, r_2) + \sum_{f_1=1}^{r_1} \binom{r_1-1}{f_1-1} \cdot \widehat{\mathrm{S}}(l-f_1, r_1-f_1) \quad (5.9)$$

with the initial condition

$$\widehat{\mathrm{S}}(l,0) = \delta_{l,0}.$$

Let us denote by $g_1 = f_1 + n$, by $g_2 = f_2 + n$. Using the identity $\sum_{n=1}^{min\{g_1,g_2\}} \binom{g_1}{n} \cdot \binom{g_2-1}{n-1} = \binom{g_1+g_2-1}{g_1-1}$ and (5.9), we get

$$\widehat{\mathrm{S}}(l,r_1) = \sum_{g_1=1}^{r_1} \binom{r_1-1}{g_1-1} \cdot \left(\widehat{\mathrm{S}}(l-g_1, r_1-g_1) \cdot \frac{X_{g_1}}{p^{g_1/2-1}} + \sum_{d=r_1-g_1}^{l-r_1} \widehat{\mathrm{S}}(d, r_1-g_1)\cdot \right.$$

$$\left. \cdot \sum_{g_2=1}^{l-d-g_1} \binom{g_1+g_2-1}{g_1-1} \cdot \frac{X_{g_1+g_2}}{p^{(g_1+g_2)/2-1}} \sum_{r_2=1}^{l-d-g_1-g_2} \binom{r_2+g_2-1}{g_2-1} \cdot \widehat{\mathrm{S}}(l-d-g_1-g_2, r_2) \right)$$

If $a_{ij} \equiv 1$ and $p = 1$, we obtain

$$\widehat{\mathrm{S}}(l,r_1) = \sum_{g_1=1}^{r_1} \binom{r_1-1}{g_1-1} \cdot \left(\widehat{\mathrm{S}}(l-g_1, r_1-g_1) + \sum_{d=r_1-g_1}^{l-r_1} \widehat{\mathrm{S}}(d, r_1-g_1)\cdot \right.$$

$$\left. \cdot \sum_{g_2=1}^{l-d-g_1} \binom{g_1+g_2-1}{g_1-1} \cdot \sum_{r_2=1}^{l-d-g_1-g_2} \binom{r_2+g_2-1}{g_2-1} \cdot \widehat{\mathrm{S}}(l-d-g_1-g_2, r_2) \right)$$

If we want to find the coefficients of $\frac{1}{p}$-expansion of $l_k^{(p)}$

$$l_k^{(p)} = \sum_{i=-k}^{k-2} \left(\sum_{r_1=0}^{k} \widehat{\mathrm{S}}(k, r_1, i) \right) \cdot \frac{1}{p^{i/2}}, \quad (5.10)$$

we can apply the method described above. Then after some calculations we get the following recurrent relations

$$\widehat{\mathrm{S}}(k,r_1,i) = \sum_{g_1=1}^{r_1} \binom{r_1-1}{g_1-1} \cdot \left(\widehat{\mathrm{S}}(k-g_1, r_1-g_1, i+2-g_1) \cdot X_{g_1} + \right.$$

$$+ \sum_{d=r_1-g_1}^{k-r_1} \sum_{j=-d}^{d-2\cdot(1-\delta_d)} \widehat{\mathrm{S}}(d, r_1-g_1, j) \cdot \sum_{g_2=1}^{k-d-g_1} \binom{g_1+g_2-1}{g_1-1} \cdot \frac{X_{g_1+g_2}}{p^{(g_1+g_2)/2-1}} \cdot$$

$$\left. \cdot \sum_{r_2=1}^{k-d-g_1-g_2} \binom{r_2+g_2-1}{g_2-1} \cdot \widehat{\mathrm{S}}(k-d-g_1-g_2, r_2, i+2-g_1-g_2-j) \right) \quad (5.11)$$

with the initial condition $\widehat{\mathrm{S}}(l,0,i) = \delta_{l,0} \cdot \delta_{i,0}$.

Similar formulas are obtained in [2]. The difference is that in [2] matrices are not normalized by $\frac{1}{\sqrt{p}}$. This leads to expressions for $\frac{1}{p}$-terms different from our (5.11).

In conclusion, let us discuss the limiting transition $p \to \infty$ in (2.8) and (2.10). Regarding the first sum of the relation for the limiting moments of the adjacency matrix (2.8), one can easily observe that the terms with $f > 1$ vanish in the limit of infinite p. Then (2.8) to gather with (2.7) lead to the recurrent relations

$$m_k = X_2 \sum_{u=0}^{k-1} m_{k-1-u} m_u$$

that certainly determine the semicircle law [22].

Let us turn to the Laplace case. In general, it is impossible to pass to the limit $p \to \infty$ in relations (2.10) because there is the term of the order $p^{1/2}$. However, if one pass to the case of random variables a with zero mean value, $X_1 = 0$, then the limit $p \to \infty$ leads to the following recurrent relations

$$\widehat{\mathrm{S}}(l, r_1) = X_2 \cdot \left((r_1 - 1) \cdot \widehat{\mathrm{S}}(l-2, r_1 - 2) + \sum_{d=r_1-1}^{l-r_1} \widehat{\mathrm{S}}(d, r_1 - 1) \cdot \sum_{r_2=1}^{l-d-2} \widehat{\mathrm{S}}(l-d-2, r_2) \right) \quad (5.12)$$

with the initial condition $\widehat{\mathrm{S}}(l, 0) = \delta_{l,0}$.

These recurrent relations obviously differ from those for the semicircle law. Using resolvent approach, we show at the end of Section 6 that the limiting moments

$$l_s = \lim_{p\to\infty} l_s^{(p)} = \sum_{i=0}^{s} \hat{S}(s, i)$$

determine the distribution known as the deformed semicircle law (see [16]).

6 Proofs of Theorems 3 and 4

Proof of Theorem 3. It is easy to see that $G_{11}^{(N,p)}(z)$ can be represented in the form

$$G_{11}^{(N,p)}(z) = \left(z + \sum_{j,k=2}^{N} \tilde{G}_{jk}^{(N-1,p)} A_{1j}^{(N,p)} A_{1k}^{(N,p)} \right)^{-1}, \quad (6.1)$$

where the matrix $\{\tilde{G}_{ij}^{(N-1,p)}(z)\}_{i,j=2}^{N}$ is the resolvent of the matrix $i\tilde{A}^{(N-1,p)}$, which can be obtained from $A^{(N,p)}$ if we replace $\{A_{1j}^{(N,p)}\}_{j=2}^{N}$, $\{A_{j1}^{(N,p)}\}_{j=2}^{N}$ by zeros. We remark here that in order to simplify formulas in this section we assume that $A_{jj}^{(N,p)} = 0$. The general case can be studied by the same way. Let us use the formula (see [1]):

$$e^{-ua^2 R} = 1 - u^{1/2}|a| \int_0^\infty dv \frac{\mathcal{J}_1(2|a|\sqrt{uv})}{\sqrt{v}} \exp\{-R^{-1}v\}, \quad (6.2)$$

which is valid for any $u \geq 0$, $\Re R > 0$. Then, on the basis of (6.1), we get

$$\begin{aligned} \exp\{-ua_1^2 G_{11}^{(N,p)}\} = 1 - u^{1/2}|a_1| \int_0^\infty dv \frac{\mathcal{J}_1(2|a_1|\sqrt{uv})}{\sqrt{v}} \\ \exp\{-zv - v \sum_{j,k=2}^{N} \tilde{G}_{ij}^{(N-1,p)} A_{1i}^{(N,p)} A_{1j}^{(N,p)}\}. \end{aligned} \quad (6.3)$$

Denote

$$\tilde{R}_N = \sum_{j\neq k} \tilde{G}_{jk}^{(N-1,p)} A_{1j}^{(N,p)} A_{1k}^{(N,p)}. \quad (6.4)$$

One can see easily that

$$\mathbf{E}\{|\tilde{R}_N|^2\} \le 2\frac{X_2^2}{N|\Re z|^2} + \frac{p^2X_1^4}{N^2|\Re z|^2} + \frac{pX_1^2X_2}{N^2|\Re z|^2}. \tag{6.5}$$

Indeed,

$$\begin{aligned} \mathbf{E}\{|\tilde{R}_N|^2\} = &\sum_{j_1\ne j_2\ne k_1\ne k_2} \mathbf{E}\Big\{ G_{j_1k_1}^{(N-1,p)}\overline{\tilde{G}_{j_2k_2}^{(N-1,p)}} A_{1j_1}^{(N,p)}A_{1j_2}^{(N,p)}A_{1k_1}^{(N,p)}A_{1k_2}^{(N,p)}\Big\} \\ &+4\sum_{j,k_1\ne k_2} \mathbf{E}\Big\{ \tilde{G}_{jk_1}^{(N-1,p)}\overline{\tilde{G}_{jk_2}^{(N-1,p)}} |A_{1j}^{(N,p)}|^2 A_{1k_1}^{(N,p)}A_{1k_2}^{(N,p)}\Big\} \\ &+2\sum_{j\ne k} \mathbf{E}\Big\{ \tilde{G}_{jk}^{(N-1,p)}\overline{\tilde{G}_{jk}^{(N-1,p)}} |A_{1j}^{(N,p)}|^2|A_{1k}^{(N,p)}|^2\Big\} = I + 4II + 2III. \end{aligned} \tag{6.6}$$

Averaging with respect to $\{A_{1,i}^{(N,p)}\}_{i=2}^N$ and using the fact that $\{\tilde{G}_{ij}^{(N-1,p)}(z)\}_{i,j=2}^N$ do not depend on $A_{1,i}^{(N,p)}$, we obtain

$$\begin{aligned} I &\le X_1^4\frac{p^2}{N^2}\mathbf{E}\Big\{\Big|N^{-1}\sum_{j,k}(z-i\tilde{A}^{(N-1,p)})_{jk}^{-1}\Big|^2\Big\} \le \frac{p^2X_1^4}{N^2|\Re z|^2}, \\ II &\le X_1^2X_2\frac{p}{N^3}\sum_{k_1\ne k_2}\mathbf{E}\Big\{[(z-i\tilde{A}^{(N-1,p)})(\overline{z}+i\tilde{A}^{(N-1,p)})]_{k_1k_2}^{-1}\Big\} \le \frac{pX_1^2X_2}{N^2|\Re z|^2}, \\ III &\le \frac{X_2^2}{N^2}\sum_k \mathbf{E}\Big\{[(z-i\tilde{A}^{(N-1,p)})(\overline{z}+i\tilde{A}^{(N-1,p)})]_{kk}^{-1}\Big\} \le \frac{X_2^2}{N|\Re z|^2}. \end{aligned}$$

Besides, since evidently

$$\Re\Big\{\sum \tilde{G}_{ij}^{(N-1,p)}A_{1i}^{(N,p)}A_{1j}^{(N,p)}\Big\} \ge 0, \quad \Re\Big\{\sum \tilde{G}_{jj}^{(N-1,p)}|A_{jj}^{(N,p)}|^2\Big\} \ge 0,$$

the inequality $|e^{-z_1}-e^{-z_2}| \le |z_1-z_2|$ (valid for $\Re z_1, \Re z_2 \ge 0$) and (6.3) imply

$$\begin{aligned} \exp\{-ua_1^2G_{11}^{(N,p)}\} = 1 - u^{1/2}|a_1|\int_0^\infty dv\frac{\mathcal{J}_1(2|a_1|\sqrt{uv})}{\sqrt{v}} \\ \exp\{-zv - v\sum \tilde{G}_{ij}^{(N-1,p)}|A_{1j}^{(N,p)}|^2\} + \tilde{r}_N(u), \end{aligned} \tag{6.7}$$

where

$$|\tilde{r}_N(u)| \le |\tilde{R}_N|u^{1/2}|a_1|\int_0^\infty dv\Big|\frac{\mathcal{J}_1(2|a_1|\sqrt{uv})}{\sqrt{v}}e^{-zv}\Big| \le C|\tilde{R}_N|u^{1/2}|a_1||\Re z|^{-3/2}.$$

Here and below we denote by C some constants (different in different formulas), which do not depend on N, z, p. Taking into account (6.5), we get

$$\mathbf{E}\{|\tilde{r}_N(u)|^2\} \le \frac{Cup}{N|\Re z|^3}. \tag{6.8}$$

Averaging with respect to a_1 and $\{A_{1,i}^{(N,p)}\}_{i=2}^N$ we obtain

$$\begin{aligned} \mathbf{E}\Big\{\exp\{-ua_1^2G_{11}^{(N,p)}\}\Big\} = 1 - u^{1/2}\int d\mu(a_1)|a_1|\int_0^\infty dv\frac{\mathcal{J}_1(2|a_1|\sqrt{uv})}{\sqrt{v}} \\ \times\mathbf{E}\Big\{\exp\{-zv - v\sum_{j=2}^N \tilde{G}_{jj}^{(N-1,p)}|A_{1j}^{(N,p)}|^2\}\Big\} + r_N(u), \end{aligned} \tag{6.9}$$

$$r_N(u) \le \frac{C(up)^{1/2}}{N^{1/2}|\Re z|^{3/2}}.$$

Taking into account that $\{\tilde G_{ij}^{(N-1,p)}(z)\}_{i,j=2}^N$ do not depend on $A_{1,i}^{(N,p)}$, we obtain

$$\begin{aligned}\mathbf{E}\Big\{\exp\{-v\sum G_{jj}^{(N-1,p)}|A_{1j}^{(N,p)}|^2\}\Big\} &= \mathbf{E}\Big\{\prod_{j=2}^N\Big((1-\frac{p}{N})+\frac{p}{N}e^{-va_{1j}^2\tilde G_{jj}^{(N-1,p)}/p}\Big)\Big\}\\ &= e^{-p}\mathbf{E}\Big\{\exp\{p\tilde f_{N-1}(v/p,z)\}\Big\}+R_N(v)\\ |R_N(v)| &\le \frac{Cp^2}{N}.\end{aligned} \tag{6.10}$$

Let us prove that $\tilde f_{N-1}(v/p,z)=\frac{1}{N}\sum e^{-va_{1j}^2\tilde G_{jj}^{(N-1,p)}/p}$ can be replaced by $f_N(v/p,z)$. To this end consider the matrices $A^{(N,p)}(t)=(1-t)A^{(N,p)}+t\tilde A^{(N-1,p)}$, $G(t,z)=(z-iA^{(N,p)}(t))^{-1}$ and the function

$$f_N(u,z,t)=\frac{1}{N}\sum_{i=1}^N e^{-ua_i^2G_{ii}(t,z)}. \tag{6.11}$$

It is easy to see that

$$\begin{aligned}&\Big|\tilde f_{N-1}(u,z)-f_N(u,z)+\frac{1}{Nz}\Big| = \Big|\frac{1}{Nz}+\frac{1}{N}\sum_{i=2}^N e^{-ua_i^2\tilde G_{ii}^{(N-1,p)}(z)}-\frac{1}{N}\sum_{i=1}^N e^{-ua_i^2G_{ii}(z)}\Big|\\ &=\Big|f_N(u,z,1)-f_N(u,z,0)\Big| = \Big|\int_0^1 dt\frac{d}{dt}f_N(u,z,t)\Big|\\ &=2\Big|\int_0^1 dt\frac{u}{N}\sum_{i=2}^N a_i^2G_{ij}(z,t)A_{j1}^{(N,p)}G_{1i}(z,t)e^{-ua_i^2G_{ii}(z,t)}\Big|\\ &\le 2\frac{u}{N|\Re z|^2}\Big[\sum_{j=2}^N|a_i|^4\Big]^{1/2}\Big[\sum_{j=2}^N|A_{j1}^{(N,p)}|^2\Big]^{1/2}\end{aligned} \tag{6.12}$$

where we have used that $||G(t,z)||\le|\Re z|^{-1}$. Therefore, for any $u\in\mathbf{R}$

$$\mathbf{E}\{|\tilde f_{N-1}(u,z)-f_N(u,z)|^2\}\le\frac{u^2X_4^{1/2}X_2^{1/2}}{|\Re z|^4N}+\frac{1}{N^2|z|^2}. \tag{6.13}$$

Hence, (6.9), (6.10) could be rewritten as

$$\begin{aligned}&\mathbf{E}\{f_N(u,z)\}=1-u^{1/2}e^{-p}\int|a_1|d\mu(a_1)\int_0^\infty dv\frac{\mathcal{J}_1(2|a_1|\sqrt{uv})}{\sqrt v}e^{-zv}\mathbf{E}\{e^{pf_N(v/p,z)}\}+r_N'(v)\\ &\mathbf{E}\{|r_N'^2(u)|\}\le\frac{Cp^2u}{|\Re z|^4N}.\end{aligned} \tag{6.14}$$

Now let us prove (2.14). Denote

$$\delta_N(z,u)=f_N(z,u)-\mathbf{E}\{f_N(z,u)\}$$

and observe that due to the symmetry of the problem

$$\begin{aligned}\mathbf{E}\{\delta_N^2(z,u)\}=\frac{N-1}{N}(&\mathbf{E}\{e^{-ua_1^2G_{11}(z)}e^{-ua_2^2G_{22}(z)}\}\\ &-\mathbf{E}\{e^{-ua_1^2G_{11}(z)}\}\mathbf{E}\{e^{-ua_2^2G_{22}(z)}\})+O(N^{-1}).\end{aligned} \tag{6.15}$$

We shall use the formulas (cf. (6.1)):

$$\begin{aligned}G_{\ell\ell}^{(N,p)}=\Big[&\Big(z+\sum_{j,k=3}^N\tilde G_{jk}^{(N-2,p)}A_{\ell j}^{(N,p)}A_{\ell k}^{(N,p)}\Big)\\ &-\Big(\sum_{j,k=3}^N\tilde G_{jk}^{(N-2,p)}A_{1j}^{(N,p)}A_{2k}^{(N,p)}\Big)^2\Big(z+\sum_{j,k=3}^N\tilde G_{jk}^{(N-2,p)}A_{\bar\ell,j}^{(N,p)}A_{\bar\ell k}^{(N,p)}\Big)^{-1}\Big]^{-1},\end{aligned} \tag{6.16}$$

Here $\ell = 1,2$, $\overline{\ell} = 3-\ell$ and $\tilde{G}^{(N-2,p)} = (z - i\tilde{A}^{(N-2,p)})^{-1}$, where $\tilde{A}^{(N-2,p)}$ can be obtained from $A^{(N,p)}$ by replacing $A_{1j}^{(N,p)}, A_{2j}^{(N,p)}$ $A_{j1}^{(N,p)}, A_{j2}^{(N,p)}$ $(j = 3, \dots N)$ by zeros. Similarly to (6.5), one can get that

$$\mathbf{E}\left\{\left|\sum \tilde{G}_{jk}^{(N-2,p)} A_{1j}^{(N,p)} A_{2k}^{(N,p)}\right|^2\right\} \le \frac{Cp}{N|\Re z|^2}, \quad \mathbf{E}\left\{\left|\sum_{j\neq k} \tilde{G}_{jk}^{(N-2,p)} A_{\ell j}^{(N,p)} A_{\ell k}^{(N,p)}\right|^2\right\} \le \frac{Cp}{N|\Re z|^2}.$$

Hence, using (6.2), similarly to (6.3)-(6.6), we get

$$\begin{aligned} &\mathbf{E}\{e^{-ua_1^2 G_{11}^{(N,p)}} e^{-ua_2^2 G_{22}^{(N,p)}}\} - \mathbf{E}\{e^{-ua_1^2 G_{11}^{(N,p)}}\}\mathbf{E}\{e^{-ua_2^2 G_{22}^{(N,p)}}\} \\ &= ue^{-2p}\int |a_1||a_2| d\mu(a_1) d\mu(a_2) \int_0^\infty \int_0^\infty dv_1 dv_2 \frac{\mathcal{J}_1(2|a_1|\sqrt{uv_1})\mathcal{J}_1(2|a_2|\sqrt{uv_2})}{\sqrt{v_1 v_2}} e^{-zv_1 - zv_2} \\ &\times \left[\mathbf{E}\{e^{p\tilde{f}_{N-2}(v_1/p,z)} e^{p\tilde{f}_{N-2}(v_2/p,z)}\} - \mathbf{E}\{e^{p\tilde{f}_{N-2}(v_1/p,z)}\}\mathbf{E}\{e^{p\tilde{f}_{N-2}(v_2/p,z)}\}\right] + \tilde{r}_N(u), \\ &\mathbf{E}\{|\tilde{r}_N(u)|\} \le \frac{Cp}{N|\Re z|^2}. \end{aligned} \tag{6.17}$$

By the same way as in (6.11)-(6.13) it is easy to prove that the estimate (6.13) remains valid if we replace $\tilde{f}_{N-1}(u,z)$ by $\tilde{f}_{N-2}(u,z)$. Thus inequalities (6.17) remain valid if we replace $\tilde{f}_{N-2}(u,z)$ by $f_N(u,z)$ in the r.h.s. of (6.17).

Besides, since

$$|f_N(v,z)| \le \max_i e^{-ua_i^2 \Re G_{ii}^{(N,p)}} \le 1, \tag{6.18}$$

we have the bound

$$\begin{aligned} &\left|\mathbf{E}\{e^{pf_N(v_1/p,z)} e^{pf_N(v_2/p,z)}\} - \mathbf{E}\{e^{pf_N(v_1/p,z)}\}\mathbf{E}\{e^{pf_N(v_2/p,z)}\}\right| \\ &\le 4e^{2p}p^2(\mathbf{E}\{|\delta_N(v_1/p,z)|^2\} + \mathbf{E}\{|\delta_N(v_2/p,z)|^2\}) \end{aligned} \tag{6.19}$$

Let us take the norm (2.16) and consider the Banach space $\mathcal{B}$ of all the functions $\phi : \mathbf{R}_+ \to \mathbf{C}$ which possess this norm. Consider $\phi_z(u) = \delta_N(u,z)$.

Then, using (6.19) and the inequality $|\mathcal{J}_1(x)| \le 1$ (see [1]), on the basis of (6.15)–(6.18) we get

$$\mathbf{E}\{||\delta_N(u,z)||^2\} \le \frac{8X_2 e^{2p} p^2 \pi}{|\Re z|}\left(1 + \frac{1}{2p|\Re z|}\right)\mathbf{E}\{||\delta_N(u,z)||^2\} + \frac{C}{N}. \tag{6.20}$$

Hence, it is evident that there exists $M > 0$, such that for any $z : \Re z > M$

$$\mathbf{E}\{||\delta_N(u,z)||^2\} \le \frac{C}{N}. \tag{6.21}$$

Thus, for any $z : \Re z > M$ equation(6.14) can be rewritten in the form

$$\begin{aligned} &\mathbf{E}\{f_N(u,z)\} = 1 - u^{1/2}e^{-p}\int |a| d\mu(a) \int_0^\infty dv \frac{\mathcal{J}_1(2|a|\sqrt{uv})}{\sqrt{v}} e^{-zv} e^{p\mathbf{E}\{f_N(v/p,z)\}} + \tilde{r}'_N(u) \\ &\mathbf{E}\{|\tilde{r}'^2_N(u)|\} \le \frac{Cp^2 u}{N}. \end{aligned} \tag{6.22}$$

Define the operator $F_z : \mathcal{B} \to \mathcal{B}$ of the form

$$F_z(\phi)(u) = 1 - u^{1/2}e^{-p}\int |a| d\mu(a) \int_0^\infty dv \frac{\mathcal{J}_1(2|a|\sqrt{uv})}{\sqrt{v}} e^{-zv} e^{p\phi(v/p)}. \tag{6.23}$$

Then for any ϕ_1, ϕ_2 $||\phi_{1,2}|| \le 1$

$$||F_z(\phi_1) - F_z(\phi_2)|| \le X_2^{1/2}\int_0^\infty \frac{dv}{\sqrt{v}} e^{-|\Re z|v} e^{v^{1/2}/p^{1/2}} ||\phi_1 - \phi_2|| \le \frac{2X_2^{1/2}\sqrt{\pi} e^{1/4p|\Re z|}}{|2\Re z|^{1/2}} ||\phi_1 - \phi_2||$$

Hence, there exists $M_1 > 0$, such that for all $z : \Re z > M_1$

$$||F_z(\phi_1) - F_z(\phi_2)|| \leq \frac{1}{2}||\phi_1 - \phi_2||.$$

Thus, if we denote by $B_{0,1} = \{\phi : ||\phi|| \leq 1\}$ the ball of radius 1 centered in the origin, then we obtain that $F_z : B_{0,1} \to B_{0,1}$ and the restriction F_z on $B_{0,1}$ is a contraction. Therefore, there exists the unique fixed point $f(u,z)$ of the mapping $F_z : B_{0,1} \to B_{0,1}$ which is a solution of (2.17), and $\mathbf{E}\{f_N(u,z)\} \to f(u,z)$, as $N \to \infty$. But since $\mathbf{E}\{f_N(u,z)\}$ for any $z : \Re z > 0$ is an analytical function, the uniqueness theorem of complex analysis guarantees that equation (2.17) has a solution for any $z : \Re z > 0$ and $\mathbf{E}\{f_N(u,z)\} \to f(u,z)$, as $N \to \infty$.

Similarly, since $\delta_N(u,z)$ is a bounded analytical function, $\mathbf{E}\{||\delta_N(u,z)||^2\} \to 0$ $(N \to \infty)$ implies that $\mathbf{E}\{||\delta_N(u,z)||^2\} \to 0$ for any $z : \Re z > 0$.

Proof of Theorem 4. The proof of theorem 4 repeats almost literally the proof of Theorem 3. We use the formula (cf. (6.3)):

$$G_{11}^{(\Delta,N,p)}(z) = \left(z + i\sum_{j=2}^{N} A_{1j}^{(N,p)} + \sum_{j,k=2}^{N} \tilde{G}_{jk}^{(\Delta,N-1,p)} A_{1j}^{(N,p)} A_{1k}^{(N,p)}\right)^{-1} \tag{6.24}$$

where $\{\tilde{G}_{ij}^{(\Delta,N-1,p)}(z)\}_{i,j=2}^{N}$ is the resolvent of the matrix $i\Delta^{(N-1,p)}$, obtained from $\Delta^{(N,p)}$ by replacing $\{A_{1j}^{(N,p)}\}_{j=2}^{N}$ $\{A_{j1}^{(N,p)}\}_{j=2}^{N}$ with zeros. Then, similarly to (6.2)-(6.22) we obtain

$$\begin{aligned}\mathbf{E}\{f_N^{(\Delta)}(u,z)\} = \hat{\mu}(-u) - u^{1/2}e^{-p}\int |a|e^{iua}d\mu(a)\int_0^\infty dv\frac{\mathcal{J}_1(2|a|\sqrt{uv})}{\sqrt{v}}e^{-zv}e^{p\mathbf{E}\{f_N^{(\Delta)}(v/p,z)\}}\\ +\tilde{r}_N'(u), \quad \mathbf{E}\{|\tilde{r}_N'^2(u)|\} \leq \frac{Cp^2u}{N}.\end{aligned} \tag{6.25}$$

Then we consider the Banach space $\mathcal{B}$ with the norm (2.16) and the operator $F_z^{(\Delta)} : \mathcal{B} \to \mathcal{B}$ of the form

$$F_z^{(\Delta)}(\phi)(u) = \hat{\mu}(-u) - u^{1/2}e^{-p}\int |a|e^{iua}d\mu(a)\int_0^\infty dv\frac{\mathcal{J}_1(2|a|\sqrt{uv})}{\sqrt{v}}e^{-zv}\cdot e^{p\phi(v/p)}. \tag{6.26}$$

It is easy to see that there exists $M_1 > 0$, such that for any $z : \Re z > M_1$ the operator $F_z^{(\Delta)} : B_{0,1} \to B_{0,1}$ and its restriction to $B_{0,1}$ is a contraction.

Hence, there exists the unique fixed point $f^{(\Delta)}(u,z)$, which is a solution of (2.24), and $\mathbf{E}\{f_N^{(\Delta)}(u,z)\} \to f^{(\Delta)}(u,z)$, as $N \to \infty$. Similarly to Theorem 3 the statement of Theorem 4 can be derived from this fact.

In conclusions let us discuss the limiting transition $p \to \infty$. Assume that $X_1 = 0$. Then in the case of the adjacency matrix, by using formula (6.1) we can write

$$G_{11}^{(N,p)}(z) = \left(z + \sum_{j=2}^{N} \tilde{G}_{jj}^{(N-1,p)}\frac{X_2}{N} + \tilde{R}_N + R_p\right)^{-1}, \tag{6.27}$$

where $\tilde{R}_N$ is defined by (6.4) and

$$R_p = \sum_{j,k=2}^{N} \tilde{G}_{jj}^{(N-1,p)}(A_{1,j}^2 - \mathbf{E}\{A_{1,j}^2\})$$

Then, since $X_1 = 0$ in view of (6.5) $\mathbf{E}\{|\tilde{R}_N|^2\} \to 0$, as $N \to \infty$ and

$$\mathbf{E}\{|R_p|^2\} \leq \sum_{j=2}^{N} |\tilde{G}_{jj}^{(N-1,p)}|^2\frac{X_4}{Np} + O(|\Re z|^{-2}N^{-1}) = O(|\Re z|^{-2}p^{-1}) + O(|\Re z|^{-2}N^{-1}).$$

Besides, similarly to the above consideration it is easy to conclude that

$$\mathbf{E}\{|N^{-1}\mathrm{Tr}\ \tilde{G}^{(N-1,p)} - N^{-1}\mathrm{Tr}\ G^{(N,p)}|^2\} \to 0,$$
$$\mathbf{E}\{|N^{-1}\mathrm{Tr}\ G^{(N,p)} - \mathbf{E}\{N^{-1}\mathrm{Tr}\ G^{(N,p)}\}|^2\} \to 0, \quad N \to \infty.$$

We remark also that this self averaging property can be obtained directly from Theorem 3 (see Section 2.2).

Thus, we get that if $g_{N,p}(z) = N^{-1}\mathrm{Tr}\ (A^{(N,p)} - z)^{-1}$, then

$$\mathbf{E}\{ig_{N,p}(-iz)\} = (z + X_2\mathbf{E}\{ig_{N,p}(-iz)\})^{-1} + o(1)$$

Similarly to the proof of Theorem 3, we conclude that for $|\Re z|$ large enough $\mathbf{E}\{g_{N,p}(-iz)\} \to g(-iz)$, as $N, p \to \infty$, where $g(z)$ is the solution, of the equation

$$g(z) = (X_2 g(z) - z)^{-1},$$

satisfying condition $\Re g(z)\Re z > 0$. So we have got once more the result of [16] that if $X_1 = 0$ and $N, p \to \infty$, then IDS of $A^{(N,p)}$ tends to the Wigner semicircle law.

By the same way, using formula (6.24), we get for the $G^{(\Delta,N,p)}(z)$

$$G_{11}^{(\Delta,N,p)}(z) = \left(z + i\sum_{j=2}^{N} A_{1j}^{(N,p)} + \frac{X_2}{N}\mathbf{E}\{\mathrm{Tr}\ G^{(\Delta,N,p)}(z)\}\right)^{-1} + O(|\Re z|^{-2}p^{-1}) + O(|\Re z|^{-2}N^{-1}). \tag{6.28}$$

Thus, since $\sum_{j=2}^{N} A_{1j}^{(N,p)}$ converge in distribution as $N, p \to \infty$ to the Gaussian random variable with zero mean and the variance X_2, we get from (6.28) the equation

$$\mathbf{E}\{ig_{N,p}^{(\Delta)}(-iz)\} = \int \frac{e^{-v^2/2}dv}{\sqrt{2\pi}}(z + iv + \mathbf{E}\{ig_{N,p}^{(\Delta)}(-iz)\})^{-1}$$

and so we conclude that there exists

$$\lim_{N,p\to\infty} \mathbf{E}\{ig_{N,p}^{(\Delta)}(iz)\} = ig^{(\Delta)}(iz),$$

where $ig^{(\Delta)}(iz)$ is defined by the equation

$$g^{(\Delta)}(-iz) = \int \frac{e^{-v^2/2}dv}{\sqrt{2\pi}}(iz - v - X_2 g_{N,p}^{(\Delta)}(-iz))^{-1} \tag{6.29}$$

and the condition $\Re g(z)\Re z > 0$. The last equation determines the Stieltjes transform of the deformed Wigner law (see [16]). The semicircle distribution is "deformed" by the normal one and this makes the support of the corresponding IDS to be infinite. The moments of this deformed Wigner law are determined by relations (5.12).

Regarding the matrix of the Laplace operator (2.3), it is easy to explain the result (6.29). The diagonal term B of (2.3) is given by the sum of approximately p independent random variables a and this sum is normalized by $\sqrt{p}$. So, if the mathematical expectation of a equals to zero, the order of magnitude of the diagonal term of (2.3) remains finite as $p \to \infty$ and this equalize it with the matrix $A^{(N,p)}$. Since the elements of these random matrices A and B become statistically independent in the limit $p \to \infty$, the limiting IDS results in the semicircle law given by A deformed by the normal distribution provided by B.

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