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Homogenization of eigenvalue problem for Laplace–Beltrami operator on Riemannian manifold with complicated 'bubble-like' microstructure

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SUMMARY

We study the asymptotic behavior of the eigenvalues and the eigenfunctions of the Laplace–Beltrami operator on a Riemannian manifold M^ε depending on a small parameter $\varepsilon>0$ and whose structure becomes complicated as $\varepsilon\to 0$. Under a few assumptions on scales of M^ε we obtain the homogenized eigenvalue problem. In addition we study the behavior of the heat equation on M^ε and investigate the large time behavior of the homogenized equation. Copyright © 2009 John Wiley & Sons, Ltd.

KEY WORDS: homogenization; eigenvalue problem; natural oscillations; Riemannian manifold; Laplace–Beltrami operator

0. INTRODUCTION

Homogenization problems on Riemannian manifolds with complicated microstructure arise in various areas of mathematical physics. For example, in [1] the authors study asymptotic behavior of colored particles moving in the domain with small obstacles when the number of obstacles tends to infinity; the problem is reduced to study of the diffusion equation on a special Riemannian manifold. In [2] the results of [1] are applied to study of some qualitative properties of the system of integro-differential equations describing transport of particles of several species. In [3] the authors study the asymptotic behavior of harmonic vector fields with given fluxes on Riemannian manifolds consisting of several copies of the Euclidean space with small holes attached edge by

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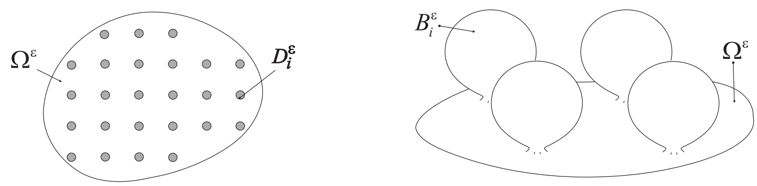

Figure 1. The domain Ω^ε and the manifold M^ε.

edge by means of thin tubes ('wormholes') when the number of holes tends to infinity; the obtained results are interpreted in terms of general relativity.

In the present paper we study the eigenvalue problem for the Laplace–Beltrami operator Δ^ε on the n-dimensional Riemannian manifold M^ε depending on small parameter $\varepsilon>0$. In the case $n=2$, M^ε can be interpreted as a membrane with complicated microstructure. It consists of a fixed bounded domain $\Omega\subset\mathbb{R}^n$ with a large number of disjoint small 'holes' D_i^ε, $i=1,\dots,N(\varepsilon)$, whose boundaries are attached to the boundaries of B_i^ε ('bubbles')—the n-dimensional spheres with small truncated segment. The fragment of this manifold is presented in Figure 1, more precise description of M^ε will be specified later.

It is supposed that the 'holes' D_i^ε are distributed periodically with the period ε. In addition, we suppose that the radii of 'bubbles' B_i^ε are of order ε (and therefore the total volume of B_i^ε is bounded from above and from below uniformly in ε).

We study the following spectral problem:

$$-\Delta^\varepsilon u^\varepsilon=\lambda u^\varepsilon,\quad \tilde{x}\in M^\varepsilon,\quad \frac{\partial u^\varepsilon}{\partial v}=0,\quad \tilde{x}\in\partial M^\varepsilon \tag{1}$$

where $\partial/\partial v$ is the outer normal derivative on the boundary ∂M^ε of M^ε.

Since M^ε has a piecewise smooth metrics (see discussion in Section 1) then problem (1) is understood in the following way:

Find $\lambda^\varepsilon\in\mathbb{R}$(eigenvalue) and $u^\varepsilon\in H^1(M^\varepsilon)$ (eigenfunction) such that

$$(\nabla^\varepsilon u^\varepsilon,\nabla^\varepsilon v)_{L_2(M^\varepsilon)}=\lambda^\varepsilon(u^\varepsilon,v)_{L_2(M^\varepsilon)}\quad \forall v\in H^1(M^\varepsilon)$$

where $(\cdot,\cdot)_{L_2(M^\varepsilon)}$ is a scalar product in $L_2(M^\varepsilon)$ (a precise definition is given below).

It is well known (see, e.g. [4]) that, for each $\varepsilon>0$, there exists a sequence of eigenvalues of problem (1), $0=\lambda_1^\varepsilon<\lambda_2^\varepsilon\leqslant\cdots\lambda_k^\varepsilon\leqslant\cdots\underset{k\to\infty}{\to}\infty$, and the sequence of the corresponding eigenfunctions can be orthonormalized in $L_2(M^\varepsilon)$.

We choose the Neumann boundary condition only for the sake of definiteness, all results are still valid for another boundary conditions.

Problem (1) describes the natural oscillations of the membrane M^ε.

The aim of this paper is to study the asymptotic behavior of problem (1) as $\varepsilon\to 0$. In the 'critical' case, i.e. when the radii d_i^ε of 'holes' D_i^ε are of order $\exp(-a/\varepsilon^2)$ ($n=2$) or $\varepsilon^{n/(n-2)}$ ($n>2$), this problem has been studied in [5]. It has been shown that the spectrum of problem (1) converges (in the Hausdorff sense) to the spectrum of the homogenized operator which has (in contrast to Δ^ε) non-empty essential spectrum.

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In the present paper we consider an ‘intermediate’ case:

$$\exp(-a/\varepsilon^2) \ll d_i^\varepsilon \ll \varepsilon, \quad n=2 \quad \text{or} \quad \varepsilon^{n/(n-2)} \ll d_i^\varepsilon \ll \varepsilon, \quad n>2$$

We prove that the eigenvalues of problem (1) converge in some sense (in particular in the Hausdorff sense) to the eigenvalues of the following problem:

$$-\Delta u = \lambda B(x) u, \quad x \in \Omega, \quad \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega \tag{2}$$

where $B(x)$ is a smooth positive function. In addition, we describe the behavior of the corresponding eigenfunctions.

It is easy to show that the spectrum of this problem has a structure similar to the structure of the spectrum of problem (1), i.e. there exists a sequence of eigenvalues $0=\lambda_1<\lambda_2\leqslant\cdots\lambda_k\leqslant\cdots \underset{k\to\infty}{\to} \infty$ and the sequence of the corresponding eigenfunctions can be orthonormalized in the Hilbert space $\mathscr{H}^0$ of real-valued functions from $L_2(\Omega)$ with the scalar product given by $(u,v)_{\mathscr{H}^0} = \int_\Omega u(x)v(x)B(x)\mathrm{d}x$.

The proof is based on the abstract scheme proposed in [6]. In the ‘critical’ case investigated in [5] this scheme violates (because the spectrum of the homogenized operator in [5] is not purely discrete).

Results of the same type can be easily obtained for classical evolution equations. We illustrate this by an example of the initial-boundary value problem for the heat equation on M^ε. It is proved that under a few assumptions on the initial function its solution converges to the solution of a parabolic equation describing the heat transfer in a medium with non-constant heat capacity. In addition, we describe the behavior of the homogenized problem as time variable tends to infinity, namely we prove that its solution converges to a constant calculated explicitly. Instead of the direct analysis of the homogenized equation we show that this fact follows from the homogenization results that allows us to ‘guess’ easily this constant.

Various spectral problems in the homogenization theory have been studied by many authors. In particular we mention the monographs [7–10] where a number spectral problems are considered. Except [5] the homogenization of the eigenvalue problems for Laplace–Beltrami operator on the Riemannian manifolds has been studied in [11], but for the manifolds with quite another structure.

The paper is organized as follows. In Section 1 we formulate our main results on the convergence of eigenvalues (Theorem 1.1) and eigenfunctions (Theorem 1.2). In Section 2 we prove the main results. Finally, in Section 3 we study the behavior of the heat equation (Theorem 3.1) and investigate the large time behavior of the homogenized equation (Proposition 3.1).

1. ASYMPTOTIC BEHAVIOR OF EIGENVALUE PROBLEM: PROBLEM SETTING AND MAIN RESULTS

Let $\Omega\subset\mathbb{R}^n$ ($n\geqslant 2$) be a bounded domain with a smooth boundary and let D_i^ε, $i=1,\ldots,N(\varepsilon)$ be a system of disjoint balls (‘holes’) of the radius d_i^ε with centers $x_i^\varepsilon = i\cdot\varepsilon$ ($i\in\mathbb{Z}^n$) such that $D_i^\varepsilon\subset\Omega$. Here $N(\varepsilon)$ stands for corresponding set of multi-indexes $i\in\mathbb{Z}^n$.

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We consider the following domain with 'holes':

$$\Omega^\varepsilon = \Omega \Big\backslash \bigcup_{i=1}^{N(\varepsilon)} D_i^\varepsilon$$

We glue to the boundary of each 'hole' D_i^ε the truncated n-dimensional sphere ('bubble') with the radius b_i^ε:

$$B_i^\varepsilon = \{\tilde{x} = (\theta_1, \theta_2, \ldots, \theta_n) : \theta_1 \in [0, 2\pi], \theta_j \in [0, \pi], j = 2, \ldots, n-1, \theta_n \in [\Theta_i^\varepsilon, \pi]\}$$

where $\theta_1, \ldots, \theta_n$ are the spherical coordinates and

$$\Theta_i^\varepsilon = \arcsin(d_i^\varepsilon / b_i^\varepsilon)$$

Namely, we identify ∂D_i^ε and the boundary $\partial B_i^\varepsilon = \{x \in B_i^\varepsilon : \theta_n = \Theta_i^\varepsilon\}$ of B_i^ε.

As a result we obtain a manifold M^ε (see Figure 1):

$$M^\varepsilon = \overline{\Omega}^\varepsilon \cup \left(\bigcup_{i=1}^{N(\varepsilon)} \overline{B_i^\varepsilon} \right)$$

with boundary which coincides with $\partial\Omega$. We denote by $\tilde{x}$ points of this manifold. If the point $\tilde{x}$ belongs to Ω^ε sometimes we write x instead of $\tilde{x}$ having in mind a corresponding point in Ω.

Clearly, M^ε can be covered by a system of charts and suitable local coordinates $\{x_1 \ldots x_n\} \mapsto \tilde{x} \in M^\varepsilon$ can be introduced.

We suppose that M^ε is equipped by the Riemannian metrics g^ε that coincides with the flat Euclidean metrics on Ω^ε and coincides with the spherical metrics on the 'bubbles'.

Let us give more precise description of this metrics at the points, where the boundary of the 'holes' is glued to the boundary of the 'bubbles'. We introduce local coordinates $(x_1, \ldots, x_n)$ in a small neighborhood of ∂D_i^ε as follows. Let $(\theta_1, \ldots, \theta_{n-1}, r)$ be the spherical coordinates in Ω with the origin at x_i^ε. Here $\theta_1, \ldots, \theta_{n-1}$ are the angular coordinates, r is the distance to x_i^ε (in particular, $r = d_i^\varepsilon$ for the points of ∂D_i^ε). We set $x_j = \theta_j$ $(j = 1, \ldots, n-1)$, $x_n = r - d_i^\varepsilon$ $(x_n \geqslant 0)$ for $\tilde{x} \in \Omega^\varepsilon$ and $x_n = -b^\varepsilon(\theta_n - \Theta_i^\varepsilon)$ $(x_n < 0)$ for $\tilde{x} \in B_i^\varepsilon$. Then the components of the corresponding metric tensor $g_{\alpha\beta}^\varepsilon(x_1, \ldots, x_n)$ have the following form:

$$g_{\alpha\beta}^\varepsilon = \begin{cases} \delta_{\alpha\beta}(x_n + d_i^\varepsilon)^2 \prod\limits_{j=\alpha+1}^{n-1} \sin^2 x_j, & x_n \geqslant 0, \\ \delta_{\alpha\beta}(b_i^\varepsilon)^2 \sin^2\left(\dfrac{x_n}{b_i^\varepsilon} - \Theta_i^\varepsilon\right) \prod\limits_{j=\alpha+1}^{n-1} \sin^2 x_j, & x_n < 0, \end{cases} \quad \alpha = \overline{1, n-1}, \quad g_{n\beta}^\varepsilon = \delta_{n\beta}$$

(for $\alpha = n-1$ we set $\prod_{j=\alpha+1}^{n-1} \sin^2 x_j := 1$). Here $\delta_{\alpha\beta}$ is Kronecker's delta.

It is clear that the components of the metric tensor $g_{\alpha\beta}^\varepsilon$ introduced above are continuous but not differentiable functions. Nevertheless this tensor can be approximated by a smooth tensor $g_{\alpha\beta}^{\varepsilon\delta}$ that differs from $g_{\alpha\beta}^\varepsilon$ only in a small neighborhood of ∂D_i^ε (see [5] for the exact construction). But in order to simplify our calculations, we will further consider the piecewise smooth tensor $g_{\alpha\beta}^\varepsilon$. However, all results are still valid for the smooth tensor $g_{\alpha\beta}^{\varepsilon\delta}$ if $\delta(\varepsilon)$ converges to 0 sufficiently fast as $\varepsilon \to 0$.

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Let $L_2(M^\varepsilon)$ be the Hilbert space of real-valued functions with the scalar product and the norm given by

$$(u,v)_{L_2(M^\varepsilon)}=\int_{M^\varepsilon} u(\tilde{x})v(\tilde{x})\,\mathrm{d}\tilde{x},\quad \|u\|_{L_2(M^\varepsilon)}=\sqrt{(u,u)_{L_2(M^\varepsilon)}}$$

where $\mathrm{d}\tilde{x}=\sqrt{\det g^\varepsilon}\mathrm{d}x_1\ldots\mathrm{d}x_n$ be the volume form on M^ε.

By $\langle u\rangle_D$ we denote the mean value of the function u over the domain D:

$$\langle u\rangle_D=\frac{1}{|D|}\int_D u(\tilde{x})\,\mathrm{d}\tilde{x}$$

where $|\cdot|$ is a volume.

By ω_n we denote the volume of the unit n-dimensional sphere.

The assumptions about the sizes of the 'holes' and the 'bubbles' are the following:

$$\lim_{\varepsilon\to 0}\varepsilon^n\mathbf{D}_n^\varepsilon=0\quad \text{where } \mathbf{D}_n^\varepsilon=\max_i\begin{cases}|\ln d_i^\varepsilon|, & n=2\\ (d_i^\varepsilon)^{2-n}, & n>2\end{cases}\tag{A}$$

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\max_i d_i^\varepsilon=0\tag{B}$$

$$b_i^\varepsilon=b(x_i^\varepsilon)\cdot\varepsilon\quad \text{where } b(x) \text{ is a smooth positive function}\tag{C}$$

Condition (A) implies that $\min_i d_i^\varepsilon\gg\exp(-a/\varepsilon^2),\forall a>0$ $(n=2)$ or $\min_i d_i^\varepsilon\gg\varepsilon^{n/(n-2)}$ $(n>2)$. Condition (C) means that the radii of the 'bubbles' are of order ε and the total volume of the 'bubbles' is bounded from above and from below uniformly in ε. Namely, $\lim_{\varepsilon\to 0}\sum_{i=1}^{N(\varepsilon)}|B_i^\varepsilon|=\int_\Omega (b(x))^n\omega_n\,\mathrm{d}x$.

We introduce the operators $Q^\varepsilon: L_2(M^\varepsilon)\to L_2(\Omega)$ such that for any $u^\varepsilon\in L_2(M^\varepsilon)$

$$(Q^\varepsilon u^\varepsilon)(x)=\begin{cases}u^\varepsilon(x), & x\in\Omega^\varepsilon\\ 0, & x\in\bigcup_{i=1}^{N(\varepsilon)} D_i^\varepsilon\end{cases}$$

Now we study the eigenvalue problem (1). Recall that Laplace–Beltrami operator Δ^ε has the following form in local coordinates:

$$\Delta^\varepsilon=\frac{1}{\sqrt{\det g^\varepsilon}}\sum_{\alpha,\beta=1}^n\frac{\partial}{\partial x_\alpha}\left(g_\varepsilon^{\alpha\beta}\sqrt{\det g^\varepsilon}\frac{\partial}{\partial x_\beta}\right)$$

where $g_\varepsilon^{\alpha\beta}$ are the components of the tensor inverse to $g_{\alpha\beta}^\varepsilon$.

Let $0=\lambda_1^\varepsilon<\lambda_2^\varepsilon\leqslant\cdots\lambda_k^\varepsilon\leqslant\cdots\to\infty$ be the eigenvalues of this problem written with account of their multiplicity and $u_1^\varepsilon,u_2^\varepsilon,\ldots,u_k^\varepsilon\ldots$ be the corresponding eigenfunctions such that $(u_k^\varepsilon,u_l^\varepsilon)_{L_2(M^\varepsilon)}=\delta_{kl}$.

Now we formulate the main results.

Theorem 1.1

Let (A)–(C) hold. Then for any $k=1,2,3,\ldots$

$$\lambda_k^\varepsilon\to\lambda_k,\quad \varepsilon\to 0\tag{3}$$

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Here $0=\lambda_1<\lambda_2\leqslant\cdots\lambda_k\leqslant\cdots$ are the eigenvalues of problem (2), where $B(x)$ is defined by the formula

$$B(x)=1+b^n(x)\omega_n \tag{4}$$

Theorem 1.2
Let $\lambda_{k-1}<\lambda_k=\lambda_{k+1}=\cdots=\lambda_{k+m-1}<\lambda_{k+m}$ (i.e. the multiplicity of the eigenvalue λ_k is equal to m). Let $N(\lambda_k)$ be the eigenspace that corresponds to λ_k. Then for any $w\in N(\lambda_k)$ there exists a linear combination $\widehat{u}^\varepsilon$ of the eigenfunctions $u_k^\varepsilon,\ldots,u_{k+m-1}^\varepsilon$ such that

$$\|Q^\varepsilon\widehat{u}^\varepsilon-w\|^2_{L_2(\Omega^\varepsilon)}+\sum_{i=1}^{N(\varepsilon)}\|\widehat{u}^\varepsilon-\langle w\rangle_{\square_i^\varepsilon}\|^2_{L_2(B_i^\varepsilon)}\to 0,\quad \varepsilon\to 0 \tag{5}$$

where $\square_i^\varepsilon$ is the cube in Ω with the center at x_i^ε and the side length ε.

2. PROOF OF THEOREMS 1.1 AND 1.2

The proof is based on the following abstract scheme.

Let $\mathscr{H}^\varepsilon,\mathscr{H}^0$ be separable Hilbert spaces with the scalar products

$$(u^\varepsilon,v^\varepsilon)_{\mathscr{H}^\varepsilon},\quad (u,v)_{\mathscr{H}^0}$$

correspondingly and let

$$\mathscr{A}^\varepsilon:\mathscr{H}^\varepsilon\to\mathscr{H}^\varepsilon,\quad \mathscr{A}^0:\mathscr{H}^0\to\mathscr{H}^0$$

be linear continuous operators, $\mathrm{Im}\mathscr{A}^0\subset\mathscr{V}\subset\mathscr{H}^0$, where $\mathscr{V}$ is a subspace in $\mathscr{H}^0$.

We suppose that the following conditions C1–C4 hold.

C1. There exist linear bounded operators $R^\varepsilon:\mathscr{H}^0\to\mathscr{H}^\varepsilon$ such that for any $f\in\mathscr{V}$

$$\|R^\varepsilon f\|^2_{\mathscr{H}^\varepsilon}\to\gamma\|f\|^2_{\mathscr{H}^0},\quad \varepsilon\to 0,\ \gamma>0$$

C2. $\mathscr{A}^\varepsilon,\mathscr{A}^0$ are positive, compact, self-adjoint operators. The norms $\|\mathscr{A}^\varepsilon\|_{\mathscr{L}(\mathscr{H}^\varepsilon)}$ are bounded uniformly in ε.

C3. For any $f\in\mathscr{V}$

$$\|\mathscr{A}^\varepsilon R^\varepsilon f-R^\varepsilon\mathscr{A}^0 f\|_{\mathscr{H}^\varepsilon}\to 0,\quad \varepsilon\to 0$$

C4. For any sequence $f^\varepsilon\in\mathscr{H}^\varepsilon$ such that $\sup_\varepsilon\|f^\varepsilon\|_{\mathscr{H}^\varepsilon}<\infty$ there exist the subsequence ε' and $w\in\mathscr{V}$ such that

$$\|\mathscr{A}^\varepsilon f^\varepsilon-R^\varepsilon w\|_{\mathscr{H}^\varepsilon}\to 0,\quad \varepsilon=\varepsilon'\to 0$$

Let $\mu_1^\varepsilon\geqslant\mu_2^\varepsilon\geqslant\cdots\geqslant\mu_k^\varepsilon\geqslant\cdots\underset{k\to\infty}{\to}0$ be the eigenvalues of $\mathscr{A}^\varepsilon$ written with account of their multiplicity and let $f_1^\varepsilon,f_2^\varepsilon,\ldots,f_k^\varepsilon\ldots$ be the corresponding eigenvectors normalized by the condition $(f_i^\varepsilon,f_j^\varepsilon)_{\mathscr{H}^\varepsilon}=\delta_{ij}$. Let $\mu_1\geqslant\mu_2\geqslant\cdots\geqslant\mu_k\geqslant\cdots\underset{k\to\infty}{\to}0$ be the eigenvalues of $\mathscr{A}^0$.

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Theorem 2.1 (*Iosifyan et al.* [*6*])
Let C1–C4 hold. Then

$$\mu_k^\varepsilon \to \mu_k, \quad \varepsilon \to 0, \ \ k=1,2,3,\ldots \tag{6}$$

and moreover if $\mu_{k-1} > \mu_k = \mu_{k+1} = \cdots = \mu_{k+m-1} > \mu_{k+m}$ then for any $w \in N(\mu_k)$ there exists the linear combination $\widehat{f}^{\,\varepsilon}$ of the eigenvectors $f_k^\varepsilon, \ldots, f_{k+m-1}^\varepsilon$ such that

$$\|\widehat{f}^{\,\varepsilon} - R^\varepsilon w\|_{\mathscr{H}^\varepsilon} \to 0, \quad \varepsilon \to 0 \tag{7}$$

Let us apply this scheme. We set $\mathscr{H}^\varepsilon := L_2(M^\varepsilon)$. Let $\mathscr{H}^0$ be the Hilbert space of real-valued functions from $L_2(\Omega)$ with the scalar product and norm given by

$$(u,v)_{\mathscr{H}^0} = \int_\Omega u(x)v(x)B(x)\,\mathrm{d}x, \quad \|u\|_{\mathscr{H}^0} = \sqrt{(u,u)_{\mathscr{H}^0}}$$

We introduce the operators $R^\varepsilon : \mathscr{H}^0 \to \mathscr{H}^\varepsilon$ by the following formula:

$$(R^\varepsilon u)(\tilde{x}) = \begin{cases} u(\tilde{x}), & \tilde{x} \in \Omega^\varepsilon \\ \langle u \rangle_{\square_i^\varepsilon}, & \tilde{x} \in B_i^\varepsilon \end{cases} \tag{8}$$

Clearly, R^ε are linear operators and due to Cauchy's inequality we have

$$\|R^\varepsilon f\|^2_{\mathscr{H}^\varepsilon} \leqslant \|f\|^2_{\Omega^\varepsilon} + \max_\Omega (b^n(x)\omega_n) \sum_{i=1}^{N(\varepsilon)} \|f\|^2_{L_2(\square_i^\varepsilon)} \leqslant \max_\Omega B(x) \cdot \|f\|^2_{L_2(\Omega)} \leqslant \max_\Omega B(x) \cdot \|f\|^2_{\mathscr{H}^0}$$

Thus, R^ε are bounded uniformly in ε.

Let $L^\varepsilon : \mathscr{H}^\varepsilon \to \mathscr{H}^\varepsilon$, $L^0 : \mathscr{H}^0 \to \mathscr{H}^0$ be the operators determined by the operations $-\Delta^\varepsilon$ and $-B^{-1}\Delta$ correspondingly and Neumann boundary conditions.

We denote by $\mathscr{A}^\varepsilon$ and $\mathscr{A}^0$ the operators inverse to $L^\varepsilon + I$ and $L + I$ correspondingly (I is the identical operator). Since $\operatorname{Im} \mathscr{A}^0$ is dense in $\mathscr{H}^0$ we have $\mathscr{V} = \mathscr{H}^0$.

As it known $\mathscr{A}^\varepsilon, \mathscr{A}^0$ are positive, compact, self-adjoint operators and the estimate

$$\|\mathscr{A}^\varepsilon\|_{\mathscr{L}(\mathscr{H}^\varepsilon)} \leqslant 1$$

is valid. Thus, C2 is valid.

We check C1 (with $\gamma = 1$). Let $f \in C^\infty(\overline{\Omega})$. Using Poincare's inequality, (B) and the fact that $|B_i^\varepsilon| = b^n(x_i^\varepsilon)\omega_n \varepsilon^n + O(d_i^{\varepsilon n})$ $(\varepsilon \to 0)$ we have

$$\begin{aligned} \|R^\varepsilon f\|^2_{\mathscr{H}^\varepsilon} &= \int_{\Omega^\varepsilon} f^2(x)\,\mathrm{d}x + \sum_{i=1}^{N(\varepsilon)} \left(\frac{1}{|\square_i^\varepsilon|} \int_{\square_i^\varepsilon} f^2(x)\,\mathrm{d}x \right) |B_i^\varepsilon| + O(\varepsilon^2) \cdot \|\nabla f\|^2_{L_2(\Omega)} \\ &= \int_{\Omega^\varepsilon} f^2(x)\,\mathrm{d}x + \sum_{i=1}^{N(\varepsilon)} \int_{\square_i^\varepsilon} f^2(x)(b(x_i^\varepsilon))^n \omega_n\,\mathrm{d}x + O(\varepsilon^2) \cdot \|\nabla f\|^2_{L_2(\Omega)} \\ &\quad + \varepsilon^{-n} \cdot O(d_i^{\varepsilon n}) \cdot \|f\|^2_{L_2(\Omega)} \underset{\varepsilon \to 0}{\to} \|f\|^2_{\mathscr{H}^0} \end{aligned}$$

The validity of C1 for any $f \in \mathscr{H}^0$ follows from the uniform in ε boundedness of the operators R^ε and the fact that $C^\infty(\overline{\Omega})$ is dense in $\mathscr{H}^0$.

Let us check condition C3. Let $f \in \mathscr{H}^0$. We denote

$$u^\varepsilon = \mathscr{A}^\varepsilon R^\varepsilon f, \quad f^\varepsilon = R^\varepsilon f$$

In order to describe the behavior of u^ε in Ω^ε as $\varepsilon \to 0$ we introduce the family of the extension operators $\{\Pi^\varepsilon\}^\varepsilon$, $\Pi^\varepsilon : H^1(M^\varepsilon) \to H^1(\Omega)$ with the following properties:

$$\begin{aligned} &(1) \qquad \Pi^\varepsilon u^\varepsilon(x) = Q^\varepsilon u^\varepsilon(x) \quad \text{on } \Omega^\varepsilon \\ &(2)\ \|\nabla \Pi^\varepsilon u^\varepsilon\|_{L_2(\Omega)} \leqslant C \|\nabla^\varepsilon u^\varepsilon\|_{L_2(\Omega^\varepsilon)} \\ &(3) \quad \|\Pi^\varepsilon u^\varepsilon\|_{L_2(\Omega)} \leqslant C \|u^\varepsilon\|_{L_2(\Omega^\varepsilon)} \end{aligned} \tag{9}$$

where the constant $C > 0$ does not depend on ε. Such operators exist, see [12].

As it known $u^\varepsilon(\tilde{x})$ minimizes the functional

$$I[u] = \int_{M^\varepsilon} (|\nabla^\varepsilon u|^2 + u^2 - 2 f^\varepsilon u)\, \mathrm{d}\tilde{x}, \quad u \in H^1(M^\varepsilon)$$

Then $I[u^\varepsilon] \leqslant I[0] = 0$ and so

$$\|\nabla^\varepsilon u^\varepsilon\|^2_{\mathscr{H}^\varepsilon} + \|u^\varepsilon\|^2_{\mathscr{H}^\varepsilon} \leqslant 2 \|f^\varepsilon\|_{\mathscr{H}^\varepsilon} \cdot \|u^\varepsilon\|_{\mathscr{H}^\varepsilon}$$

Using this inequality and properties (9) of the operators Π^ε we conclude that the functions $\Pi^\varepsilon u^\varepsilon$ are bounded in $H^1(\Omega)$ uniformly in ε and therefore there exists a subsequence (still denoted by ε) and $u^0 \in H^1(\Omega)$ such that

$$\Pi^\varepsilon u^\varepsilon \underset{\varepsilon \to 0}{\to} u^0 \text{ weakly in } H^1(\Omega) \text{ and strongly in } L_2(\Omega) \tag{10}$$

Let us show that

$$u^0 = \mathscr{A}^0 f \tag{11}$$

In order to prove (11) we need the following lemmas.

Lemma 2.1
Let D be a convex domain in $\mathbb{R}^n$, b be the diameter of D, X and Y be an arbitrary measurable subsets of D. Then for any $u \in H^1(D)$ the following inequality holds:

$$[\langle u \rangle_X - \langle u \rangle_Y]^2 \leqslant C \|\nabla u\|^2_{L_2(D)} \frac{b^{n+2}}{|X| \cdot |Y|}$$

where the constant C does not depend on u. This inequality is also valid if D is the n-dimensional sphere of the radius b.

Lemma 2.2
For any $u \in H^1(M^\varepsilon)$ the following inequality holds:

$$[\langle \Pi^\varepsilon u \rangle_{\square_i^\varepsilon} - \langle u \rangle_{B_i^\varepsilon}]^2 \leqslant C (\|\nabla \Pi^\varepsilon u^\varepsilon\|^2_{L_2(\square_i^\varepsilon)} + \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(B_i^\varepsilon)}) \cdot (\mathbf{D}_n^\varepsilon + \varepsilon^{2-n}) \tag{12}$$

where the constant C does not depend on ε.

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Proof of Lemmas

We restrict ourselves to the proof of Lemma 2.2. Lemma 2.1 can be proved in a similar way as Lemma 4.9, in [7, p. 117].

Let $\widehat{B}_i^\varepsilon = \{\tilde{x} = (\theta_1, \dots, \theta_n) \in B_i^\varepsilon : \ \Theta_i^\varepsilon < \theta_n < \pi - \Theta_i^\varepsilon\}$, $R_i^\varepsilon = \{\tilde{x} \in \Omega^\varepsilon : d_i^\varepsilon \leqslant |x - x_i^\varepsilon| < \varepsilon/2\}$.

At first we estimate the difference $\langle u \rangle_{R_i^\varepsilon} - \langle u \rangle_{\widehat{B}_i^\varepsilon}$. Let $\tilde{x} = (\theta_1, \dots, \theta_{n-1}, r) \in R_i^\varepsilon$, $\tilde{y} = (\theta_1, \dots, \theta_{n-1}, \theta_n) \in \widehat{B}_i^\varepsilon$. Then

$$u(\tilde{x}) - u(\tilde{y}) = \int_0^{l^\varepsilon(\tilde{x}, \tilde{y})} \frac{\mathrm{d}u}{\mathrm{d}\tau}(\xi(\tau))\,\mathrm{d}\tau \tag{13}$$

Here

$$\xi(\tau) = \begin{cases} (\theta_1, \dots, \theta_{n-1}, \theta_n - \tau b_i^{\varepsilon-1}) \in \widehat{B}_i^\varepsilon, & \tau \in [0, \ (\theta_n - \Theta_i^\varepsilon) b_i^\varepsilon] \\ (\theta_1, \dots, \theta_{n-1}, \tau - (\theta_n - \Theta_i^\varepsilon) b_i^\varepsilon + d_i^\varepsilon) \in R_i^\varepsilon, & \tau \in [(\theta_n - \Theta_i^\varepsilon) b_i^\varepsilon, \ l^\varepsilon(\tilde{x}, \tilde{y})] \end{cases} \tag{14}$$

where $l^\varepsilon(\tilde{x}, \tilde{y}) = (\theta_n - \Theta_i^\varepsilon) b_i^\varepsilon + r - d_i^\varepsilon$. In fact τ is a natural parameter on the geodesic curve $\gamma : \tau \mapsto \xi(\tau)$ which connects $\tilde{x}$ and $\tilde{y}$, $l^\varepsilon(\tilde{x}, \tilde{y})$ is the length of this curve.

Multiplying (13) by $r^{n-1} (b_i^\varepsilon)^n \prod_{k=1}^n \sin^{n-k} \theta_{n-k+1}$, integrating by θ_1 from 0 to 2π, by θ_k $(k = 2, \dots, n-1)$ from 0 to π, by θ_n from Θ_i^ε to $\pi - \Theta_i^\varepsilon$, by r from d_i^ε to $\varepsilon/2$, and finally dividing by $|\widehat{B}_i^\varepsilon| \cdot |R_i^\varepsilon|$ we obtain:

$$[\langle u \rangle_{R_i^\varepsilon} - \langle u \rangle_{\widehat{B}_i^\varepsilon}]^2 \leqslant C \cdot \frac{I^\varepsilon}{|\widehat{B}_i^\varepsilon|^2 \cdot |R_i^\varepsilon|^2} \tag{15}$$

where C does not depend on ε and

$$I^\varepsilon = \left(\int_{d_i^\varepsilon}^{\varepsilon/2} \int_{\Theta_i^\varepsilon}^{\pi - \Theta_i^\varepsilon} \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \int_0^{l^\varepsilon(\tilde{x}, \tilde{y})} \frac{\mathrm{d}u}{\mathrm{d}\tau}(\xi(\tau)) r^{n-1} (b_i^\varepsilon)^n \right.$$
$$\left. \times \prod_{k=1}^n \sin^{n-k} \theta_{n-k+1} \mathrm{d}\tau \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \mathrm{d}r \right)^2, \quad \xi(\tau) \text{ is defined by (14)}$$

Using Cauchy's inequality and (A)–(C), it is easy to obtain

$$\begin{aligned} I^\varepsilon &\leqslant C \left(\varepsilon^{3n+2} \|\nabla^\varepsilon u\|^2_{L_2(\widehat{B}_i^\varepsilon)} \int_{\Theta_i^\varepsilon}^{\pi - \Theta_i^\varepsilon} \frac{\mathrm{d}\theta}{\sin^{n-1}\theta} + \varepsilon^{4n} \|\nabla^\varepsilon u\|^2_{L_2(R_i^\varepsilon)} \int_{d_i^\varepsilon}^{\varepsilon/2} \frac{\mathrm{d}r}{r^{n-1}} \right) \\ &\leqslant C_1 \cdot \varepsilon^{4n} \cdot \|\nabla^\varepsilon u\|^2_{L_2(R_i^\varepsilon \cup \widehat{B}_i^\varepsilon)} \cdot \mathbf{D}_n^\varepsilon \end{aligned} \tag{16}$$

Thus, it follows from (15), (16) that

$$[\langle u \rangle_{L_2(R_i^\varepsilon)} - \langle u \rangle_{L_2(\widehat{B}_i^\varepsilon)}]^2 \leqslant C \|\nabla^\varepsilon u\|^2_{L_2(R_i^\varepsilon \cup \widehat{B}_i^\varepsilon)} \cdot \mathbf{D}_n^\varepsilon \tag{17}$$

From Lemma 2.1 one has

$$\begin{aligned} [\langle \Pi^\varepsilon u \rangle_{\square_i^\varepsilon} - \langle u \rangle_{R_i^\varepsilon}]^2 &\leqslant C \cdot \|\nabla^\varepsilon \Pi^\varepsilon u\|^2_{L_2(\square_i^\varepsilon)} \cdot \varepsilon^{2-n} \\ [\langle u \rangle_{B_i^\varepsilon} - \langle u \rangle_{\widehat{B}_i^\varepsilon}]^2 &\leqslant C \cdot \|\nabla^\varepsilon \Pi_i^\varepsilon u\|^2_{L_2(\mathbf{B}_i^\varepsilon)} \cdot \varepsilon^{2-n} \end{aligned} \tag{18}$$

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Here $\mathbf{B}_i^\varepsilon$ is the n-dimensional sphere that contains the truncated spheres B_i^ε and $\widehat{B}_i^\varepsilon$, Π_i^ε is an extension operator from $H^1(B_i^\varepsilon)$ to $H^1(\mathbf{B}_i^\varepsilon)$ that has the same properties as the operator Π^ε with B_i^ε instead of Ω^ε and $\mathbf{B}_i^\varepsilon$ instead of Ω (the existence of such operator can be proved in a similar way as the existence of operator Π^ε).

The inequality (12) directly follows from (17), (18) and the properties of the operators Π_i^ε. Lemma 2.2 is proved. □

We continue the verification of condition C3. For any $w^\varepsilon \in H^1(M^\varepsilon)$ we have:

$$\int_{M^\varepsilon} [(\nabla^\varepsilon u^\varepsilon, \nabla^\varepsilon w^\varepsilon) + (u^\varepsilon - f^\varepsilon) w^\varepsilon]\, \mathrm{d}\tilde{x} = 0 \tag{19}$$

Let w be an arbitrary function from $C^\infty(\overline{\Omega})$. Let $\varphi(r):[0,\infty)\to\mathbb{R}$ be a smooth function equal to 1 as $0\leqslant r\leqslant 1$ and equal to 0 as $r\geqslant 2$. We denote $w_i^\varepsilon := w(x_i^\varepsilon)$.

We choose the following test-function w^ε:

$$w^\varepsilon(\tilde{x}) = \begin{cases} w(x) + \sum\limits_{i=1}^{N(\varepsilon)} (w_i^\varepsilon - w(x))\varphi_i^\varepsilon(x), & \tilde{x}\in\Omega^\varepsilon \\ w_i^\varepsilon, & \tilde{x}\in B_i^\varepsilon \end{cases} \tag{20}$$

where $\varphi_i^\varepsilon(x) = \varphi(|x - x_i^\varepsilon|/d_i^\varepsilon)$.

At first let us investigate the integrals in (19) over Ω^ε (we denote them I_1^ε):

$$I_1^\varepsilon = \int_{\Omega^\varepsilon} [(\nabla u^\varepsilon, \nabla w) + (u^\varepsilon - f) w]\, \mathrm{d}x + \sum_{i=1}^{N(\varepsilon)} \int_{\mathrm{supp}(\varphi_i^\varepsilon)\cap\Omega^\varepsilon} [(\nabla u^\varepsilon, \nabla((w_i^\varepsilon - w)\varphi_i^\varepsilon)) + (u^\varepsilon - f)(w_i^\varepsilon - w)\varphi_i^\varepsilon]\, \mathrm{d}x$$

The last sum of the integrals tends to zero as $\varepsilon\to 0$ because the function

$$\sum_{i=1}^{N(\varepsilon)} [(\nabla((w_i^\varepsilon - w)\varphi_i^\varepsilon))^2 + ((w_i^\varepsilon - w)\varphi_i^\varepsilon)^2]$$

is bounded uniformly in ε and in view of (B) $\sum_{i=1}^{N(\varepsilon)} |\mathrm{supp}(\varphi_i^\varepsilon)| \to 0, \varepsilon\to 0$. In addition, it follows from (B) that $\sum_{i=1}^{N(\varepsilon)} |D_i^\varepsilon| \to 0, \varepsilon\to 0$. Therefore, one has

$$\lim_{\varepsilon\to 0} I_1^\varepsilon = \int_\Omega [(\nabla u^0, \nabla w) + (u^0 - f) w]\, \mathrm{d}x \tag{21}$$

Now we investigate the integrals in (19) over the union $\bigcup_i B_i^\varepsilon$ of 'bubbles' (we denote them I_2^ε). We have

$$I_2^\varepsilon = \sum_{i=1}^{N(\varepsilon)} w_i^\varepsilon \cdot [\langle \Pi^\varepsilon u^\varepsilon \rangle_{\square_i^\varepsilon} - \langle f \rangle_{\square_i^\varepsilon}] \cdot |B_i^\varepsilon| + \sum_{i=1}^{N(\varepsilon)} w_i^\varepsilon \cdot [\langle u^\varepsilon \rangle_{B_i^\varepsilon} - \langle \Pi^\varepsilon u^\varepsilon \rangle_{\square_i^\varepsilon}] \cdot |B_i^\varepsilon|$$

In view of Lemma 2.2 and (C) the second sum is estimated by

$$C(w)\cdot \|\nabla^\varepsilon u^\varepsilon\|_{L_2(M^\varepsilon)} \cdot \sqrt{\varepsilon^n \mathbf{D}_n^\varepsilon + \varepsilon^2}$$

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and tends to zero in view of (A), while in view of (C) the first sum tends to

$$\lim_{\varepsilon\to 0} I_2^\varepsilon = \int_\Omega (u^0(x) - f(x))\omega_n b^n(x)\,\mathrm{d}x \tag{22}$$

It is easy to see that (11) follows from (19), (21), (22).

Finally, we have:

$$\|\mathscr{A}^\varepsilon R^\varepsilon f - R^\varepsilon \mathscr{A}^0 f\|^2_{\mathscr{H}^\varepsilon} = \|u^\varepsilon - R^\varepsilon u^0\|^2_{L_2(\Omega^\varepsilon)} + \|u^\varepsilon - R^\varepsilon u^0\|^2_{L_2(\bigcup_i B_i^\varepsilon)}$$

We have just proved that the first term tends to zero. Let us estimate the second term. We have:

$$\|u^\varepsilon - R^\varepsilon u^0\|^2_{L_2(\bigcup_i B_i^\varepsilon)} = \sum_{i=1}^{N(\varepsilon)} \int_{B_i^\varepsilon} [(u^\varepsilon - \langle u^\varepsilon\rangle_{B_i^\varepsilon}) + (\langle u^\varepsilon\rangle_{B_i^\varepsilon} - \langle \Pi^\varepsilon u^\varepsilon\rangle_{\square_i^\varepsilon}) + (\langle \Pi^\varepsilon u^\varepsilon\rangle_{\square_i^\varepsilon} - \langle u^0\rangle_{\square_i^\varepsilon})]^2\,\mathrm{d}\tilde{x}$$

Owing to Poincare's inequality and (C) we have

$$\sum_{i=1}^{N(\varepsilon)} \int_{B_i^\varepsilon} (u^\varepsilon - \langle u^\varepsilon\rangle_{B_i^\varepsilon})^2\,\mathrm{d}\tilde{x} \leqslant C\varepsilon^2 \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(\bigcup B_i^\varepsilon)} \to 0, \quad \varepsilon\to 0$$

In view of (A), (C) and Lemma 2.2 we have

$$\sum_{i=1}^{N(\varepsilon)} \int_{B_i^\varepsilon} (\langle u^\varepsilon\rangle_{B_i^\varepsilon} - \langle \Pi^\varepsilon u^\varepsilon\rangle_{\square_i^\varepsilon})^2\,\mathrm{d}\tilde{x} \leqslant C\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(M^\varepsilon)}(\varepsilon^n \mathbf{D}_n^\varepsilon + \varepsilon^2) \to 0, \quad \varepsilon\to 0$$

And finally in view of (C), (10) and Cauchy's inequality we have

$$\sum_{i=1}^{N(\varepsilon)} \int_{B_i^\varepsilon} (\langle \Pi^\varepsilon u^\varepsilon\rangle_{\square_i^\varepsilon} - \langle u^0\rangle_{\square_i^\varepsilon})^2\,\mathrm{d}\tilde{x} \leqslant C\|\Pi^\varepsilon u^\varepsilon - u^0\|^2_{L_2(\Omega)} \to 0, \quad \varepsilon\to 0$$

It follows from the last three estimated that $\|u^\varepsilon - R^\varepsilon u^0\|^2_{L_2(\bigcup_i B_i^\varepsilon)} \to 0, \varepsilon\to 0$ and thus condition C3 holds.

And in final let us check condition C4. Let $f^\varepsilon \in \mathscr{H}^\varepsilon$ and $\sup_\varepsilon \|f^\varepsilon\|_{\mathscr{H}^\varepsilon} < \infty$. Then the functions $u^\varepsilon = \mathscr{A}^\varepsilon f^\varepsilon$ are bounded in $H^1(M^\varepsilon)$ uniformly in ε and therefore there exists a subsequence ε' and $w \in H^1(\Omega)$ such that

$$\Pi^\varepsilon u^\varepsilon \to w, \quad \varepsilon = \varepsilon' \to 0$$

We have

$$\|\mathscr{A}^\varepsilon f^\varepsilon - R^\varepsilon w\|^2_{\mathscr{H}^\varepsilon} = \|\mathscr{A}^\varepsilon f^\varepsilon - R^\varepsilon w\|^2_{L_2(\Omega^\varepsilon)} + \|\mathscr{A}^\varepsilon f^\varepsilon - R^\varepsilon w\|^2_{L_2(\bigcup B_i^\varepsilon)}$$

The first term converges to zero by the definition of w, the second term converges to zero in view of Lemma 2.2 (see the same arguments in the end of the proof of C3). Thus, C4 also holds.

We have verified the fulfilment of conditions C1–C4. Thus, by virtue of Theorem 2.1 the eigenvalues μ_k^ε and the eigenfunctions f_k^ε of $\mathscr{A}^\varepsilon$ converge in the sense (6), (7) to the eigenvalues μ_k and the eigenfunctions f_k of $\mathscr{A}^0$. But due to the relations

$$\lambda_k^\varepsilon = \frac{1}{\mu_k^\varepsilon} - 1, \quad \lambda_k = \frac{1}{\mu_k} - 1, \quad u_k^\varepsilon = f_k^\varepsilon, \quad u_k = f_k$$

and the definition of the operators R^ε it follows that (3), (5) hold.

Theorems 1.1, 1.2 are proved.

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3. HOMOGENIZATION OF HEAT EQUATION AND LARGE TIME BEHAVIOR OF HOMOGENIZED EQUATION

The results of the same type as in Section 1 can be easily obtained for the classical evolution equations. We illustrate this on the example of the heat equation.

Let M^ε be the manifold which constructed above in Section 1. We consider on this manifold the following initial-boundary value problem:

$$\begin{aligned} &\frac{\partial u^\varepsilon}{\partial t} - \Delta^\varepsilon u^\varepsilon = 0, \quad \tilde{x} \in M^\varepsilon, \quad t > 0 \\ &u^\varepsilon(x,0) = f^\varepsilon(x), \quad \frac{\partial u^\varepsilon}{\partial \nu} = 0, \quad \tilde{x} \in \partial\Omega \end{aligned} \tag{23}$$

where $f^\varepsilon \in L_2(M^\varepsilon)$. It is well known (see, e.g. [4]) that there exists the unique solution $u^\varepsilon(\tilde{x},t)$ of problem (23) such that for any interval $[0,T]$

$$u^\varepsilon(\tilde{x},t) \in C^0([0,T], L_2(M^\varepsilon)) \cap L^2([0,T], H^1(M^\varepsilon))$$

Theorem 3.1
Suppose that

(i) $f^\varepsilon \in H^1(M^\varepsilon)$, $\|\nabla^\varepsilon f^\varepsilon\|^2_{L_2(M^\varepsilon)} + \|f^\varepsilon\|^2_{L_2(M^\varepsilon)} \leqslant C$, where C does not depend on ε,
(ii) there exists $f \in L_2(\Omega)$ such that

$$\lim_{\varepsilon \to 0} \|Q^\varepsilon f^\varepsilon - f\|_{L_2(\Omega^\varepsilon)} = 0$$

Then for any interval $[0,T]$ we have that

$$\lim_{\varepsilon\to 0} \left\{ \max_{t\in[0,T]} \|Q^\varepsilon u^\varepsilon - u\|^2_{L_2(\Omega^\varepsilon)} + \max_{t\in[0,T]} \sum_{i=1}^{N(\varepsilon)} \|u^\varepsilon - \langle u\rangle_{\square_i^\varepsilon}\|^2_{L_2(B_i^\varepsilon)} \right\} = 0 \tag{24}$$

where $u(x,t)$ is the solution of the problem

$$\begin{aligned} &B(x)\frac{\partial u}{\partial t} - \Delta u = 0, \quad x \in \Omega, \quad t > 0 \\ &u(x,0) = f(x), \quad \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega \end{aligned} \tag{25}$$

where $B(x)$ is defined by formula (4).

Proof
We give only a sketch of the proof since the main ideas are similar to the proof of Theorems 1.1, 1.2.

The solution of problem (23) satisfies the equality:

$$\frac{1}{2}\|u^\varepsilon(t)\|^2_{L_2(M^\varepsilon)} + \int_0^t \|\nabla^\varepsilon u^\varepsilon(\tau)\|^2_{L_2(M^\varepsilon)} \mathrm{d}\tau = \frac{1}{2}\|f^\varepsilon\|^2_{L_2(M^\varepsilon)} \quad \forall t > 0 \tag{26}$$

Moreover, since $f^\varepsilon \in H^1(M^\varepsilon)$ we have

$$u^\varepsilon \in C^0([0,T], H^1(M^\varepsilon)), \quad \partial_t u^\varepsilon \in L_2([0,T], L_2(M^\varepsilon))$$

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and

$$\frac{1}{2}\|\nabla^\varepsilon u^\varepsilon(t)\|^2_{L_2(M^\varepsilon)}+\int_0^t \|\partial_t u^\varepsilon(\tau)\|^2_{L_2(M^\varepsilon)}\,\mathrm{d}\tau=\frac{1}{2}\|\nabla^\varepsilon f^\varepsilon\|^2_{L_2(M^\varepsilon)}\quad \forall t>0 \tag{27}$$

It follows from (26)–(27) and (i) that the functions u^ε are bounded in $H^1(M^\varepsilon\times[0,T])$ uniformly in ε. Let $\{\Pi^\varepsilon\}^\varepsilon$ be the family of linear extension operators which are introduced in Section 2 and which satisfies (9). It is easy to prove that

$$\Pi^\varepsilon[u^\varepsilon(t)]\in C^0([0,T],H^1(\Omega)),\quad \partial_t\Pi^\varepsilon[u^\varepsilon(t)]\in L_2([0,T],L_2(\Omega)),\quad \partial_t[\Pi^\varepsilon u^\varepsilon]=\Pi^\varepsilon[\partial_t u^\varepsilon] \tag{28}$$

and then the functions $\Pi^\varepsilon u^\varepsilon$ are bounded in $H^1(\Omega\times[0,T])$ uniformly in ε. Therefore, there exists a function $u(x,t)\in H^1(\Omega\times[0,T])$ and a subsequence (still denoted by ε) such that

$$\Pi^\varepsilon u^\varepsilon \underset{\varepsilon\to 0}{\to} u \text{ weakly in } H^1(\Omega\times[0,T]) \text{ and strongly in } L_2(\Omega\times[0,T])$$

In addition, in view of (26)–(28) the set of functions $\Pi^\varepsilon u^\varepsilon(t)\in C^0([0,T],L_2(M^\varepsilon))$ is equicontinuous and pointwise compact (i.e. compact in $L_2(\Omega)$ for any fixed t). Therefore, due to Arzelá–Ascoli theorem $\{\Pi^\varepsilon u^\varepsilon(t)\}^\varepsilon$ is a compact set in $C^0([0,T],L_2(\Omega))$ and thus $\Pi^\varepsilon u^\varepsilon$ converges to u in $C^0([0,T],L_2(\Omega))$.

Let $\gamma^\varepsilon(\tilde{x},t)\in C^\infty(\overline{M^\varepsilon}\times[0,T])$ such that $\gamma^\varepsilon(\tilde{x},T)=0$. Then the solution $u^\varepsilon(\tilde{x},t)$ of (23) satisfies the following equality:

$$(f^\varepsilon,\gamma^\varepsilon(0))_{L_2(M^\varepsilon)}+\int_0^T\{-(u^\varepsilon,\partial_t\gamma^\varepsilon)_{L_2(M^\varepsilon)}+(\nabla^\varepsilon u^\varepsilon,\nabla^\varepsilon\gamma^\varepsilon)_{L_2(M^\varepsilon)}\}\,\mathrm{d}t=0 \tag{29}$$

Let $g(t)$ and $w(x)$ be arbitrary functions from $C^\infty([0,T])$ and $C^\infty(\overline{\Omega})$ correspondingly, $g(T)=0$. We choose the following test function γ^ε:

$$\gamma^\varepsilon(\tilde{x},t)=w^\varepsilon(\tilde{x})g(t)$$

where $w^\varepsilon(\tilde{x})$ is constructed by formula (20).

Plugging this γ^ε into equality (29) and passing to the limit as $\varepsilon\to 0$ one can prove that $u(x,t)$ satisfies the following equality:

$$\begin{aligned}\int_\Omega B(x)f(x)w(x)g(0)\,\mathrm{d}x+\int_0^T\int_\Omega\{&-B(x)u(x,t)w(x)\partial_t g(t)\\ &+(\nabla u(x,t),\nabla w(x))g(t)\}\,\mathrm{d}x\,\mathrm{d}t=0\end{aligned} \tag{30}$$

The proof is similar to the proof of C3 in Section 2.

It follows from (30) that $u(x,t)$ is a solution of (25). Since (25) has the unique solution from $C^0([0,T],L_2(\Omega))\cap L_2([0,T],H^1(\Omega))$ than the whole sequence $\Pi^\varepsilon u(x,t)$ converges in $C^0([0,T],L_2(\Omega))$ to $u(x,t)$.

Finally, using the same arguments as in the end of C3 proof in Section 2 we conclude that u^ε converges to u in the sense of (24). Theorem 3.1 is proved. □

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Problem (25) describes the heat transfer in a medium with non-constant heat capacity $B(x)$ (the solution $u(x,t)$ is a temperature). It is well known that in the case $B(x)=\text{const}$ $u(x,t)$ converges to the mean value of the initial function f as $t\to\infty$. Using the results obtained above we can easily investigate the large time behavior of $u(x,t)$ in the case of an arbitrary smooth $B(x)>0$ and an arbitrary initial function $f(x)\in L_2(\Omega)$.

Proposition 3.1
The solution $u(x,t)$ of problem (25) converges to the constant $L=(\int_\Omega B(x)f(x)\,\mathrm{d}x)\cdot(\int_\Omega B(x)\,\mathrm{d}x)^{-1}$ strongly in $L_2(\Omega)$ as $t\to\infty$.

Of course it is possible to prove this proposition directly analyzing problem (25), but we show that it immediately follows from the homogenization result. This approach allows us to 'guess' easily the constant L. Similar method has been used in [2] in order to investigate the large time behavior of the linear reaction–diffusion system describing transport of particles of several species.

Proof
Since the solution of (25) depends continuously on the initial data (namely, $\|u(t)\|_{L_2(M^\varepsilon)}\leqslant C\|f\|_{L_2(M^\varepsilon)}$, $\forall t>0$) then it is sufficient to prove the proposition only for smooth initial functions $f(x)$.

At first we suppose that $\min_\Omega B(x)>1$.

We consider the manifold M^ε constructed in Section 1 with $b(x)=((B(x)-1)/\omega_n)^{1/n}$.

Let f^ε be the function from $C^\infty(M^\varepsilon)$ constructed by formula (20) with f instead of w. It is easy to see that

$$\text{(i)}\ \|\nabla^\varepsilon f^\varepsilon\|^2_{L_2(M^\varepsilon)}+\|f^\varepsilon\|^2_{L_2(M^\varepsilon)}\leqslant C,\quad \text{(ii)}\ \lim_{\varepsilon\to 0}\|Q^\varepsilon f^\varepsilon-f\|_{L_2(\Omega^\varepsilon)}=0 \tag{31}$$

Let us consider on $M^\varepsilon\times[0,T]$ problem (23) with f^ε defined above.

From Gronwall's lemma we obtain that the solution u^ε of this problem satisfies the inequality

$$\|u^\varepsilon(t)-L^\varepsilon\|^2_{L_2(M^\varepsilon)}\leqslant\|f^\varepsilon-L^\varepsilon\|^2_{L_2(M^\varepsilon)}\cdot\exp[-2\lambda_2^\varepsilon t] \tag{32}$$

where $L^\varepsilon=(1/|M^\varepsilon|)\int_{M^\varepsilon}f^\varepsilon(\tilde{x})\,\mathrm{d}x$ is the mean value of f^ε over M^ε, λ_2^ε is the second eigenvalue of problem (1).

Using the form of the function f^ε and Theorem 1.1 we obtain

$$L^\varepsilon\to L,\quad \|f^\varepsilon-L^\varepsilon\|_{L_2(M^\varepsilon)}\to\|B^{1/2}(f-L)\|_{L_2(\Omega)},\quad \lambda_2^\varepsilon\to\lambda_2\quad(\varepsilon\to 0) \tag{33}$$

where λ_2 is the second eigenvalue of problem (2). Obviously $\lambda_2\neq 0$.

Since the initial functions f^ε satisfy the conditions of Theorem 3.1, $\Pi^\varepsilon u^\varepsilon$ converges to u in $C^0([0,T],L_2(\Omega))$. Thus, since T is arbitrary, then for any $t>0$ we have:

$$\|\Pi^\varepsilon u^\varepsilon(t)-u(t)\|_{L_2(M^\varepsilon)}\to 0,\quad \varepsilon\to 0 \tag{34}$$

By virtue of (32) one has that for any $t>0$:

$$\|\Pi^\varepsilon u^\varepsilon(t)-L^\varepsilon\|^2_{L_2(\Omega)}\leqslant C\|u^\varepsilon(t)-L^\varepsilon\|^2_{L_2(M^\varepsilon)}\leqslant C\|f^\varepsilon-L^\varepsilon\|^2_{L_2(M^\varepsilon)}\cdot\exp[-2\lambda_2^\varepsilon t] \tag{35}$$

Using (33), (34) we pass to the limit in (35) as $\varepsilon\to 0$ and conclude that for any $t>0$

$$\|u(t)-L\|^2_{L_2(\Omega)}\leqslant C\|B^{1/2}(f-L)\|^2_{L_2(\Omega)}\cdot\exp[-2\lambda_2 t]\to 0,\quad t\to\infty$$

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Finally, if $\min_\Omega B(x) \leqslant 1$ it is sufficient to make a substitution $t = \tau \cdot \alpha$, $0 < \alpha < \min_\Omega B(x)$ and reduce the problem to the previous case. Proposition is proved. □

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