ON THE NORM OF RANDOM MATRICES

by

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Abstract

We consider symmetric $n \times n$ matrices $\mathbb{H}_n$ known as the deformed Wigner ensemble and having the form of sum of a non-random matrix and a random matrix with independent identically and symmetrically distributed entries. We prove that if tails of the probability distribution of entries decay as $\exp\{-c|x|^{2\alpha}\}$ and the ensemble admits the integrated density of states (I.D.S.), i.e. the limiting eigenvalue distribution, with z^* being the endpoint of its support, then the probability that $\|\mathbb{H}_n\|$ exceeds $z^* + \varepsilon$ is bounded above by $\exp\Big\{ -\text{const}.\varepsilon^{\frac{3+\alpha}{2+\alpha}} n^{\frac{1}{2+\alpha}} \Big\}$. Similar result is obtained also for certain block random matrices for large size of blocks.

1. INTRODUCTION

Consider the ensemble of random matrices

$$\mathbb{J}_n = \Big\{ \frac{\mathcal{J}(i,j)}{\sqrt{n}} \Big\}_{i,j=1}^{n} \tag{1.1}$$

where $\mathcal{J}(i,j) = \mathcal{J}(j,i)$ $(i,j = 1,2,...)$ are independent identically and symmetrically distributed random variables with zero mean and variance $\mathcal{J}^2$. We call this ensemble the Wigner ensemble. It plays an important role in many problems of spectral theory (see e.g. reviews [1], [5] and references therein) and statistical physics in particular in the theory of disordered spin systems (see e.g. review [2] and references therein), where the random matrix $\mathbb{J}_n$ determines the interaction in the Sherrington-Kirkpatrick spin glasses model.

It is known [1], [2] that if we define the integrated density of states (I.D.S.) of matrices (1.1) as

$$N_n(\lambda) = \frac{1}{n} \sum_{\lambda_i \le \lambda} 1$$

where $\{\lambda_i\}_{i=1}^n$ are eigenvalues of $\mathbb{J}_n$, then $N_n(\lambda)$ tends in probability to the non-random limit $N(\lambda)$ as $n \to \infty$ and for each λ

$$N'(\lambda) \equiv \rho(\lambda) = \begin{cases} \frac{\sqrt{4\mathcal{J}^2 - \lambda^2}}{2\pi\mathcal{J}^2}, & \text{if } |\lambda| \le 2\mathcal{J} \\ 0, & \text{if } |\lambda| > 2\mathcal{J}. \end{cases}$$

This implies that the number of eigenvalues of $\mathbb{J}_n$ lying outside of interval $[-2\mathcal{J}, 2\mathcal{J}]$ being divided by n, tends to zero in probability as $n \to \infty$. However, in many problems of random matrix theory and statistical physics we need more precise information on behaviour of λ_n. In particular, it is important to know the values of the extreme eigenvalues, i.e. the norm of respective random matrix.

In this paper we give a large deviation type bound for

$$\Pr\{\|J_n\| > 2\mathcal{J} + \varepsilon\} \tag{1.2}$$

In particular, our bound (see Theorem 1 below) is such that

$$\sum_{n=1}^{\infty} \Pr\{\|J_n\| > 2\mathcal{J} + \varepsilon\} < \infty. \tag{1.3}$$

Thus, according to the Borel-Cantelli lemma if n is large enough, then with probability 1 all eigenvalues are inside of $[2\mathcal{J} - \varepsilon, 2\mathcal{J} + \varepsilon]$.

We consider also the same question for the so-called deformed Wigner ensemble [2]:

$$H_n(i,j) = (\mathbb{H}_n^{(0)} + \mathbb{J}_n)(i,j) \qquad (i,j = 1, ..., n) \tag{1.4}$$

where $\mathbb{J}_n$ has the form (1.1) and $\mathbb{H}_n^{(0)}$ has the limiting I.D.S. $N^{(0)}(\lambda)$ as $n \to \infty$, i.e. for any interval (a,b):

$$\lim_{n\to\infty} \int_a^b dN_n^{(0)}(\lambda) = \int_a^b dN^{(0)}(\lambda). \tag{1.5}$$

For ensemble (1.4) it is also known [2] that its I.D.S. $N_n(\lambda)$ converges in probability to the non-random limit $N(\lambda)$ and that its Stieltjes transform

$$g(z) = \int \frac{dN(\lambda)}{\lambda - z}, \qquad \operatorname{Im} z \neq 0$$

is the unique solution of the equation

$$g(z) = \int (\lambda - \mathcal{J}^2 g(z) - z)^{-1} dN^{(0)}(\lambda) \tag{1.6}$$

such that $\operatorname{Im} g(z) \cdot \operatorname{Im} z > 0$, $\operatorname{Im} z \neq 0$.

Let z^* be the right-hand endpoint of the support of $N(\lambda)$. Then the analog of (1.2) for this ensemble is

$$\Pr\{\|\mathbb{H}_n\| > z^* + \varepsilon\} \tag{1.7}$$

and we give an upper bound for this probability (Theorem 2 below) of the same type.

And the last ensemble which we will be interested in is the Wegner ensemble [2]

$$H_{n,\Lambda}(x,i;y,j) = \big(\mathbb{H}_\Lambda^{(0)} + \mathbb{J}_{n,\Lambda}\big)(x,i;y,j) \qquad (x,y \in \Lambda \subset \mathbb{Z}^d,\ \ i,j = 1,\dots,n) \tag{1.8}$$

where Λ is a cube of side length L centered at the origin. Unlike ensembles (1.1) and (1.4) this ensemble consist of matrices acting in $\bigotimes_{x\in\Lambda} \mathbb{R}^n$ i.e. having the "block" structure. Here the matrix $\mathbb{H}_\Lambda^{(0)}$ has entries

$$H_\Lambda^{(0)}(x,i;y,j) = H_\Lambda^{(0)}(x-y)\delta_{ij} \qquad (x,y \in \Lambda,\ \ i,j = 1,...,n) \tag{1.9}$$

and corresponds to the interaction between the "blocks". We impose the periodic boundary conditions. In terms of $H_\Lambda^{(0)}$ they mean that $H_\Lambda^{(0)}(x+Lr) = H_\Lambda^{(0)}(x), \quad r \in \mathbb{Z}^d$. The matrix $\mathbb{J}_{n,\Lambda}$,

$$\mathbb{J}_{n,\Lambda}(x,i;y,j) = \delta(x-y)\mathcal{J}(x;i,j)\frac{1}{\sqrt{n}}, \tag{1.10}$$

corresponds to an interaction inside each "block". Here

$$\mathcal{J}(x;i,j) = \mathcal{J}(x;j,i) \qquad (x \in \mathbb{Z}^d,\ \ i,j = 1,2,...)$$

are independent identically and symmetrically distributed random variables with zero mean and variance $\mathcal{J}^2$. Some results of the rigorous study of this ensemble can be found in the review [2]. The I.D.S. of this ensemble is defined as

$$N_{n,\Lambda}(\lambda) = (|\Lambda| n)^{-1} \sum_{\lambda_i \le \lambda} 1$$

where λ_i are eigenvalues of $H_{n,\Lambda}$. It can be shown [3] that with probability 1 there exists the non-random limit

$$\lim_{\substack{n\to\infty \\ |\Lambda|\to\infty}} N_{n,\Lambda}(\lambda) = N(\lambda) \tag{1.11}$$

which coincides with that for ensemble (1.4) if $H_n^{(0)}$ and $H_\Lambda^{(0)}$ have the same limiting I.D.S. Again we will estimate

$$\Pr\{\|H_{n,\Lambda}\| > z^* + \varepsilon\},$$

where z^* has the same meaning as in (1.7).

For the case of Gaussian $\mathcal{J}(i,j)$ the norm of $\mathbb{J}_n$ (1.1) was estimated in [4]. The bound obtained is

$$\Pr\{\|\mathbb{J}_{n,\Lambda}\| > 2\mathcal{J} + \varepsilon\} \le \exp\{-\text{const}.\varepsilon.\, N^{\frac{2}{3}}\}.$$

In [5] it was proven that if $\mathcal{J}(i,j)$ have finite 4^{th} moment, then with probability 1:

$$\lim_{n\to\infty} \|\mathbb{J}_n\| = 2\mathcal{J}.$$

In present paper we will assume only that

$$\mathrm{E}\{\mathcal{J}^{2k+2}(i,j)\} \le C\mathcal{J}^2 k^\alpha \mathrm{E}\{\mathcal{J}^{2k}(i,j)\}. \tag{1.12}$$

Here and below symbol $\mathrm{E}\{\ldots\}$ means averaging with respect to the random variables.

The method which we are using here is rather similar to that used in the pioneer paper by Wigner [6].

2. THE MAIN RESULTS

Theorem 1. *Let us consider random matrices* (1.1) *satisfying conditions* (1.12). *Then*

$$\Pr\left\{\|\mathbb{J}_n\| > 2\mathcal{J}(1+\varepsilon)\right\} \le n\exp\left\{-M\varepsilon^{\frac{3+\alpha}{2+\alpha}} n^{\frac{1}{2+\alpha}}\right\} \tag{2.1}$$

where M does not depend on n and ε.

The proof of Theorem 1 is based on the following lemma:

Lemma 1. *Let*

$$a_k^{(n)} = a_k^{(n)}(\mathcal{J}) = \mathrm{E}\left\{\frac{1}{n}\mathrm{tr}\,\mathbb{J}_n^{2k}\right\}$$

then for $1 < k \le \left(\frac{n\varepsilon}{64C}\right)^{(\alpha+2)^{-1}}$, *where C is specified by* (1.12), *we have*

$$a_{k+1}^{(n)} = \mathcal{J}^2 \sum_{\ell=0}^{k} a_\ell^{(n)} a_{k-\ell}^{(n)} + \mathcal{O}\left(\frac{1}{n}\right) \tag{2.2}$$

$$a_{k+1}^{(n)} \le \mathcal{J}^2(1+\varepsilon)\sum_{\ell=0}^{k} a_\ell^{(n)} a_{k-\ell}^{(n)}. \tag{2.3}$$

Proof of Theorem 1

We will prove Lemma 1 later. Now let us derive (2.1) from (2.2) and (2.3). To this end we introduce the sequence of numbers a_k^* by recurrence formula:

$$a_{k+1}^* = \mathcal{J}^2(1+\varepsilon)\sum_{\ell=0}^{k} a_\ell^* a_{k-\ell}^* \tag{2.4}$$

and initial conditions

$$a_0^* = 1 \qquad a_1^* = \mathcal{J}^2(1+\varepsilon). \tag{2.5}$$

It is easy to check that

$$a_0^{(n)} = a_0^* \qquad a_1^{(n)} < a_1^*.$$

Therefore from inequality (2.3) one can derive by induction that for $k \le \left(\frac{n\varepsilon}{64C}\right)^{(\alpha+2)^{-1}}$ we have

$$a_k^{(n)} \le a_k^*. \tag{2.6}$$

On the other hand from relations (2.2) it follows that if for finite k we take the limit in the r.h.s. and l.h.s. of (2.2), then we prove that there exists

$$a_k(\mathcal{J}) = \lim_{n\to\infty} a_k^{(n)}(\mathcal{J})$$

and if we set $\bar{\mathcal{J}} = \sqrt{1+\varepsilon}\mathcal{J}$, then $a_k(\bar{\mathcal{J}})$ satisfy relations (2.4), (2.5). Therefore

$$a_k(\bar{\mathcal{J}}) = a_k^*.$$

Besides, according to [2]

$$a_k(\bar{\mathcal{J}}) = \frac{1}{2\pi\bar{\mathcal{J}}^2}\int_{-2\bar{\mathcal{J}}}^{2\bar{\mathcal{J}}} \lambda^{2k}\sqrt{4\bar{\mathcal{J}}^2 - \lambda^2}d\lambda \le (2\bar{\mathcal{J}})^{2k}$$

and from the inequality (2.6) it follows that for $k \le \left(\frac{n\varepsilon}{64C}\right)^{(\alpha+2)^{-1}}$

$$a_k^{(n)} \le \left(2\bar{\mathcal{J}}\right)^{2k} = \left(2\mathcal{J}\sqrt{1+\varepsilon}\right)^{2k}. \tag{2.7}$$

Now let us use the simple inequality

$$a_k^{(n)} = \mathrm{E}\Big\{\frac{1}{n}\operatorname{tr}\mathbb{J}_n^{2k}\Big\} \ge \frac{1}{n}\int_0^\infty \lambda^{2k}dP(\lambda) \ge \left[2\mathcal{J}(1+\varepsilon)\right]^{2k}P(2\mathcal{J}(1+\varepsilon))\frac{1}{n}.$$

Here $P(\lambda) = \Pr\{\|J_n\| > \lambda\}$. We obtain for $k = \left(\frac{n\varepsilon}{64C}\right)^{(\alpha+2)^{-1}}$

$$P(2\mathcal{J}(1+\varepsilon)) \le n\left(\frac{2\mathcal{J}\sqrt{1+\varepsilon}}{2\mathcal{J}(1+\varepsilon)}\right)^{2k} \le n\exp\Big\{-M\varepsilon^{\frac{3+\alpha}{2+\alpha}}n^{\frac{1}{2+\alpha}}\Big\}.$$

This estimate proves Theorem 1.

Proof of Lemma 1

By definition

$$a_k^{(n)} = \sum_{i_1,\dots,i_{2k}} \frac{1}{n^k}\mathrm{E}\{\mathcal{J}(i_1,i_2)\mathcal{J}(i_2,i_3)\cdots\mathcal{J}(i_{2k},i_1)\} \tag{2.8}$$

Since $\mathcal{J}(i,j)$ are independent with zero mean, we have non-zero terms in this sum only if for some ℓ:

$$(i_\ell, i_{\ell+1}) = (i_1, i_2) \qquad \text{or} \qquad (i_\ell, i_{\ell+1}) = (i_2, i_1). \tag{2.9}$$

Thus we can write the representation

$$a_k^{(n)} = \Sigma_1(k) + \Sigma_2(k) = \Sigma_1^1(k) + \Sigma_2^1(k) + \Sigma_3(k) \tag{2.10}$$

where

$$\Sigma_1(k) = \sum_{\ell=0}^{2k-2} \mathrm{E}\Big\{ \frac{1}{n} \sum_{i_1,i_2} \frac{\mathcal{J}(i_1,i_2)}{\sqrt{n}} \Big(\mathbb{J}_n^{2\ell}\Big)(i_2,i_1) \frac{\mathcal{J}(i_1,i_2)}{\sqrt{n}} \Big(\mathbb{J}_n^{2k-2-2\ell}\Big)(i_2,i_1) \Big\},$$

$$\Sigma_2(k) = \sum_{\ell=0}^{2k-2} \mathrm{E}\Big\{ \frac{1}{n} \sum_{i_1,i_2} \frac{\mathcal{J}(i_1,i_2)}{\sqrt{n}} \Big(\mathbb{J}_n^{2\ell}\Big)(i_2,i_2) \frac{\mathcal{J}(i_2,i_1)}{\sqrt{n}} \Big(\mathbb{J}_n^{2k-2-2\ell}\Big)(i_1,i_1) \Big\}.$$

$\Sigma_1^1(k)$ and $\Sigma_2^1(k)$ include those terms of $\Sigma_1(k)$ and $\Sigma_2(k)$ which contain only two $\mathcal{J}(i_1,i_2)$ or $\mathcal{J}(i_2,i_1)$ and $\Sigma_3(k)$ is the remainder, which contains more than three of these variables (three is impossible due to the symmetry of distribution of $\mathcal{J}(i_1,i_2)$). It is important also that due to the symmetry of distributions of all $\mathcal{J}(i,j)$ all terms of (2.10) are positive.

Now let us estimate the r.h.s. of (2.10):

$$\Sigma_1^1(k) = \mathcal{J}^2 \sum_{\ell=0}^{2k-2} \mathrm{E}\Big\{ \sum_{i_1,i_2} \frac{1}{n^2} (\mathbb{J}_n^{2\ell})(i_2,i_1)(\mathbb{J}_n^{2k-2-2\ell})(i_2,i_1) \Big\} \le \frac{2k}{n} \mathcal{J}^2 a_{k-1}^{(n)}, \tag{2.11}$$

$$\Sigma_2^1(k) \le \sum_{\ell=0}^{2k-2} \mathcal{J}^2 \mathrm{E}\Big\{ \frac{1}{n} \operatorname{tr}\big(\mathbb{J}_n^{2\ell}\big) \frac{1}{n} \operatorname{tr}(\mathbb{J}_n^{2k-2-2\ell}) \Big\}. \tag{2.12}$$

But for every $\ell + m \le 2k-2$

$$\begin{aligned} &\mathrm{E}\Big\{ \frac{1}{n} \operatorname{tr}(\mathbb{J}_n^{\ell}) \operatorname{tr}(\mathbb{J}_n^{m}) \Big\} - \mathrm{E}\Big\{ \frac{1}{n} \operatorname{tr}(\mathbb{J}_n^{\ell}) \Big\} \cdot \mathrm{E}\Big\{ \frac{1}{n} \operatorname{tr}(\mathbb{J}_n^{m}) \Big\} \\ &\le \ell m \mathrm{E}\Big\{ \frac{1}{n^3} \sum_{j_1,j_2} \mathcal{J}(j_1,j_2)(\mathbb{J}_n^{\ell-1})\mathcal{J}(j_2,j_1)\mathcal{J}(j_1,j_2)(\mathbb{J}_n^{m-1})\mathcal{J}(j_2,j_1) \Big\} \\ &\le \frac{4k^2 \cdot k^\alpha}{n^2} \mathcal{J}^2 C a_{(\ell+m-2)/2}^{(n)} \end{aligned} \tag{2.13}$$

where we have used inequality (1.12). From (2.12) and (2.13) it follows that

$$\Sigma_2^1(k) \le \mathcal{J}^2 \sum_{\ell=0}^{k-1} a_\ell^{(n)} a_{k-1-\ell}^{(n)} + \frac{8k^{3+\alpha}}{n^2} C \mathcal{J}^2 a_{k-2} \tag{2.14}$$

It remains to estimate $\Sigma_3(k)$. To this end we single out such factors in $\mathbb{J}_n^{k-1}$ whose indices are (i_1, i_2) or (i_2, i_1). We obtain

(2.15)

$$\Sigma_3(k) \le \frac{1}{n^3} \sum_{\ell_1+\ell_2+\ell_3+\ell_4=2k-4}$$

$$\mathrm{E}\Big\{ \sum_{i_1,i_2} \mathcal{J}(i_1,i_2)\big(\mathbb{J}_n^{\ell_1}\big)(i_1,i_2)\mathcal{J}(i_1,i_2)\cdot \big(\mathbb{J}_n^{\ell_2}\big)(i_2,i_1)\mathcal{J}(i_1,i_2)\cdot \big(\mathbb{J}_n^{\ell_3}\big)(i_2,i_1)\mathcal{J}(i_1,i_2)\cdot \big(\mathbb{J}_n^{\ell_4}\big)(i_2,i_1)$$

$$+ \sum_{i_1,i_2} \mathcal{J}(i_1,i_2)\big(\mathbb{J}_n^{\ell_1}\big)(i_1,i_2)\mathcal{J}(i_2,i_1)\cdot \big(\mathbb{J}_n^{\ell_2}\big)(i_1,i_2)\mathcal{J}(i_2,i_1)\cdot \big(\mathbb{J}_n^{\ell_3}\big)(i_1,i_2)\mathcal{J}(i_2,i_1)\cdot \big(\mathbb{J}_n^{\ell_4}\big)(i_1,i_1)$$

$$+ \sum_{i_1,i_2} \mathcal{J}(i_1,i_2)\big(\mathbb{J}_n^{\ell_1}\big)(i_2,i_2)\mathcal{J}(i_2,i_1)\cdot \big(\mathbb{J}_n^{\ell_2}\big)(i_1,i_1)\mathcal{J}(i_1,i_2)\cdot \big(\mathbb{J}_n^{\ell_3}\big)(i_2,i_2)\mathcal{J}(i_2,i_1)\cdot \big(\mathbb{J}_n^{\ell_4}\big)(i_1,i_1) + \cdots \Big\}$$

where we have written only 3 of 8 internal sums. Let us estimate the first of these internal sums in (2.15). Others can be estimated similarly. We have:

$$\mathrm{E}\Big\{ \sum_{i_1,i_2} \frac{\mathcal{J}^4(i_1,i_2)}{n^3} \big(\mathbb{J}_n^{\ell_1}\big)(i_2,i_1)\big(\mathbb{J}_n^{\ell_2}\big)(i_2,i_1)\big(\mathbb{J}_n^{\ell_3}\big)(i_2,i_1)\big(\mathbb{J}_n^{\ell_4}\big)(i_2,i_1)\Big\} \le$$

$$\le \frac{\mathcal{J}^2 C k^\alpha}{n^2} \mathrm{E}\Big\{ \sum_{i_1,i_2,i_3,i_4} \frac{\mathcal{J}^2(i_1,i_2)}{\sqrt{n}} \big(\mathbb{J}_n^{\ell_1}\big)(i_2,i_3)\big(\mathbb{J}_n^{\ell_2}\big)(i_2,i_3)\big(\mathbb{J}_n^{\ell_3}\big)(i_1,i_4)\big(\mathbb{J}_n^{\ell_4}\big)(i_4,i_1)\Big\}$$

$$\le \frac{\mathcal{J}^2 C k^\alpha}{n^2} \mathrm{E}\Big\{ \sum_{i_1,i_2} \frac{\mathcal{J}(i_1,i_2)}{\sqrt{n}} \big(\mathbb{J}_n^{\ell_1+\ell_2}\big)(i_2,i_2)\frac{\mathcal{J}(i_1,i_2)}{\sqrt{n}} \big(\mathbb{J}_n^{\ell_3+\ell_4}\big)(i_1,i_1)\Big\}$$

and similar bounds for other sums in (2.15). Inserting these bounds in (2.15) we obtain:

$$\Sigma_3(k) \le$$

$$\le \frac{8k^\alpha C \mathcal{J}^2}{n^2} \mathrm{E}\Big\{ \sum_{\ell_1+\ell_2+\ell_3+\ell_4=2k-4} \frac{\mathcal{J}(i_1,i_2)}{\sqrt{n}} \big(\mathbb{J}_n^{\ell_1+\ell_2}\big)(i_2,i_2)\frac{\mathcal{J}(i_2,i_1)}{\sqrt{n}} \big(\mathbb{J}_n^{\ell_3+\ell_4}\big)(i_2,i_1)\Big\}$$

$$\le \frac{64 k^{\alpha+2} C \mathcal{J}^2}{n} \Sigma_2(k-1).$$

But according to (2.10)

$$\Sigma_2(k-1) \le a_{k-1}^{(n)} + \Sigma_3(k-1).$$

Therefore for $k \le \left(\frac{n\varepsilon}{64C}\right)^{(\alpha+2)^{-1}}$

$$\Sigma_3(k) \le \mathcal{J}^2\varepsilon\Big(a_{k-1}^{(n)} + \Sigma_3(k-1)\Big) \le \ldots \le \mathcal{J}^2\varepsilon \sum_{\ell=1}^{k} (\mathcal{J}^2\varepsilon)^\ell a_{k-\ell}^{(n)} \tag{2.16}$$

and since

$$a_\ell^{(n)} > \mathrm{E}\Big\{\frac{1}{n^\ell}\sum_{i_1,\dots,i_\ell}\mathcal{J}(i_1,i_2)\mathcal{J}(i_2,i_1)\mathcal{J}(i_1,i_3)\mathcal{J}(i_3,i_1)\cdots\mathcal{J}(i_\ell,i_1)\Big\} = \mathcal{J}^{2\ell},$$

it follows from (2.16) that for $\varepsilon \le 1$ we have:

$$\Sigma_3(k) \le \mathcal{J}^2\varepsilon\sum_{\ell=1}^{k} a_{k-\ell}^{(n)}a_k^{(n)}. \tag{2.17}$$

Now inequalities (2.11), (2.14) and (2.16) give us inequality (2.3) of Lemma 1. Relation (2.2) follows from (2.10), (2.11), and (2.14)-(2.16). Lemma 1 is proved.

Now we will consider the deformed Wigner ensemble (1.4).

Theorem 2. *Let $\mathbb{H}_n$ has the form* (1.4), *where $\mathbb{H}_n^{(0)}$ satisfies* (1.5) *and $H_n^{(0)}(i,j) \ge 0$, $\mathbb{J}_n$ has the form* (1.1) *with $\mathcal{J}(i,j)$ satisfying condition* (1.12). *Let also $z^*(\mathbb{H}_n^{(0)},\mathcal{J})$ be a right-hand endpoint of the support of $N(\lambda)$ whose Stieltjes transform is determined by equation* (1.6). *Then*

$$\Pr\big\{\|\mathbb{H}\| > z^*(\mathbb{H}_n^{(0)},\mathcal{J}) + \varepsilon\big\} \le n\exp\Big\{-M\varepsilon^{\frac{3+\alpha}{2+\alpha}}n^{\frac{1}{2+\alpha}}\Big\}$$

where M does not depend on n and ε.

The idea of the proof of Theorem 2 is the same as that for Theorem 1. It is based on Lemma 2 which is an analog of Lemma 1.

Lemma 2. *Let $a_{m,k}^{(n)} = \frac{1}{n}\mathrm{E}\Big\{\operatorname{tr}\big(\mathbb{H}_n^{(0)}\big)^m\mathbb{H}_n^k\Big\}$. Then for all m and $1 \le k \le \Big(\frac{n\varepsilon M_1}{64C}\Big)^{(\alpha+2)^{-1}}$*

$$a_{m,k+1}^{(n)} = a_{m+1,k}^{(n)} + \mathcal{J}^2\sum_{\ell=0}^{k-2} a_{m,\ell}^{(n)}a_{0,k-\ell-2}^{(n)} + \mathcal{O}\Big(\frac{1}{n}\Big), \tag{2.18}$$

$$a_{m,k+1}^{(n)} \le a_{m+1,k}^{(n)} + \mathcal{J}^2(1+M_1\varepsilon)\sum_{\ell=0}^{k-2} a_{m,\ell}^{(n)}a_{0,k-\ell-2}^{(n)}. \tag{2.19}$$

Here

$$\begin{cases} M_1 = \Big(\mathcal{J}\frac{\partial z^*}{\partial \mathcal{J}}\Big)^{-1} & \text{if } \frac{\partial z^*}{\partial \mathcal{J}} \ne 0 \\ M_1 = 1 & \text{if } \frac{\partial z^*}{\partial \mathcal{J}} = 0. \end{cases}$$

The proof of Lemma 2 is almost literally the same as that of Lemma 1. Therefore we will show only how to derive Theorem 2 from (2.18) and (2.19).

As in Theorem 1 let us introduce $a^*_{m,k}$ by the recurrence relations:

$$a^*_{m,k+1} = a^*_{m+1,k} + \mathcal{J}^2(1+M_1\varepsilon)\sum_{\ell=0}^{k-2} a^*_{m,\ell}a^*_{0,k-\ell-2} \tag{2.20}$$

and initial conditions

$$a^*_{m,0} = \frac{1}{n}\mathrm{E}\Big\{\operatorname{tr}\Big(\mathbb{H}_n^{(0)}\Big)^m\Big\}. \tag{2.21}$$

By the same arguments as that in Theorem 1, we have

$$a^{(n)}_{m,k} \le a^*_{m,k} \qquad \Big(k \le \Big(\tfrac{n\varepsilon M_1}{64C}\Big)^{(\alpha+2)^{-1}}\Big)$$

in particular

$$a^{(n)}_{0,k} \le a^*_{0,k}.$$

Now using as in the proof of Theorem 1 the relations (2.18) and results of [2] we find that for $k \le \Big(\frac{n\varepsilon M_1}{64C}\Big)^{(\alpha+2)^{-1}}$

$$a^*_{0,k} \le \Big[z^*\big(\mathbb{H}_n^{(0)}, \mathcal{J}\sqrt{1+M_1\varepsilon}\big)\Big]^k$$

and therefore as in Theorem 1

$$\begin{aligned}\Pr\{\|\mathbb{H}_n\| > z^*+\varepsilon\} &\le n\cdot\left[\frac{z^*\big(\mathbb{H}_n^{(0)}, \mathcal{J}\sqrt{1+M_1\varepsilon}\big)}{z^*\big(\mathbb{H}_n^{(0)}, \mathcal{J}\big)+\varepsilon}\right]^k \\ &\le n\cdot\exp\Big\{-M_1 n^{\frac{1}{2+\alpha}}\cdot\varepsilon^{\frac{3+\alpha}{2+\alpha}}\Big\}.\end{aligned}$$

Theorem 2 is proved.

Theorem 3. *Let us consider the Wegner ensemble* (1.8) *with*

$$H_\Lambda^{(0)}(x) \ge 0 \quad \textit{and} \quad \sum_{x\in\Lambda} H_\Lambda^{(0)}(x) < \infty \quad \textit{uniformly on } L.$$

Let also $\mathcal{J}(x;i,j)$ *satisfies conditions* (1.12). *Then*

$$\Pr\{\|\mathbb{H}_n\| > z^*+\varepsilon\} \le |\Lambda|\cdot\exp\Big\{-M_1 n^{(2+\alpha)^{-1}}\cdot\varepsilon^{\frac{3+\alpha}{2+\alpha}}\Big\},$$

where $z^* = z^*\big(\mathbb{H}_\Lambda^{(0)}, \mathcal{J}\big)$ *and* M_1 *are the same as in Theorem 2 if the limiting I.D.S. of* $\mathbb{H}_\Lambda^{(0)}$ *coincides with that of* $\mathbb{H}_n^{(0)}$.

To prove Theorem 3 we start from Lemma 3 which is an analog of Lemmas 1 and 2:

Lemma 3. *Let* $a_{m,k}^{(n,\Lambda)} = \frac{1}{|\Lambda| n}\mathrm{E}\Big\{ \mathrm{tr}\big(\mathbb{H}_\Lambda^{(0)}\big)^m \mathbb{H}_{n,\Lambda}^k\Big\}$. *Then for all* m *and all* k *such that* $1 \le k \le \left(\frac{n\varepsilon M_1}{64C}\right)^{(\alpha+2)^{-1}}$

$$a_{m,k+1}^{(n,\Lambda)} = a_{m+1,k}^{(n,\Lambda)} + \mathcal{J}^2 \sum_{\ell=0}^{k-2} a_{m,\ell}^{(n,\Lambda)} a_{0,k-\ell-2}^{(n,\Lambda)} + \mathcal{O}\Big(\frac{1}{n}\Big), \tag{2.18}$$

$$a_{m,k+1}^{(n,\Lambda)} \le a_{m+1,k}^{(n,\Lambda)} + \mathcal{J}^2(1+M_1\varepsilon) \sum_{\ell=0}^{k-2} a_{m,\ell}^{(n,\Lambda)} a_{0,k-\ell-2}^{(n,\Lambda)}$$

where M_1 *is the same as in Lemma 2.*

The proof of Lemma 3 is the same as that of Lemma 1. The only difference is in the estimate of Σ_2^1 because in this case instead of the inequality (2.12) for Σ_1^2 we obtain inequality:

$$\Sigma_2^1 \le \frac{1}{|\Lambda| n^2} \sum_{\ell_1+\ell_2=2k-2} \sum_{(x,i_1),(x,i_2)} \mathrm{E}\Big\{ \Big(\big(\mathbb{H}_\Lambda^{(0)}\big)^m \mathbb{H}_{n,\Lambda}^{\ell_1}\Big)(x,i_1;x,i_1) H_{n,\Lambda}^{\ell_2}(x,i_2;x,i_2)\Big\}$$

Thus we have to check that for all m, ℓ and $x \in \Lambda$ we have

$$\mathrm{E}\Big\{ \Big(\big(\mathbb{H}_\Lambda^{(0)}\big)^m \mathbb{H}_{n,\Lambda}^{\ell}\Big)(x,i;x,i)\Big\} = a_{m,\ell}^{(n,\Lambda)}.$$

But since $\mathbb{H}_\Lambda^{(0)}$ is translationaly invariant with periodic boundary condition the last relation is obvious. Then we estimate the difference

$$\frac{1}{|\Lambda| n^2} \sum_{(x,i_1),(x,i_2)} \mathrm{E}\Big\{ \Big(\big(\mathbb{H}_\Lambda^{(0)}\big)^m \mathbb{H}_{n,\Lambda}^{\ell_1}\Big)(x,i_1;x,i_1) H_{n,\Lambda}^{\ell_2}(x,i_2;x,i_2)\Big\} - a_{m,\ell_1}^{(n,\Lambda)} a_{\ell_2}$$

in the same manner as in Lemma 1, and all other estimates of Lemma 1 may be repeated almost literally.

The derivation of Theorem 3 from Lemma 3 is the same as that of Theorem 2.

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