On spectrum of Riemannian manifolds with attached thin handles

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Abstract. The behavior as $\varepsilon \to 0$ of the spectrum of the Laplace-Beltrami operator Δ^ε is studied on Riemannian manifolds M^ε depending on a small parameter ε. They consist of a fixed compact manifold with attached handles whose radii tend to zero as $\varepsilon \to 0$. We consider two cases: when the number of the handles is fixed and their lengthes are also fixed and when the number of the handles tend to infinity and their lengthes tend to zero as $\varepsilon \to 0$. For these cases we obtain the operators whose spectrum attracts the spectrum of Δ^ε as $\varepsilon \to 0$.

Keywords: homogenization, Laplace-Beltrami operator, spectrum, Riemannian manifold

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Introduction

The aim of this paper is to study the behavior as $\varepsilon \to 0$ of the spectrum of the Laplace-Beltrami operator Δ^ε on Riemannian manifolds M^ε depending on a small parameter ε. We consider two different problems.

In Section 1 we consider a manifold M^ε that consists of a fixed 2-dimensional compact Riemannian manifold without boundary Ω and an attached "thin"manifold Γ^ε. The last one consists of several tubes with fixed lengthes and radii ε (see Fig.1 below). Thus Γ^ε "converges"to some graph Γ as $\varepsilon \to 0$.

Let Δ_Ω be the Laplace-Beltrami operator on Ω and $\mathbf{L}$ be the Laplace operator on Γ, i.e. $\mathbf{L}$ is defined by the operation $\frac{d^2}{ds^2}$ on the edges of Γ (s is a natural parameter on the edge), Dirichlet boundary conditions on the ends of Γ and Kirchhoff conditions on the vertices of Γ. We prove that the spectrum of Δ^ε converges in some suitable sense to the union of the spectrum of Δ_Ω and the spectrum of $\mathbf{L}$. Also we investigate the behaviour of corresponding eigenvalues.

This results generalize the results of C.Anne [1]. In [1] the behavior of the spectrum is studied on a manifold with *one* attached handle having a fixed length and a vanishingly small radius. In [2] these results are extended to the case of the Laplacian acting on differential p-forms. The convergence of spectra on manifolds which collapse to a graph has been studied in [6].

In Section 2 we consider the manifold M^ε whose topological genus increases as $\varepsilon \to 0$. It is constructed in the following way. Let Ω be a compact 2-dimensional Riemannian manifold without a boundary and $D_i^\varepsilon, i = 1...N(\varepsilon) = 3N_1(\varepsilon)$ be a system of nonintersecting balls ("holes") in Ω depending on ε. Let $\Omega^\varepsilon = \Omega \backslash \bigcup\limits_{i=1}^{N(\varepsilon)} D_i^\varepsilon$. Suppose that the set $\{1...N(\varepsilon)\}$ is divided on subsets that consist of 3 elements. If the indexes i, j, k lie on one subset we connect the "holes" $D_i^\varepsilon, D_j^\varepsilon, D_k^\varepsilon$ by means of a manifold that consists of tubes $G_i^\varepsilon, G_j^\varepsilon, G_k^\varepsilon$ and a truncated sphere B_{ijk}^ε (see Fig.2 below). As a result we obtain the manifold

$$M^\varepsilon = \Omega^\varepsilon \bigcup_{i,j,k} \left[G_i^\varepsilon \cup G_j^\varepsilon \cup G_k^\varepsilon \cup B_{ijk}^\varepsilon\right].$$

We suppose that the number of "holes"increases as $\varepsilon \to 0$, while their radii tend to 0. It is supposed that the radii of the "holes"are much smaller than the distances between them. We also suppose that in contrast to the manifold Γ^ε in Section 1 and in contrast to [1] the metrics is such that the lengthes of the tubes converge to 0.

We obtain the following result: if some conditions on a distribution of the "holes"and on the metrics on the tubes and the truncated spheres are hold then the spectrum of the operator $-\Delta^\varepsilon$ converges in some suitable sense to the spectrum of the operator $\mathcal{L}$ defined by the formula

$$[\mathcal{L}u](x) = -\Delta_\Omega u(x) + \int_\Omega W(x,y)(u(x) - u(y))dy.$$

Here $W(x,y)$ is a positive symmetric function. We present an example for which $W(x,y)$ is calculated explicitly.

The behavior of the spectrum of manifolds with complex microstructure has been also studied in [5],[8] for another types of manifolds. We note that the behavior of the spectrum of the manifold with attached *one* handle having a vanishingly small radius and (in contrast to [1]) a vanishingly small length has been studied in [4].

The proof of main results is based on the abstract scheme proposed in [7].

Through the paper we denote by C various constants independent on ε.

1 Riemannian manifold with attached "graph"

1.1 Problem setting and main result

Let Ω be a 2-dimensional compact Riemannian manifold without a boundary and with a metrics g. By Δ_Ω we denote the corresponding Laplace-Beltrami operator.

Let $D_i^\varepsilon, i = 1...N$ be the system of balls in Ω with the centers $x_i \in \Omega$ and the radii ε. We consider the following domain with holes

$$\Omega^\varepsilon = \Omega \backslash \bigcup_{i=1}^{N} D_i^\varepsilon.$$

We glue to Ω^ε the manifold Γ^ε illustrated on Fig.1 and constructed in the following way.

Let Γ be the graph in $\mathbb{R}^3$. We denote by $p_i, i = 1...m$ $(m \geq N)$ the vertices of this graph and by γ_{ij}^ε the edges of this graph. γ_{ij}^ε connects the vertices p_i and p_j. We introduce the symmetric matrix $\{A_{ij}\}_{i,j=1}^m$ such that $A_{ij} = 1$ if p_i^ε and p_j^ε are connected and $A_{ij} = 0$ otherwise. We suppose that for the first N vertices $p_i, i = 1...N$ there is only one p_j such that $A_{ij} = 1$. This are the ends of the graph.

Let z_{ij} be the natural parameter on γ_{ij}, $z_{ij} \in [0, l_{ij}]$. We denote by $p(z_{ij})$ the point on γ_{ij} that corresponds to the natural parameter z_{ij}.

We denote by G_{ij}^ε the cylinder with the axis $\widehat{\gamma}_{ij} = \{p(z_{ij}) \in \gamma_{ij} : z_{ij} \in [\delta^\varepsilon, l_{ij} - \delta^\varepsilon], \delta^\varepsilon \geq 0\}$ and with the radius ε. The length of G_{ij}^ε is equal to $l_{ij}^\varepsilon = l_{ij} - 2\delta^\varepsilon$. We choose the standard cylindrical coordinates on G_{ij}^ε

$$G_{ij}^\varepsilon = \left\{ (\varphi_{ij}, z_{ij}) : \varphi_{ij} \in [0, 2\pi], z_{ij} \in [\delta^\varepsilon, l_{ij} - \delta^\varepsilon] \right\}.$$

Clearly δ^ε can be chosen such that

1. G_{ij}^ε are pairwise disjoint,
2. $|\delta^\varepsilon| \leq C \cdot \varepsilon$.

The boundary of G_{ij}^ε consists of two circles S_{ij}^ε and S_{ji}^ε. Here we suppose that S_{ij}^ε is closer to the vertex p_i and S_{ji}^ε is closer to the vertex p_j.

Let for $i \in \{N+1...m\}$ $\mathcal{B}_i^\varepsilon$ be the sphere of the radius $b^\varepsilon = \sqrt{\varepsilon^2 + \delta^{\varepsilon 2}}$ with the center p_i. It is clear that $S_{ij}^\varepsilon \subset \mathcal{B}_i^\varepsilon$. Let $\mathcal{D}_{ij}^\varepsilon$ be the part of $\mathcal{B}_i^\varepsilon$ that lie inside of the cylinder G_{ij}^ε and let

$$B_i^\varepsilon = \mathcal{B}_i^\varepsilon \backslash \bigcup_{j : A_{ij} = 1} \mathcal{D}_{ij}.$$

We obtain the 2-dimensional manifold (see Fig.1):

$$\Gamma^\varepsilon = \bigcup_{i=1}^{m} \left[\bigcup_{i,j : A_{ij} = 1, i < j} G_{ij}^\varepsilon \right] \bigcup_{i=N+1}^{m} B_i^\varepsilon.$$

The boundary of Γ^ε consists of S_{ij}^ε, $i, j : i = 1...N, A_{ij} = 1$.

Now we suppose that $S_{ij}^\varepsilon, i, j : i = 1...N, A_{ij} = 1$ are diffeomorphic to ∂D_i^ε. Using this diffeomorphisms we glue Γ^ε to Ω^ε and obtain the manifold without a boundary:

$$M^\varepsilon = \Omega^\varepsilon \cup \Gamma^\varepsilon.$$

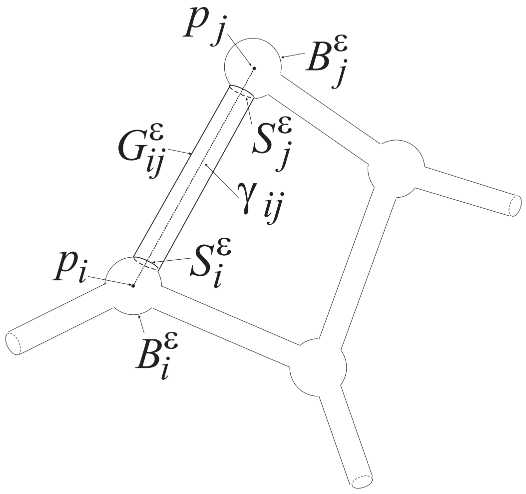

Figure 1: Manifold Γ^ε.

We denote by $\tilde{x}$ the points of this manifold. Clearly M^ε can be covered by a system of charts and suitable local coordinates $\{x_1, x_2\}$ can be introduced.

It is supposed that M^ε is equipped by the metrics g^ε that coincides with the metrics g on Ω^ε and coincides with the Euclidean metrics induced from $\mathbb{R}^3$ on Γ^ε. By $g^\varepsilon_{\alpha\beta}$ we denote the components of the metric tensor in local coordinates.

Let $L_2(B)$ be the Hilbert space of the real-valued functions on $B \subset M^\varepsilon$ with the scalar product and the norm

$$(u,v)_{L_2(B)} = \int\limits_B u(\tilde{x}) \cdot v(\tilde{x}) d\tilde{x}, \ \|u\|^2_{L_2(B)} = \int\limits_B \big(u(\tilde{x})\big)^2 d\tilde{x}$$

where $d\tilde{x} = \sqrt{\det g^\varepsilon_{\alpha\beta}} dx_1 dx_2$ be the volume form.

We denote $\mathcal{H}^\varepsilon := L_2(M^\varepsilon), \ \mathcal{H}_0 := L_2(\Omega) \times L_2(\Gamma)$.

Let Δ^ε be the Laplace-Beltrami operator on M^ε. It is well-known that the spectrum of the operator $-\Delta^\varepsilon$ is purely discrete. Let $0 = \lambda_1^\varepsilon < \lambda_2^\varepsilon \le \lambda_3^\varepsilon \le ... \le \lambda_k^\varepsilon \underset{k\to\infty}{\to} \infty$ be the eigenvalues of $-\Delta^\varepsilon$ written with account of their multiplicity, $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon ...$ be the corresponding eigenvectors normalized by the condition $(u_i^\varepsilon, u_j^\varepsilon)_{\mathcal{H}^\varepsilon} = \delta_{ij}$.

In this section we study the behavior of λ_k^ε as $\varepsilon \to 0$.

Let $\mathbf{L} : L_2(\Gamma) \to L_2(\Gamma)$ be the operator Laplace on a graph Γ with Dirichlet boundary conditions, i.e. $\mathbf{L}$ is defined by the operation

$$[\mathbf{L}u](x) = -\frac{d^2 u}{dz_{ij}}(x), \ x = p(z_{ij}) \in \gamma_{ij}$$

and the definitional domain that consists of the functions $u \in H^2(\gamma_{ij})\ \forall i, j$ and such that if we denote by u_{ij} the restriction of u on γ_{ij} then:

$$\text{for } i = \overline{1, N}: \ u(x_i) = 0,$$

$$\text{for } i = \overline{N+1, m}: \begin{cases} u_{ij}(p_i) \text{ are equivalent for all } j: A_{ij} = 1, \\ \sum\limits_{j: A_{ij}=1} \dfrac{\partial u_{ij}}{\partial \nu}(p_i) = 0, \end{cases}$$

where $\dfrac{\partial}{\partial \nu}$ means the derivative in the direction outward to γ_{ij}. In short, u is a continuous function on Γ that satisfies to the Dirichlet conditions on the ends of the graph and satisfies to the Kirchhoff conditions in the vertices (for more precise description of differential operators on the graphs and its properties see, e.g., [6]).

In order to describe the behavior of the eigenfunctions we introduce the operator $R^\varepsilon : \mathcal{H}_0 \to \mathcal{H}^\varepsilon$:

$$[R^\varepsilon f](\tilde{x}) = \begin{cases} f_0(\tilde{x}), \tilde{x} \in \Omega^\varepsilon, \\ f_{ij}(z_{ij})\varepsilon^{-1/2}, \tilde{x} = (z_{ij}, \varphi_{ij}) \in G^\varepsilon_{ij}, \\ 0, \tilde{x} \in B^\varepsilon_i, \end{cases} \quad f = (f_0, f_{ij}, i, j: A_{ij} = 1) \in L_2(\Omega) \times L_2(\Gamma).$$

Let $\mathcal{L} : \mathcal{H}_0 \to \mathcal{H}_0$:

$$\mathcal{L} = \begin{pmatrix} -\Delta_\Omega & 0 \\ 0 & \mathbf{L} \end{pmatrix}.$$

and let $\lambda_0, \lambda_1, \lambda_2 ...$ be the eigenvalues of $\mathcal{L}$ written with account of their multiplicity. It is clear that *the spectrum of $\mathcal{L}$ is the union of the eigenvalues of the operator $-\Delta_\Omega$ and the eigenvalues of the operator $\mathbf{L}$ that are taken with account of their multiplicity.*

Theorem 1.1. *For any k=1,2,3...*

$$\lambda^\varepsilon_k \to \lambda_k, \ \varepsilon \to 0.$$

Theorem 1.2. *Let $\lambda_k < \lambda_{k+1} = \lambda_{k+2} = ... = \lambda_{k+m} < \lambda_{k+m+1}$ (i.e. the multiplicity of λ_{k+1} is equal to m). Let $N(\lambda_{k+1})$ be the eigenspace of the eigenvalue λ_{k+1}. Then for any $w \in N(\lambda_{k+1})$ there exists the linear combination $\bar{u}^\varepsilon$ of the eigenfunctions $u^\varepsilon_{k+1} ... u^\varepsilon_{k+m}$ such that*

$$\|\bar{u}^\varepsilon - R^\varepsilon w\|_{\mathcal{H}^\varepsilon} \to 0, \ \varepsilon \to 0. \tag{1.1.1}$$

1.2 Proof of Theorems 1.1 and 1.2

We prove Theorems 1.1 and 1.2 for the case $N = 3, m = 4$, i.e. Γ^ε consists of three tubes $G^\varepsilon_{14}, G^\varepsilon_{24}, G^\varepsilon_{34}$ and the truncated sphere B^ε_4 that connect these tubes. For the general case the theorems are proved in a similar way.

We introduce the new notations:

$$l^\varepsilon_i := l^\varepsilon_{i4},\ z_i := z_{i4},\ \varphi_i := \varphi_{i4},\ G^\varepsilon_i := G^\varepsilon_{i4},$$
$$S^\varepsilon_j := S^\varepsilon_{4j},\ C^\varepsilon_i := S^\varepsilon_{i4},\ \mathcal{B}^\varepsilon := \mathcal{B}^\varepsilon_4,\ B^\varepsilon := B^\varepsilon_4,\ i, j = 1, 2, 3.$$

(i.e. $\partial\Gamma^\varepsilon = \bigcup\limits_{i=1,2,3} C^\varepsilon_i$).

For simplicity we suppose that metrics g is Euclidean in some neighbourhood of the holes D^ε_i (and thus g^ε is continuous). For the general case the proof needs small modifications.

We denote by $\mathcal{A}^\varepsilon$ and $\mathcal{A}_0$ the operators inverse to $-\Delta^\varepsilon + \mathrm{I}$ and $\mathcal{L} + \mathrm{I}$ correspondingly (I is the identical operator).

Now we study the behavior of $\mathcal{A}^\varepsilon$ as $\varepsilon \to 0$.

Theorem 1.3. *The following conditions are fulfilled:*

C1. *For any* $f \in \mathcal{H}_0$

$$\|R^\varepsilon f\|_{\mathcal{H}^\varepsilon} \to \|f\|_{\mathcal{H}_0}, \varepsilon \to 0. \tag{1.2.1}$$

C2. *Operators* $\mathcal{A}^\varepsilon, \mathcal{A}_0$ *are positive, compact, self-adjoint and bounded in* $\mathcal{L}(\mathcal{H}^\varepsilon)$ *uniformly with respect to* ε.

C3. *For any* $f \in \mathcal{H}_0$

$$\|\mathcal{A}^\varepsilon R^\varepsilon f - R^\varepsilon \mathcal{A}_0 f\|_{\mathcal{H}^\varepsilon} \to 0, \varepsilon \to 0. \tag{1.2.2}$$

C4. *For any sequence* $f^\varepsilon \in \mathcal{H}^\varepsilon$ *such that* $\sup \|f^\varepsilon\|_{\mathcal{H}^\varepsilon} < \infty$ *there exist the subsequence* ε' *and* $w \in \mathcal{H}_0$ *such that*

$$\|\mathcal{A}^\varepsilon f^\varepsilon - R^\varepsilon w\|_{\mathcal{H}^\varepsilon} \to 0, \varepsilon = \varepsilon' \to 0. \tag{1.2.3}$$

Proof **1.** Condition C1 is directly follows from the definition of the operator R^ε.

2. Condition C2 follows easily from the properties of the resolvent, namely the following estimate is valid

$$\|\mathcal{A}^\varepsilon\|_{\mathcal{L}(H^\varepsilon)} \leq 1.$$

3. Let $f \in \mathcal{H}$. We denote $u^\varepsilon = \mathcal{A}^\varepsilon R^\varepsilon f,\ f^\varepsilon = R^\varepsilon f$. In order to describe the behavior of u^ε on Ω^ε we introduce the operator $\Pi^\varepsilon_0 : H^1(M^\varepsilon) \to H^1(\Omega)$ with the following properties:

1) $\|\Pi^\varepsilon u^\varepsilon\|_{\mathcal{H}_0} + \|\nabla^\varepsilon \Pi^\varepsilon u^\varepsilon\|_{\mathcal{H}_0} \leq C\Big\{\|u^\varepsilon\|_{\mathcal{H}^\varepsilon} + \|\nabla^\varepsilon u^\varepsilon\|_{\mathcal{H}^\varepsilon}\Big\}$, $C > 0$,

2) $\Pi_0^\varepsilon u^\varepsilon(\tilde{x}) = u^\varepsilon(\tilde{x})$ on Ω^ε.

(Here $\|\nabla^\varepsilon u^\varepsilon\|_{\mathcal{H}^\varepsilon} := \int\limits_{M^\varepsilon} \sum\limits_{\alpha,\beta=1}^{2} g_\varepsilon^{\alpha\beta} \frac{\partial u}{\partial x_\alpha} \frac{\partial u}{\partial x_\beta} d\tilde{x}$, where $g_\varepsilon^{\alpha\beta}$ are the components of the tensor inverse to g^ε). Such operator exists, see, e.g, [3].

Due to C1-C2 we have $\|u^\varepsilon\|_{\mathcal{H}^\varepsilon} \leq \|f^\varepsilon\|_{\mathcal{H}^\varepsilon} \to \|f\|_{\mathcal{H}_0}$. Moreover using variational methods we obtain

$$\|\nabla^\varepsilon u^\varepsilon\|^2_{\mathcal{H}^\varepsilon} \leq 2\|f^\varepsilon\|_{\mathcal{H}^\varepsilon} \cdot \|u^\varepsilon\|_{\mathcal{H}^\varepsilon}.$$

Using these inequalities and the properties of the operator Π_0^ε we conclude that $\Pi_0^\varepsilon u^\varepsilon$ is bounded in $H^1(\Omega)$ uniformly with respect to ε and therefore there exists a subsequence (still denoted by ε) such that

$$\Pi_0^\varepsilon u^\varepsilon \underset{\varepsilon\to 0}{\to} u_0 \in H^1(\Omega) \text{ weakly in } H^1(\Omega) \text{ and strongly in } L_2(\Omega). \tag{1.2.4}$$

In order to describe the behavior of u^ε on the tubes $G_i^\varepsilon, i = 1, 2, 3$ we represent u^ε in the form

$$u^\varepsilon(\varphi_i, z_i) = P_i^\varepsilon u^\varepsilon(z_i) + Q_i^\varepsilon u^\varepsilon(\varphi_i, z_i) \tag{1.2.5}$$

where

$$P_i^\varepsilon u^\varepsilon(z_i) = \frac{1}{2\pi} \int\limits_0^{2\pi} u^\varepsilon(\varphi_i, z_i) d\varphi_i.$$

Let $\Pi_i^\varepsilon : H^1(G_i^\varepsilon) \to H^1([0, l_i])$ that defined by the formula

$$\Pi_i^\varepsilon u^\varepsilon(z_i) = \begin{cases} \sqrt{\varepsilon} P_i^\varepsilon(z_i), \ z_i \in [\delta^\varepsilon, l_i - \delta^\varepsilon], \\ \sqrt{\varepsilon} P_i^\varepsilon(\delta^\varepsilon), \ z_i \in [0, \delta^\varepsilon), \\ \sqrt{\varepsilon} P_i^\varepsilon(l_i - \delta^\varepsilon), \ z_i \in (l_i - \delta^\varepsilon, l_i]. \end{cases}$$

We have the following estimates:

$$\left\| \frac{d}{dz_i} \Pi_i^\varepsilon u^\varepsilon \right\|^2_{L_2[0,l_i]} = \int\limits_{\delta^\varepsilon}^{l_i - \delta^\varepsilon} \left(\frac{\partial}{\partial z_i} \frac{1}{2\pi} \int\limits_0^{2\pi} u^\varepsilon(\varphi_i, z_i) \sqrt{\varepsilon} d\varphi_i \right)^2 dz_i \leq$$

$$\leq \frac{1}{2\pi} \int\limits_{\delta^\varepsilon}^{l_i - \delta^\varepsilon} \int\limits_0^{2\pi} \left(\frac{\partial}{\partial z_i} u^\varepsilon(\varphi_i, z_i) \right)^2 \varepsilon d\varphi_i dz_i \leq (2\pi)^{-1} \|\nabla^\varepsilon u^\varepsilon\|^2_{0\varepsilon}, \tag{1.2.6}$$

$$\|\Pi_i^\varepsilon u^\varepsilon\|^2_{L_2[0,l_i]} \leq \delta^\varepsilon \Big[(\Pi_i^\varepsilon u^\varepsilon(\delta^\varepsilon))^2 + (\Pi_i^\varepsilon u^\varepsilon(l_i - \delta^\varepsilon))^2 \Big] + (2\pi)^{-1} \|u^\varepsilon\|^2_{L_2(G_i^\varepsilon)}. \tag{1.2.7}$$

Further

$$(\Pi_i^\varepsilon u^\varepsilon(\delta^\varepsilon))^2 \le 2\left((\Pi_i^\varepsilon u^\varepsilon(z_i))^2 + l_i \int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon} \left|\frac{\partial}{\partial z_i}\Pi_i^\varepsilon u^\varepsilon(\varphi_i, z_i)\right|^2 dz\right)$$

Integrating this estimate on z_i from δ^ε to $l_i - \delta^\varepsilon$ one has

$$(\Pi_i^\varepsilon u^\varepsilon(\delta^\varepsilon))^2 \le C\left(\|\Pi_i^\varepsilon u^\varepsilon\|^2_{L_2[\delta^\varepsilon, l_i-\delta^\varepsilon]} + \left\|\frac{d}{dz_i}\Pi_i^\varepsilon u^\varepsilon\right\|^2_{L_2[\delta^\varepsilon, l_i-\delta^\varepsilon]}\right) \qquad (1.2.8)$$

and similarly

$$(\Pi_i^\varepsilon u^\varepsilon(l_i-\delta^\varepsilon))^2 \le C\left(\|\Pi_i^\varepsilon u^\varepsilon\|^2_{L_2[\delta^\varepsilon, l_i-\delta^\varepsilon]} + \left\|\frac{d}{dz_i}\Pi_i^\varepsilon u^\varepsilon\right\|^2_{L_2[\delta^\varepsilon, l_i-\delta^\varepsilon]}\right) \qquad (1.2.9)$$

It follows from (1.2.6)-(1.2.9) that $\Pi_i u^\varepsilon$ is bounded in $H^1([0, l_i])$ and therefore for $i = 1, 2, 3$ there exists a subsequence (still denoted by ε) such that

$$\Pi_i^\varepsilon u^\varepsilon \underset{\varepsilon\to 0}{\to} u_i \in H^1([0, l_i]) \text{ weakly in } H^1([0, l_i]) \text{ and strongly in } L_2([0, l_i]). \qquad (1.2.10)$$

The following lemma say that u^ε is vanishingly small in B^ε.

Lemma 1.1. *Let $u^\varepsilon \in H^1(M^\varepsilon)$. Then*

$$\|u^\varepsilon\|^2_{L_2(B^\varepsilon)} \le C\Big(\varepsilon^2\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(B^\varepsilon)} + \varepsilon\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(\cup_i G_i^\varepsilon)} + \varepsilon^2|\ln\varepsilon|\big(\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} + \|u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)}\big)\Big).$$

Proof. At first we note, that u^ε can be extended to whole ball $\mathcal{B}^\varepsilon$ in such a way that $\|\nabla^\varepsilon u^\varepsilon\|_{L_2(\mathcal{B}^\varepsilon)} \le C\|\nabla^\varepsilon u^\varepsilon\|_{L_2(B^\varepsilon)}$ (see [9, p. 118, Example 4.10]). Let us fixed i from $\{1, 2, 3\}$. We introduce the spherical coordinates $\varphi \in [0, 2\pi], \theta \in [0, \pi]$ on $\mathcal{B}^\varepsilon$ such that the points of S_i^ε have the coordinates $\varphi \in [0, 2\pi], \theta = \arcsin(\varepsilon/b^\varepsilon) =: \theta^\varepsilon$.

So we extend u^ε to $\mathcal{B}^\varepsilon$ and have:

$$u^\varepsilon(\varphi, \theta) = u^\varepsilon(\varphi, \theta^\varepsilon) + \int\limits_{\theta^\varepsilon}^{\theta} \frac{\partial u^\varepsilon(\varphi, \psi)}{\partial\psi} d\psi.$$

Further

$$\int\limits_0^{2\pi}\int\limits_{\theta^\varepsilon}^{\pi-\theta^\varepsilon} (u^\varepsilon(\varphi, \theta))^2 b^{\varepsilon 2} \sin\theta d\theta d\varphi \le$$

$$\le C\varepsilon^2\left(\int\limits_{\theta^\varepsilon}^{\pi-\theta^\varepsilon}\left(\frac{\partial u^\varepsilon(\varphi, \psi)}{\partial\psi}\right)^2 \sin\psi d\psi d\varphi \cdot \int\limits_{\theta^\varepsilon}^{\pi-\theta^\varepsilon} (\sin\psi)^{-1} d\psi + \int\limits_0^{2\pi} (u^\varepsilon(\varphi, \theta^\varepsilon))^2 d\varphi\right). \qquad (1.2.11)$$

Since $C_1 \le \theta^\varepsilon \le C_2$ the first term is estimated by $C\varepsilon^2\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(B^\varepsilon)}$. Now we estimate the second term. Representing the corresponding integral in the cylindrical coordinates one has

$$u^\varepsilon(\varphi,\theta^\varepsilon) \equiv u^\varepsilon(\varphi_i,l_i) = u^\varepsilon(\varphi_i,0) + \int\limits_0^{l_i} \frac{\partial u^\varepsilon(\varphi_i,z_i)}{\partial z_i} dz_i$$

Let D^ε and R^ε be the balls in Ω with the radiuses d^ε and r^ε ($r^\varepsilon > d^\varepsilon$). Then for any $u \in H^1(R^\varepsilon \setminus D^\varepsilon)$ the following estimate is valid (see [?]):

$$\|u\|^2_{L_2(\partial D^\varepsilon)} \le C d^\varepsilon \left[|\ln d^\varepsilon| \cdot \|\nabla u\|^2_{L_2(R^\varepsilon\setminus D^\varepsilon)} + \frac{1}{(r^\varepsilon)^2}\|u\|^2_{L_2(R^\varepsilon\setminus D^\varepsilon)} \right]. \qquad (1.2.12)$$

Using (1.2.12) we have

$$\varepsilon^2 \int\limits_0^{2\pi} (u^\varepsilon(\varphi_i,l_i))^2 \, d\varphi_i \le C\varepsilon^2 \left[\left(\int\limits_0^{2\pi} u^\varepsilon(\varphi_i,0) d\varphi_i \right)^2 + \int\limits_0^{2\pi}\int\limits_0^{l_i} \left(\frac{\partial u^\varepsilon(\varphi_i,z_i)}{\partial z_i} \right)^2 dz_i \right] \le$$

$$\le C\Big[\varepsilon^2 |\ln\varepsilon| \left(\|u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} + \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} \right) + \varepsilon \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(G_i^\varepsilon)} \Big]$$

We denote $\mathcal{D}_i^{*\varepsilon} = \{(\varphi,\theta) \in \mathcal{B}^\varepsilon : \theta \in [\pi - \theta^\varepsilon, \theta^\varepsilon]\}$. It follows from (1.2.11) that for $i = 1,2,3$:

$$\|u^\varepsilon\|^2_{L_2(\mathcal{B}^\varepsilon\setminus(D_i^\varepsilon\cup D_i^{*\varepsilon}))} \le C\Big(\varepsilon^2 \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(B^\varepsilon)} + \varepsilon \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(G_i^\varepsilon)} + \varepsilon^2 |\ln\varepsilon| \big(\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} + \|u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} \big) \Big).$$

The Lemma is proved since $\bigcup\limits_{i=1,2,3} [\mathcal{B}^\varepsilon \setminus (D_i^\varepsilon \cup D_i^{*\varepsilon})] = \mathcal{B}^\varepsilon$.

We return to the proof of Theorem 1.3. We denote $u := (u_0,u_1,u_2,u_3)$. Let us prove that $u = \mathcal{A}_0 f$. It is easy to see that this is equal to the fulfilment of the following conditions:

$$\textbf{I}.\ u_i(0) = 0, i = 1,2,3, \qquad (1.2.13)$$

$$\textbf{II}.\ u_1(l_1) = u_2(l_2) = u_1(l_3), \qquad (1.2.14)$$

$$\textbf{III}.\ (\nabla u_0, \nabla w)_{\mathcal{H}_0} + (u_0, w)_{\mathcal{H}_0} = (f_0, w)_{\mathcal{H}_0},\ \forall w \in H^1(\Omega), \qquad (1.2.15)$$

$$\textbf{IV}.\ \sum_{i=1}^3 \int\limits_0^{l_i} (u_i(z))'(w_i(z))' dz + \sum_{i=1}^3 \int\limits_0^{l_i} u_i(z) w_i(z) dz =$$

$$= \sum_{i=1}^3 \int\limits_0^{l_i} f_i(z) w_i(z) dz,\ \forall w_i \in H^1[0,l_i]. \qquad (1.2.16)$$

Let us verify the fulfilment of these conditions.

I. Using trace theorem we have for $i = 1, 2, 3$

$$u_i(0) = \lim_{\varepsilon\to 0} \Pi_i^\varepsilon u^\varepsilon(0) = \lim_{\varepsilon\to 0} \Pi_i^\varepsilon u^\varepsilon(\delta^\varepsilon) = \sqrt{\varepsilon} \lim_{\varepsilon\to 0} \bar{u}_i^\varepsilon. \tag{1.2.17}$$

It follows from (1.2.12)-(1.2.17) that (1.2.13) fulfils.

II. One has for $i, j = 1, 2, 3$:

$$|u_i(l_i) - u_j(l_j)| = \lim_{\varepsilon\to 0} \sqrt{\varepsilon} |\hat{u}_i^\varepsilon - \hat{u}_j^\varepsilon|, \tag{1.2.18}$$

where $\hat{u}_i^\varepsilon$ is the average value of u^ε over the circle S_i^ε.

We denote $v^\varepsilon(\tilde{x}) := u^\varepsilon(\tilde{x}) - U^\varepsilon$, where U^ε is the average value of u^ε over B^ε and we denote by $\hat{v}_i^\varepsilon$ the average value of v^ε over the circle S_i^ε.

Using the inequality of the type (1.2.12) and Poincare's inequality one has

$$|v_i^\varepsilon|^2 \leq C \left[\left| \ln \tan \frac{\theta^\varepsilon}{2} \right| \cdot \|\nabla^\varepsilon v^\varepsilon\|^2_{L_2(B_i^\varepsilon)} + \frac{1}{(b_i^\varepsilon)^2} \|v^\varepsilon\|^2_{L_2(B_i^\varepsilon)} \right] \leq \tag{1.2.19}$$

$$\leq C \left| \ln \tan \frac{\theta^\varepsilon}{2} \right| \cdot \|\nabla^\varepsilon v^\varepsilon\|^2_{L_2(B_i^\varepsilon)}. \tag{1.2.20}$$

Using (1.2.19) we have:

$$|\hat{u}_i^\varepsilon - \hat{u}_j^\varepsilon| = |\hat{v}_i^\varepsilon - \hat{v}_j^\varepsilon| \leq |\hat{v}_i^\varepsilon| + |\hat{v}_j^\varepsilon| \leq C \sqrt{\left| \ln \tan \frac{\theta^\varepsilon}{2} \right|} \cdot \|\nabla^\varepsilon v^\varepsilon\|_{L_2(B^\varepsilon)}. \tag{1.2.21}$$

Since $\left| \ln \tan \frac{\theta^\varepsilon}{2} \right| < C$ it follows from (1.2.18), (1.2.21) that (1.2.14) fulfils.

III. Clearly it is sufficient to prove (1.2.15) for w such that

$$\exists \delta > 0 \ \forall i = 1 ... N : \ \rho_g(\text{supp}(w), x_i^\varepsilon) \geq \delta,$$

where ρ_g is the distance on Ω generated by the metric g (because the set of such w is dense in $H^1(\Omega)$). Then for such w and for sufficiently small ε $\text{supp}(w) \subset \Omega^\varepsilon$. Let $w^\varepsilon(\tilde{x}) \in L_2(M^\varepsilon)$: $w^\varepsilon(\tilde{x}) = w(\tilde{x})$ in Ω^ε and $w^\varepsilon = 0$ in $M^\varepsilon \setminus \Omega^\varepsilon$. Clearly $w^\varepsilon \in H^2(M^\varepsilon)$ for ε small enough.

We have

$$0 = \lim_{\varepsilon\to 0} \Big((\nabla^\varepsilon u^\varepsilon, \nabla^\varepsilon w^\varepsilon)_{\mathcal{H}^\varepsilon} + (u^\varepsilon, w^\varepsilon)_{\mathcal{H}^\varepsilon} - (f^\varepsilon, w^\varepsilon)_{\mathcal{H}^\varepsilon} \Big) =$$

$$= \lim_{\varepsilon\to 0} \Big((\nabla \Pi^\varepsilon u^\varepsilon, \nabla w)_{L_2(\Omega)} + (\Pi^\varepsilon u^\varepsilon, w)_{L_2(\Omega)} - (f, w)_{L_2(\Omega)} \Big) =$$

$$= (\nabla u_0, \nabla w)_{\mathcal{H}_0} + (u_0, w)_{\mathcal{H}_0} - (f_0, w)_{\mathcal{H}_0}.$$

and (1.2.15) fulfils.

IV. It is clear that it is sufficient to prove (1.2.16) for such w_i that

$$\exists\delta>0\ \forall z\in[0,\delta]:\ w_i(z)=0 \text{ and } \forall z\in[l_i-\delta,l_i]:\ w_i(z)=w_i(l_i),$$

because the set of such w_i is dense in the set of test functions mentioned above. For sufficiently small ε $\delta^\varepsilon\le\delta$ and therefore such w_i satisfies to the conditions: $\forall z\in[0,\delta^\varepsilon]:\ w_i(z)=0$ and $\forall z\in[l_i-\delta^\varepsilon,l_i]:\ w_i(z)=w_i(l_i)$.

At first, let us estimate the reminder $Q^\varepsilon u^\varepsilon$ on G_i^ε. Using Poincare's inequality we have:

$$\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\int\limits_0^{2\pi}(Q_i^\varepsilon u^\varepsilon(\varphi_i,z_i))^2d\varphi_i dz_i\le C\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\int\limits_0^{2\pi}\left(\frac{\partial Q_i^\varepsilon u^\varepsilon}{\partial\varphi_i}\right)^2d\varphi_i dz_i=$$
$$=C\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\int\limits_0^{2\pi}\left(\frac{\partial u^\varepsilon}{\partial\varphi_i}\right)^2d\varphi dz_i\le C\varepsilon\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(G_i^\varepsilon)}. \tag{1.2.22}$$

Using this and representation (1.2.5) we have

$$\sum_{i=1}^3\int\limits_0^{l_i}(u_i(z_i))'(w_i(z_i))'dz_i=\sum_{i=1}^3\lim_{\varepsilon\to0}\frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\frac{\partial\Pi_i^\varepsilon u^\varepsilon}{\partial z_i}\frac{\partial w_i^\varepsilon}{\partial z_i}dz_i d\varphi_i=$$
$$=\sum_{i=1}^3\lim_{\varepsilon\to0}\frac{1}{2\pi}\left[\int\limits_0^{2\pi}\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\frac{\partial u^\varepsilon}{\partial z_i}\cdot\frac{\partial}{\partial z_i}\left(\frac{w_i}{\sqrt{\varepsilon}}\right)\varepsilon dz_i d\varphi_i+\int\limits_0^{2\pi}\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\sqrt{\varepsilon}Q_i^\varepsilon u^\varepsilon\cdot\frac{\partial^2 w_i}{dz_i^2}dz_i d\varphi_i\right]$$

In view of (1.2.22) the second integral tends to zero. Therefore we have

$$\sum_{i=1}^3\int\limits_0^{l_i}(u_i(z))'(w_i(z))'dz=\frac{1}{2\pi}\lim_{\varepsilon\to0}(\nabla^\varepsilon u^\varepsilon,\nabla w^\varepsilon)_{\mathcal{H}^\varepsilon},$$

where $w^\varepsilon\in H^1(M^\varepsilon)$

$$w^\varepsilon(x)=\begin{cases}w_i(z_i)\varepsilon^{-1/2},\ \tilde{x}=(z_i,\varphi_i)\in G_i,\\ 0,\ \tilde{x}\in\Omega^\varepsilon,\\ w_i(\delta^\varepsilon)\varepsilon^{-1/2},\ \tilde{x}\in B^\varepsilon.\end{cases}$$

In the same way we obtain

$$\sum_{i=1}^3\left(\int\limits_0^{l_i}u_i(z)w_i(z)dz-\int\limits_0^{l_i}f_i(z)w_i(z)dz\right)=\frac{1}{2\pi}\lim_{\varepsilon\to0}\Big((u^\varepsilon,w^\varepsilon)_{\mathcal{H}^\varepsilon}-(f^\varepsilon,w^\varepsilon)_{\mathcal{H}^\varepsilon}\Big).$$

The proof is similar to the proof of (1.2.24). The last two equalities imply the condition(1.2.16).

Thus we prove that $u = \mathcal{A}_0 f$.

It is easy to see that (1.2.2) follows from (1.2.4),(1.2.10),(1.2.22) and Lemma 1.1. The condition C3 is fulfilled.

4. It remains to verify the fulfilment of the condition C4. Let $f^\varepsilon \in \mathcal{H}^\varepsilon$ be such that $\sup \|f^\varepsilon\|_{\mathcal{H}^\varepsilon} < \infty$. We denote $u^\varepsilon = \mathcal{A}^\varepsilon f^\varepsilon$. It is clear that the norms $\|u^\varepsilon\|^2_{L_2(M^\varepsilon)} + \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(M^\varepsilon)}$ are uniformly bounded with respect to ε. In the same way as in the item **3** it can be proved that there exists a subsequence (still denoted by ε) such that the following limits exist

$$w_0 = \lim_{\varepsilon\to 0} \Pi_0^\varepsilon u^\varepsilon \in H^1(\Omega) \text{ strongly in } L_2(\Omega), \tag{1.2.23}$$

$$w_i = \lim_{\varepsilon\to 0} \Pi_i^\varepsilon u^\varepsilon \in H^1[0,l_i], i = 1,2,3 \text{ strongly in } L_2[0,l_i]. \tag{1.2.24}$$

By means Lemma 1.1 we have

$$\|u^\varepsilon\|^2_{L_2(B^\varepsilon)} \to 0, \varepsilon \to 0. \tag{1.2.25}$$

It is easy to see that the fulfilment of the condition C4 (with $w = (w_0, w_1, w_2, w_3)$) follows from (1.2.22)-(1.2.25).

Theorem 1.3 is proved.

We continue the proof of Theorems 1.1 and 1.2. Let $\mu_1^\varepsilon \geq \mu_2^\varepsilon \geq \mu_3^\varepsilon \geq ... \geq \mu_k^\varepsilon \underset{k\to\infty}{\to} 0$ be the eigenvalues of $\mathcal{A}^\varepsilon$ written with account of their multiplicity and let $f_1^\varepsilon, f_2^\varepsilon ...$ be the corresponding eigenvectors normalized by the condition $(f_i^\varepsilon, f_j^\varepsilon)_{\mathcal{H}^\varepsilon} = \delta_{ij}$. Let $\mu_1 \geq \mu_2 \geq \mu_3 \geq ... \geq \mu_k \underset{k\to\infty}{\to} 0$ be the eigenvalues of $\mathcal{A}$.

It is proved in [7] that the conditions C1-C4 imply

$$\mu_k^\varepsilon \to \mu_k,\ \varepsilon \to 0,\ k = 1,2,3...$$

and moreover if $\mu_k \geq \mu_{k+1} = \mu_{k+2} = ... = \mu_{k+m} > \mu_{k+m+1}$ then for any $w \in N(\mu_{k+1})$ such that $\|w\|_{\mathcal{H}_0} = 1$ there exists a linear combination $\bar{f}^\varepsilon$ of the eigenfunctions $f_{k+1}^\varepsilon ... f_{k+m}^\varepsilon$ such that

$$\|\bar{f}^\varepsilon - R^\varepsilon w\|_{\mathcal{H}^\varepsilon} \to 0,\ \varepsilon \to 0.$$

Since $\lambda_k^\varepsilon = \frac{1}{\mu_k^\varepsilon} - 1$, $\lambda_k = \frac{1}{\mu_k} - 1$, $u_k^\varepsilon = f_k^\varepsilon$ (and so $N(\lambda_k) = N(\mu_k)$) it follows that Theorems 1.1 and 1.2 are proved.

2 Riemannian manifold of increasing genus

2.1 Setting of the problem and main result

Let Ω be a 2-dimensional compact Riemannian manifold without a boundary and with a metrics g. By Δ_Ω we denote the corresponding Laplace-Beltrami operator. Let $D_i^\varepsilon, i = 1...N(\varepsilon) = 3N_1(\varepsilon)$ be the system of balls in Ω with the centers $x_i^\varepsilon \in \Omega$ and the radii d^ε. We consider the following domain with holes

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{i=1}^{N(\varepsilon)} D_i^\varepsilon.$$

Let G_i^ε, $i = 1...N(\varepsilon)$ be the set of tubes

$$G_i^\varepsilon = \{\tilde{x} = (\varphi_i, z_i) : \varphi_i \in [0, 2\pi], z_i \in [0, 1]\}.$$

We suppose that

$$C_i^\varepsilon = \{\tilde{x} = (\varphi_i, z_i) \in G_i^\varepsilon : z_i = 0\} \subset \partial G_i^\varepsilon$$

is diffeomorphic to ∂D_i^ε, using this diffeomorphism we glue $G_i^\varepsilon, i = 1...N(\varepsilon)$ to Ω^ε. By S_i^ε we denote the "ends" of G_i^ε:

$$S_i^\varepsilon = \{\tilde{x} = (\varphi_i, z_i) \in G_i^\varepsilon : z_i = 1\} \subset \partial G_i^\varepsilon.$$

We divide the set $\{1...N(\varepsilon)\}$ on subsets that consist of 3 elements. For each three indexes i, j, k we introduce the number A_{ijk} and set $A_{ijk} = 1$ if i, j, k lie on one subset and we set $A_{ijk} = 0$ otherwise. If $A_{ijk} = 1$ we say that the corresponding holes $D_i^\varepsilon, D_j^\varepsilon, D_k^\varepsilon$ are connected.

For any $i, j, k : A_{ijk} = 1$ we consider the sphere $\mathcal{B}_{ijk}^\varepsilon \subset \mathbb{R}^3$ of radius b^ε. Let $\mathcal{D}_i^\varepsilon, \mathcal{D}_j^\varepsilon, \mathcal{D}_k^\varepsilon$ be the geodesic balls on $\mathcal{B}_{ijk}^\varepsilon$ of the radii $b^\varepsilon \arcsin\left(\dfrac{d^\varepsilon}{b^\varepsilon}\right)$. It is clear that the radii of the circles $\partial D_i^\varepsilon, \partial D_j^\varepsilon, \partial D_k^\varepsilon$ are equal to d^ε. Let

$$B_{ijk}^\varepsilon = \mathcal{B}_{ijk}^\varepsilon \setminus (\mathcal{D}_i^\varepsilon \cup \mathcal{D}_j^\varepsilon \cup \mathcal{D}_k^\varepsilon).$$

It is clear that $\partial D_i^\varepsilon, \partial D_j^\varepsilon, \partial D_k^\varepsilon$ are diffeomorphic to $S_i^\varepsilon, S_j^\varepsilon, S_k^\varepsilon$ correspondingly. Using these diffeomorphisms we glue B_{ijk}^ε to $G_i^\varepsilon \cup G_j^\varepsilon \cup G_k^\varepsilon$. Thus we obtain the manifold (see Fig.2)

$$M^\varepsilon = \Omega^\varepsilon \cup \left[\bigcup_{i,j,k: A_{ijk}=1} (B_{ijk}^\varepsilon \cup G_i^\varepsilon \cup G_j^\varepsilon \cup G_k^\varepsilon) \right].$$

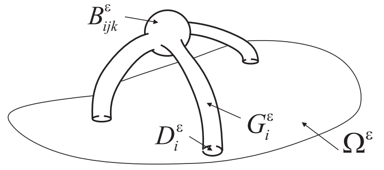

Figure 2: Manifold M^ε.

We denote by $\tilde{x}$ the points of this manifold. Clearly M^ε can be covered by a system of charts and suitable local coordinates $\{x_1, x_2\}$ can be introduced. It is supposed that M^ε is equipped by the metrics g^ε that coincides with the metrics g on Ω^ε, coincides with the metrics induced from $\mathbb{R}^3$ on B^ε_{ijk}, and on G^ε_i the metrics is defined by the formula for the square of the element of length:

$$ds^2 = q^\varepsilon_i dz_i^2 + (d^\varepsilon)^2 d\varphi_i^2,\ q^\varepsilon_i > 0.$$

By $g^\varepsilon_{\alpha\beta}$ we denote the components of the metric tensor in local coordinates.

We denote $r^\varepsilon_i = \min\limits_j \rho_g(x^\varepsilon_i, x^\varepsilon_j)$, where ρ_g denote the distance on Ω generated by the metrics g. It is supposed that the following properties are valid

(i) $|\ln d^\varepsilon|^{-1} \leq C(r^\varepsilon_i)^2$, $r^\varepsilon_i = O(\varepsilon)$,

(ii) $q^\varepsilon_i \leq q^\varepsilon \to 0,\ \varepsilon \to 0$, i.e. the lengthes of the cylinders G^ε_i tend to zero.

(iii) $(b^\varepsilon)^2 \left(|\ln d^\varepsilon| + \left|\ln \tan \frac{\theta^\varepsilon}{2}\right| + \dfrac{\sqrt{q^\varepsilon}}{d^\varepsilon} \right) \to 0,\ \varepsilon \to 0,\ \theta^\varepsilon = \arcsin \frac{d^\varepsilon}{b^\varepsilon}$.

Let Δ^ε be the Laplace-Beltrami operator on M^ε. Let $0 = \lambda^\varepsilon_1 < \lambda^\varepsilon_2 \leq \lambda^\varepsilon_3 \leq \ldots \leq \lambda^\varepsilon_k \underset{k\to\infty}{\to} \infty$ be the eigenvalues of $-\Delta^\varepsilon$ written with account of their multiplicity, $u^\varepsilon_1, u^\varepsilon_2, u^\varepsilon_3 \ldots$ be the corresponding eigenvectors normalized by the condition $(u^\varepsilon_i, u^\varepsilon_j)_{\mathcal{H}^\varepsilon} = \delta_{ij}$.

In order to describe the behavior of λ^ε_k as $\varepsilon \to 0$ we introduce the following notations

$$R^\varepsilon_i = \{\tilde{x} \in \Omega^\varepsilon : d^\varepsilon \leq \rho_g(\tilde{x}, x^\varepsilon_i) \leq r^\varepsilon_i/2\},\ \widehat{C}^\varepsilon_i = \{\tilde{x} \in \Omega^\varepsilon : \rho_g(\tilde{x}, x^\varepsilon_i) = r^\varepsilon_i/2\},$$
$$\Gamma^\varepsilon_{ijk} = G^\varepsilon_i \cup G^\varepsilon_j \cup G^\varepsilon_k \cup B^\varepsilon_{ijk},\ \widehat{\Gamma}^\varepsilon_{ijk} = R^\varepsilon_i \cup R^\varepsilon_j \cup R^\varepsilon_k \cup \Gamma^\varepsilon_{ijk}.$$

We consider the following problem for $i, j, k : A_{ijk} = 1$

$$\Delta^\varepsilon v = 0 \text{ in } \widehat{\Gamma}^\varepsilon_{ijk}, \quad v = 1 \text{ on } \widehat{C}^\varepsilon_i \text{ and } v = 0 \text{ on } \widehat{C}^\varepsilon_j \cup \widehat{C}^\varepsilon_k. \tag{2.1.1}$$

We denote by v^ε_{ijk} the solution of (2.1.1). It is clear that $v^\varepsilon_{ijk} = v^\varepsilon_{ikj}$.

For $i, j, k : A_{ijk} = 1$ we denote

$$W^{\varepsilon}_{ijk} = -\int\limits_{\widehat{\Gamma}^{\varepsilon}_{ijk}} (\nabla^{\varepsilon} v^{\varepsilon}_{ijk}, \nabla^{\varepsilon} v^{\varepsilon}_{jik}) d\tilde{x},$$

(here $(\nabla^{\varepsilon} u, \nabla^{\varepsilon} v) := \sum\limits_{\alpha,\beta=1}^{2} g^{\alpha\beta}_{\varepsilon} \frac{\partial u}{\partial x_{\alpha}} \frac{\partial v}{\partial x_{\beta}}$) otherwise we set $W^{\varepsilon}_{ijk} = 0$ (i.e. $W^{\varepsilon}_{ijk} = -(\nabla^{\varepsilon} v^{\varepsilon}_{ijk}, \nabla^{\varepsilon} v^{\varepsilon}_{jik})_{L_2(\widehat{\Gamma}^{\varepsilon}_{ijk})}$).

We introduce the generalized function

$$W^{\varepsilon}(x, y) = \sum_{i,j,k=1\ldots N(\varepsilon)} W^{\varepsilon}_{ijk} \delta(x - x^{\varepsilon}_{i}) \delta(y - x^{\varepsilon}_{j}) \in \mathcal{D}'(\Omega \times \Omega).$$

We suppose that the following limit exists

(iv) $\exists \lim\limits_{\varepsilon \to 0} W^{\varepsilon}(x, y) = W(x, y) \in L_{\infty}(\Omega \times \Omega)$ - positive symmetric function.

We denote $\mathcal{H}^{\varepsilon} := L_2(M^{\varepsilon})$, $\mathcal{H}_0 := L_2(\Omega)$.

Theorem 2.1. *For any k=1,2,3...*

$$\lambda^{\varepsilon}_{k} \to \lambda_{k}, \ \varepsilon \to 0,$$

where $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ *be the eigenvalues of the operator* $\mathcal{L} : L_2(\Omega) \to L_2(\Omega)$*:*

$$[\mathcal{L}u](x) = -\Delta_{\Omega} u(x) + \int\limits_{\Omega} W(x, y)(u(x) - u(y)) dy.$$

Theorem 2.2.*Let* $R^{\varepsilon} : \mathcal{H}_0 \to \mathcal{H}^{\varepsilon}$*:*

$$[R^{\varepsilon} f](\tilde{x}) = \begin{cases} f(\tilde{x}), & \tilde{x} \in \Omega^{\varepsilon}, \\ 0, & \tilde{x} \in \bigcup\limits_{i,j,k:A_{ijk}=1} \Gamma^{\varepsilon}_{ijk}. \end{cases}$$

Then the eigenfunctions of $-\Delta^{\varepsilon}$ *converges in the sense (1.1.1) to the eigenfunctions of the operator* $\mathcal{L}$.

2.2 Proof of Theorems 2.1 and 2.2

We denote by $\mathcal{A}^{\varepsilon}$ and $\mathcal{A}$ the operators inverse to $-\Delta^{\varepsilon}$+I and $\mathcal{L}$+I correspondingly. Just as in the previous section Theorems 2.1,2.2 follow from

Theorem 2.3.*Conditions C1-C4 are fulfilled.*

Proof. Conditions C1-C2 are trivial. Let us check the condition C3.

Let $f \in \mathcal{H}$. We denote $u^\varepsilon = \mathcal{A}^\varepsilon R^\varepsilon f$, $f^\varepsilon = R^\varepsilon f$, $u_0 = \mathcal{A}_0 f$. Note that the following estimates are valid

$$\|u^\varepsilon\|_{L_2(M^\varepsilon)} \leq \|f^\varepsilon\|_{L_2(M^\varepsilon)}, \ \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(M^\varepsilon)} \leq 2\|f^\varepsilon\|_{L_2(M^\varepsilon)} \cdot \|u^\varepsilon\|_{L_2(M^\varepsilon)}. \quad (2.2.1)$$

It is well known, that u^ε minimizes the functional

$$J^\varepsilon[u^\varepsilon] = \int\limits_{M^\varepsilon} \left(|\nabla^\varepsilon u^\varepsilon|^2 + (u^\varepsilon)^2 - 2f^\varepsilon u^\varepsilon\right) d\tilde{x} \quad (2.2.2)$$

in the class of functions $H^1(M^\varepsilon)$, while u_0 minimizes the functional

$$J_0[u] = \int\limits_\Omega \left(|\nabla u|^2 + u^2 - 2fu\right) dx + \int\limits_\Omega \int\limits_\Omega \frac{1}{2} W(x,y) \left(u(x) - u(y)\right)^2 dxdy \quad (2.2.3)$$

in the class $H^1(\Omega)$. The converse assertions are also true.

In order to prove that u^ε converges to u_0 we consider the following abstract scheme.

Let H^ε be a Hilbert space depending on the parameter $\varepsilon > 0$, $(u^\varepsilon, v^\varepsilon)_\varepsilon$, $\|u^\varepsilon\|_\varepsilon$ be the scalar product and norm in this space, F^ε be continuous linear functionals in H^ε which are uniformly bounded with respect to ε. Let H be a Hilbert space with the scalar product (u, v) and norm $\|u\|$, F be a continuous linear functional in H.

Consider the following two problem of minimization:

$$\|u^\varepsilon\|^2_\varepsilon + F^\varepsilon[u^\varepsilon] \to \inf, \quad u^\varepsilon \in H^\varepsilon \quad (2.2.4)$$

$$\|u\|^2 + F[u] \to \inf, \quad u \in H. \quad (2.2.5)$$

Let u^ε and u_0 be the minimizants of the problems (2.2.4) and (2.2.5). The following theorem is proved in [3].

Theorem 2.4. Let M be a dense subset of H, and let $\Pi^\varepsilon : H^\varepsilon \to H$ and $P^\varepsilon : M \to H^\varepsilon$ be operators satisfying the following conditions:

(a) $\|\Pi^\varepsilon w^\varepsilon\| \leq C\|w^\varepsilon\|, \forall w^\varepsilon \in H^\varepsilon$;

(b_1) $\Pi^\varepsilon P^\varepsilon w \to w$ weakly in H as $\varepsilon \to 0, \forall w \in M$;

(b_2) $\lim\limits_{\varepsilon \to 0} \|P^\varepsilon w\|_\varepsilon = \|w\|, \forall w \in M$;

(b_3) for any sequence $\gamma^\varepsilon \in H^\varepsilon$ such that $\Pi^\varepsilon \gamma^\varepsilon \to \gamma$ weakly as $\varepsilon \to 0$, for any $w \in M$ one has:

$$\lim_{\varepsilon \to 0} |(P^\varepsilon w, \gamma^\varepsilon)_\varepsilon| \leq C\|w\|\|\gamma\|;$$

(c) for any sequence $\gamma^\varepsilon \in H^\varepsilon$, such that $\Pi^\varepsilon \gamma^\varepsilon \to \gamma$ weakly as $\varepsilon \to 0$ we have

$$\lim_{\varepsilon \to 0} F^\varepsilon[\gamma^\varepsilon] = F[\gamma].$$

Then

$$\Pi^\varepsilon u^\varepsilon \underset{\varepsilon\to 0}{\to} u_0 \text{ weakly in } H.$$

Note, that Theorem 2.4 holds true if conditions (b_3) and (c) hold only for such sequences γ^ε that the norms $\|\gamma^\varepsilon\|_\varepsilon$ are uniformly bounded with respect to ε (because, in the proof of Theorem 2.4 conditions (b_3) and (c) are used only with such sequences).

We now realize our abstract scheme. Let H^ε be the Hilbert space $H^1(M^\varepsilon)$ of the functions on M^ε with the scalar product

$$(u^\varepsilon, v^\varepsilon)_\varepsilon = \int\limits_{M^\varepsilon} \left[(\nabla^\varepsilon u^\varepsilon, \nabla^\varepsilon v^\varepsilon) + u^\varepsilon v^\varepsilon\right] d\tilde{x},$$

and let F^ε be the linear functional defined by the formula

$$F^\varepsilon[u^\varepsilon] = \int\limits_{M^\varepsilon} -2 f^\varepsilon u^\varepsilon d\tilde{x}$$

Let H be the Hilbert space $H^1(\Omega)$ with scalar product

$$(u, v) = \int\limits_\Omega \left[(\nabla u, \nabla v) + uv\right] dx + \int\limits_\Omega \int\limits_\Omega \frac{1}{2} W(x,y)(u(x) - u(y))(v(x) - v(y)) dx dy$$

and f be the linear functional on it defined by the formula

$$F[u] = \int\limits_\Omega -2 f u dx.$$

It is clear that functionals F^ε are uniformly bounded with respect to ε.

Now we introduce operators Π^ε and P^ε satisfying conditions (a)-(c) of Theorem 3.

The existence of the operator $\Pi^\varepsilon : H^1(M^\varepsilon) \to H^1(\Omega)$ that have the properties

$$\|\nabla \Pi^\varepsilon u^\varepsilon\|^2_{L_2(\Omega)} + \|\Pi^\varepsilon u^\varepsilon\|^2_{L_2(\Omega)} \le C\bigg(\|\nabla u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} + \|u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)}\bigg), \tag{2.2.6}$$

$$\Pi^\varepsilon u^\varepsilon(\tilde{x}) = u^\varepsilon(\tilde{x}),\ \tilde{x} \in \Omega^\varepsilon \tag{2.2.7}$$

is proved in [9, p. 118, Example 4.10]).

Clearly (a) follows from (2.2.6).

We introduce operator P^ε. Let $\varphi(r)$ be a twice continuously differentiable non-negative function on the half-line $[0,\infty)$, equal to 1 for $r \in [0,1/4]$ and equal to 0 for $r \geq 1/2$. We set

$$\varphi_i^\varepsilon(x) = \varphi\left(\frac{\rho_g(x,x_i^\varepsilon)}{r_i^\varepsilon}\right), \varphi_{0i}^\varepsilon(x) = \varphi\left(\frac{\rho_g(x,x_i^\varepsilon)}{d_0^\varepsilon}\right).$$

where $d_{0i}^\varepsilon = \exp(-|\ln d^\varepsilon|^{1/2})$.

Let $M = C^2(\Omega)$, M is dense in $H^1(\Omega)$. Let $w \in M$. We define the operator P^ε by the equality:

$$[P^\varepsilon w](\tilde{x}) = \begin{cases} w(\tilde{x}) + (w_i^\varepsilon - w(\tilde{x}))\varphi_{0i}^\varepsilon(\tilde{x}) + \Big((v_{ijk}^\varepsilon(\tilde{x}) - 1)w_i^\varepsilon + \\ \quad v_{jik}^\varepsilon(\tilde{x})w_j^\varepsilon + v_{kij}^\varepsilon(\tilde{x})w_k^\varepsilon\Big)\varphi_i^\varepsilon(\tilde{x}),\ \tilde{x} \in R_i^\varepsilon,\ j,k : A_{ijk} = 1, \\ v_{ijk}^\varepsilon(\tilde{x})w_i^\varepsilon + v_{jik}^\varepsilon(\tilde{x})w_j^\varepsilon + v_{kij}^\varepsilon(\tilde{x})w_k^\varepsilon,\ \tilde{x} \in \Gamma_{ijk}^\varepsilon, \end{cases}$$

where $w_i^\varepsilon = w(x_i^\varepsilon)$.

To see that conditions (b$_1$)-(b$_3$) holds we use the following estimates of the solution v_{ijk}^ε to the problem (2.1.1).

Lemma 2.1 Let $R_{0q}^\varepsilon = \{\tilde{x} \in \Omega^\varepsilon :\ d_{0q}^\varepsilon \leq \rho_g(x,x_q^\varepsilon) \leq r_q^\varepsilon/2\}$.Then for $i,j,k : A_{ijk} = 1$ and $q \in \{i,j,k\}$:

$$|D^\alpha(v_{ijk}^\varepsilon(\tilde{x}) - \delta_{iq})| \leq C\left|\frac{D^\alpha(\ln \rho_g(x_q^\varepsilon,\tilde{x}))}{\ln d^\varepsilon}\right|,\ \tilde{x} \in R_{0q}^\varepsilon\ |\alpha| = 0,1.$$

The proof of this lemma is carried out in the same way as that of Lemma 2.4 in [9, p.44] using the inequality $0 \leq v_{ijk}^\varepsilon \leq 1$, which follows from the maximum principle.

Lemma 2.2 Let $u^\varepsilon \in H^1(M^\varepsilon)$. Then for any $i,j,k : A_{ijk} = 1$

$$\|u^\varepsilon\|_{B_{ijk}^\varepsilon}^2 \leq C\left[(b^\varepsilon)^2\left|\ln\tan\frac{\theta^\varepsilon}{2}\right| \cdot \|\nabla^\varepsilon u^\varepsilon\|_{L_2(B_{ijk}^\varepsilon)}^2 + \frac{\sqrt{q^\varepsilon}(b^\varepsilon)^2}{d^\varepsilon}\|\nabla^\varepsilon u^\varepsilon\|_{L_2(G_i^\varepsilon\cup G_j^\varepsilon\cup G_k^\varepsilon)}^2 + \right.$$
$$\left. +(b^\varepsilon)^2\left(|\ln d^\varepsilon| \cdot \|\nabla^\varepsilon u^\varepsilon\|_{L_2(R_i^\varepsilon\cup R_j^\varepsilon\cup R_k^\varepsilon)}^2 + \frac{1}{r_i^{\varepsilon 2}}\|u^\varepsilon\|_{L_2(R_i^\varepsilon\cup R_j^\varepsilon\cup R_k^\varepsilon)}^2\right)\right],$$
$$\|u^\varepsilon\|_{G_i^\varepsilon}^2 \leq C\left[q^\varepsilon\|\nabla^\varepsilon u^\varepsilon\|_{L_2(G_i)}^2 + d^\varepsilon\sqrt{q^\varepsilon}\big(|\ln d^\varepsilon| \cdot \|\nabla^\varepsilon u^\varepsilon\|_{L_2(R_i^\varepsilon)}^2 + \frac{1}{r_i^{\varepsilon 2}}\|u^\varepsilon\|_{L_2(R_i^\varepsilon)}\big)\right].$$

The proof of this lemma is carried out in the same way as the proof of Lemma 1.1.

We verify that condition (b$_2$) holds. We denote $\widehat{R}_i^\varepsilon = \{\tilde x \in \Omega^\varepsilon : r_i^\varepsilon/4 \le \rho_g(\tilde x, x_i^\varepsilon) \le r_i^\varepsilon/2\}$. Let $w \in M$. Then

$$\begin{aligned}\|P^\varepsilon w\|_\varepsilon^2 = \int\limits_{\Omega^\varepsilon}\Big[|\nabla w|^2+w^2\Big]dx &+ \sum_{i<j<k:A_{ijk}=1}\int\limits_{\widehat\Gamma^\varepsilon_{ijk}}\Big[{w_i^\varepsilon}^2|\nabla v^\varepsilon_{ijk}|^2 + {w_j^\varepsilon}^2|\nabla v^\varepsilon_{jik}|^2 + {w_k^\varepsilon}^2|\nabla v^\varepsilon_{kij}|^2 + \\ &+2w_i^\varepsilon w_j^\varepsilon(\nabla v_{ijk}^\varepsilon,\nabla v_{jik}^\varepsilon)+2w_j^\varepsilon w_k^\varepsilon(\nabla v^\varepsilon_{jik},\nabla v^\varepsilon_{kij})+2w_i^\varepsilon w_k^\varepsilon(\nabla v^\varepsilon_{ijk},\nabla v^\varepsilon_{kij})\Big]d\tilde x+\delta(\varepsilon).\end{aligned} \tag{2.2.8}$$

[1] Here $\delta(\varepsilon)$ are remaining integrals estimated as follow

$$|\delta(\varepsilon)| \le C(w)\sum_{i,j,k:A_{ijk}=1}\left[J^\varepsilon_{ijk}+E^\varepsilon_{ijk}+I^\varepsilon_{ijk}+Y^\varepsilon_{ijk}+(d_0^\varepsilon)^2\right].$$

where

$$J^\varepsilon_{ijk} = \int\limits_{\widehat R_i^\varepsilon\cup\widehat R_j^\varepsilon\cup\widehat R_k^\varepsilon}\left(|\nabla^\varepsilon v^\varepsilon_{ijk}|^2+\frac{1}{{r_i^\varepsilon}^2}|v^\varepsilon_{ijk}|^2\right)d\tilde x,\ E^\varepsilon_{ijk} = \int\limits_{R^\varepsilon_{0i}\cup R^\varepsilon_{0j}\cup R^\varepsilon_{0k}}\left(|\nabla^\varepsilon v^\varepsilon_{ijk}|+\frac{1}{r_i^\varepsilon}|v^\varepsilon_{ijk}|\right)d\tilde x,$$

$$I^\varepsilon_{ijk} = \int\limits_{R_i^\varepsilon}|v^\varepsilon_{ijk}-1|^2 d\tilde x + \int\limits_{R_j^\varepsilon\cup R_k^\varepsilon}|v^\varepsilon_{ijk}|^2 d\tilde x,\ Y^\varepsilon_{ijk} = \int\limits_{\Gamma^\varepsilon_{ijk}}|v^\varepsilon_{ijk}|^2 d\tilde x$$

Using Lemma 2.1 and maximum principle for v^ε_{ijk} we have

$$J^\varepsilon_{ijk} \le C|\ln d^\varepsilon|^{-2}(1+|\ln r_i^\varepsilon|^2), \tag{2.2.9}$$

$$I^\varepsilon_{ijk} \le C\big[(d_0^\varepsilon)^2(1+|\ln d^\varepsilon|^{-1})+(r_i^\varepsilon\ln r_i^\varepsilon/\ln d^\varepsilon)^2\big], \tag{2.2.10}$$

$$E^\varepsilon_{ijk} \le C|\ln d^\varepsilon|^{-1}\big(r_i^\varepsilon|\ln r_i^\varepsilon|+d_0^\varepsilon|\ln d_0^\varepsilon|\big), \tag{2.2.11}$$

$$Y^\varepsilon_{ijk} \le C\cdot|\Gamma^\varepsilon_{ijk}| \le C(d^\varepsilon\sqrt{q^\varepsilon}+(b^\varepsilon)^2) \tag{2.2.12}$$

and since $\sum\limits_{i=1}^{N(\varepsilon)}|\ln d^\varepsilon|^{-1} \le C\sum\limits_{i=1}^{N(\varepsilon)}(r_i^\varepsilon)^2 \le C|\Omega|$, where $|\cdot|$ means the volume, using (i)-(iii) we conclude that

$$\delta(\varepsilon)\to 0,\ \varepsilon\to 0. \tag{2.2.13}$$

We denote $V^\varepsilon_{ijk} = \int\limits_{\widehat\Gamma^\varepsilon_{ijk}}|\nabla^\varepsilon v^\varepsilon_{ijk}|^2 d\tilde x$, where $i,j,k : A_{ijk=1}$. Since $v^\varepsilon_{ijk}+v^\varepsilon_{jik}+v^\varepsilon_{kij}=1$ for any $i,j,k : A_{ijk}=1$ we have $V^\varepsilon_{ijk} = W^\varepsilon_{ijk}+W^\varepsilon_{ikj}$. Therefore (2.2.8)

---

[1] The sum $\sum\limits_{i<j<k:A_{ijk}=1}$ means that each 3 indexes $\{i,j,k\}$ appear one and only one in this sum.

can be rewritten in the form

$$\|P^\varepsilon w\|^2 = \int\limits_\Omega \Big(|\nabla w|^2 + w^2\Big)dx + \sum_{i,j,k=1\dots N(\varepsilon)} W^\varepsilon_{ijk}\Big((w(x^\varepsilon_i))^2 - w(x^\varepsilon_i)w(x^\varepsilon_j)\Big) + \delta(\varepsilon) =$$

$$= \int\limits_\Omega \Big(|\nabla w|^2 + w^2\Big)dx + \frac{1}{2}\sum_{i,j,k=1\dots N(\varepsilon)} W^\varepsilon_{ijk}\Big(w(x^\varepsilon_i) - w(x^\varepsilon_j)\Big)^2 + \delta(\varepsilon). \quad (2.2.14)$$

It is easy to see that (b$_2$) follows from (iv),(2.2.13) and (2.2.14).

We verify condition (b$_1$). Let $w \in M$. In view of the conditions (a) and (b$_2$) the norms $\|\Pi^\varepsilon P^\varepsilon w\|_\varepsilon$ are uniformly bounded with respect to ε and in the same way as in (b$_2$) one can prove that $\Pi^\varepsilon P^\varepsilon w \to w$ strongly in $L^2(\Omega)$. Thus the condition (b$_1$) also holds .

We verify condition (b$_3$). Let $w \in M$ and the sequence $\gamma^\varepsilon \in H^\varepsilon$ is such that the norms $\|\gamma^\varepsilon\|_\varepsilon$ are uniformly bounded with respect to ε and $\Pi^\varepsilon\gamma^\varepsilon \to \gamma$ weakly in H as $\varepsilon \to 0$. Integrating by parts, we have:

$$(P^\varepsilon w, \gamma^\varepsilon)_\varepsilon = (-\Delta_\Omega w + w, \Pi^\varepsilon\gamma^\varepsilon)_{L_2(\Omega)} + \delta(\varepsilon), \quad (2.2.15)$$

where $\delta(\varepsilon)$ are the remaining integrals. Using Lemma 2.1 in the same way as in (b$_2$) we obtain the estimate

$$\lim_{\varepsilon\to 0}|\delta(\varepsilon)| \le C \lim_{\varepsilon\to 0}\left\{\left(\sum_{i=1}^{N(\varepsilon)}(w^\varepsilon_i)^2|R^\varepsilon_i|\right)^{1/2}\|\Pi^\varepsilon\gamma^\varepsilon\|_{L_2(\Omega)}\right\}.$$

Since $\Pi^\varepsilon\gamma^\varepsilon$ converges weakly to γ in H then $\Pi^\varepsilon\gamma^\varepsilon$ converges strongly to γ in $L_2(\Omega)$ and therefore we have

$$\lim_{\varepsilon\to 0}|\delta(\varepsilon)| \le C\|w\|_{L_2(M^\varepsilon)} \cdot \|\gamma\|_{L_2(\Omega)} \le C\|w\| \cdot \|\gamma\|. \quad (2.2.16)$$

It follows from (2.2.15)-(2.2.16) that (b$_3$) holds.

And, finally, we verify, that condition (c) holds. Let the sequence $\gamma^\varepsilon \in H^\varepsilon$ be such that $\Pi^\varepsilon\gamma^\varepsilon \to \gamma$ weakly in H. Then $\Pi^\varepsilon\gamma^\varepsilon \to \gamma$ strongly in $L_2(\Omega)$. We have:

$$|F^\varepsilon[\gamma^\varepsilon] - F[\gamma]| = \left|\int\limits_{\Omega^\varepsilon} f\cdot(\Pi^\varepsilon\gamma^\varepsilon - \gamma)d\tilde{x}\right| + \left|\int\limits_{\Omega\setminus\Omega^\varepsilon} f\gamma d\tilde{x}\right| \to 0, \varepsilon \to 0.$$

and so condition (c) holds.

Thus all the conditions of Theorem 2.4 holds. Hence $\Pi^\varepsilon u^\varepsilon \to u$ weakly in H. Therefore by embedding theorem $\Pi^\varepsilon u^\varepsilon \to u_0$ strongly in $L^2(\Omega)$. Finally we have

$$\|\mathcal{A}^\varepsilon R^\varepsilon f - R^\varepsilon\mathcal{A}_0 f\|^2_{\mathcal{H}^\varepsilon} = \|u^\varepsilon\|^2_{L_2(\cup\Gamma^\varepsilon_{ijk})} + \|\Pi^\varepsilon u^\varepsilon - u_0\|^2_{L_2(\Omega^\varepsilon)}.$$

In view of Lemma 2.2 and (2.2.1) $\|u^\varepsilon\|^2_{L_2(\cup\Gamma^\varepsilon_{ijk})} \to 0, \varepsilon \to 0$. Thus C3 is proved.

And finally we verify the fulfilment of the condition C4. Let $f^\varepsilon \in \mathcal{H}^\varepsilon$ be such that $\sup\|f^\varepsilon\|_{\mathcal{H}^\varepsilon} < \infty$. Let $u^\varepsilon = \mathcal{A}^\varepsilon f^\varepsilon$. In view of the estimate (2.2.1) $\Pi^\varepsilon u^\varepsilon$ is weakly compact in $H^1(\Omega)$ and so there exists the subsequence ε' and $w \in H^1(\Omega)$ such that $\Pi^\varepsilon u^\varepsilon \to w$ strongly in $L_2(\Omega)$. This and Lemma 2.2 imply C4.

Theorem 2.3 and therefore Theorems 2.1, 2.2 are proved.

2.3 Example

We consider an example of the manifold M^ε and calculate the function $W(x,y)$ explicitly.

Let Ω contain the subset K - a flat square with the side equal to l. Let $\varepsilon > 0$ and let $n^\varepsilon = \left[\frac{1}{\varepsilon}\right]^{1/3}$.

We divide K into squares K^ε_α, $\alpha = 1 ... n^{\varepsilon 2}$ with side length l/n^ε. Within each square K^ε_α we cut out $n^{\varepsilon 4}$ holes D^ε_i with the radius $d^\varepsilon = \exp\left(-n^{\varepsilon 6}/l^6\right)$ and such that their centers form periodic lattice with period $\frac{l}{n^{\varepsilon 3}}$. It is clear that $|\ln d^\varepsilon|^{-1} = l^4 (r^\varepsilon_i)^2$. The total number of D^ε_i is equal to $N(\varepsilon) = n^{\varepsilon 6}$.

For each hole D^ε_i we denote by $\alpha(i)$ the number of square K^ε_α containing this hole. Since the number of holes within the square K^ε_α is equal to $n^{\varepsilon 2} \cdot n^{\varepsilon 2}$ we can assign to each hole $D^\varepsilon_i \subset K^\varepsilon_\alpha$ the pair $(\beta(i), \gamma(i))$, $\beta(i), \gamma(i) \in \{1 ... n^{\varepsilon 2}\}$. So each hole D^ε_i is characterized by $(\alpha(i), \beta(i), \gamma(i))$.

If $\alpha(i) = \beta(j) = \gamma(k), \alpha(j) = \beta(k) = \gamma(i), \alpha(k) = \beta(i) = \gamma(j)$ and only in this case we join the boundaries of the holes $D^\varepsilon_i, D^\varepsilon_k, D^\varepsilon_j$ by means of the manifold $\Gamma^\varepsilon_{ijk} = G^\varepsilon_i \cup G^\varepsilon_j \cup G^\varepsilon_k \cup B^\varepsilon_{ijk}$.

We set

$$q^\varepsilon_i = [q \cdot |\ln d^\varepsilon| \cdot d^\varepsilon]^2,\ q > 0$$

and choose such b^ε that (iii) fulfils and

$$\ln\left(\tan\frac{\theta^\varepsilon}{2}\right) / \ln d^\varepsilon \to 0,\ \varepsilon \to 0,\ \theta^\varepsilon = \arcsin\frac{d^\varepsilon}{b^\varepsilon}$$

(for example $d^\varepsilon \sim C b^\varepsilon$).

In order to calculate $W(x,y)$ we find a suitable approximation for the solution

v^ε_{ijk} to (2.1.1). Namely we represent it in the form $v^\varepsilon_{ijk} = \widehat{v}^\varepsilon_{ijk} + w^\varepsilon_{ijk}$ where

$$\widehat{v}^\varepsilon_{ijk}(\tilde{x}) = \begin{cases} a^\varepsilon_i \ln|\tilde{x} - x^\varepsilon_i| + b^\varepsilon_i, \ \tilde{x} \in R^\varepsilon_i, \\ A^\varepsilon_i z + B^\varepsilon_i, \ \tilde{x} = (z_i, \varphi_i) \in G^\varepsilon_i, \\ a^\varepsilon_j \ln|\tilde{x} - x^\varepsilon_j| + b^\varepsilon_j, \ \tilde{x} \in R^\varepsilon_j, \\ A^\varepsilon_j z + B^\varepsilon_j, \ \tilde{x} = (z_j, \varphi_j) \in G^\varepsilon_j, \\ a^\varepsilon_k \ln|\tilde{x} - x^\varepsilon_k| + b^\varepsilon_k, \ \tilde{x} \in R^\varepsilon_k, \\ A^\varepsilon_k z + B^\varepsilon_k, \ \tilde{x} = (z_k, \varphi_k) \in G^\varepsilon_k, \\ C^\varepsilon_{ijk}, \ \tilde{x} \in B^\varepsilon_{ijk}. \end{cases}$$

We chose the constants $a^\varepsilon_i, b^\varepsilon_i ... A^\varepsilon_k, B^\varepsilon_k, C^\varepsilon_{ijk}$ such that:

1) $\widehat{v}^\varepsilon_{ijk}$ is a harmonic function in $G^\varepsilon_i \cup R^\varepsilon_i$, $G^\varepsilon_j \cup R^\varepsilon_j$, $G^\varepsilon_k \cup R^\varepsilon_k$,

2) $\widehat{v}^\varepsilon_{ijk} = 1$ on $\widehat{C}^\varepsilon_i$, $\widehat{v}^\varepsilon_{ijk} = 0$ on $\widehat{C}^\varepsilon_j \cup \widehat{C}^\varepsilon_k$,

3) $\widehat{v}^\varepsilon_{ijk}|_{S^\varepsilon_i} = \widehat{v}^\varepsilon_{ijk}|_{S^\varepsilon_j} = \widehat{v}^\varepsilon_{ijk}|_{S^\varepsilon_k} = M$, where M is a constant.

4) $\dfrac{\partial \widehat{v}^\varepsilon_{ijk}}{\partial \vec{n}}|_{S^\varepsilon_i} + \dfrac{\partial \widehat{v}^\varepsilon_{ijk}}{\partial \vec{n}}|_{S^\varepsilon_j} + \dfrac{\partial \widehat{v}^\varepsilon_{ijk}}{\partial \vec{n}}|_{S^\varepsilon_k} = 0$, $\vec{n}$ is the outward(or inward) normal[2].

As a result we obtain

$$a^\varepsilon_i = \frac{2|\ln d^\varepsilon|^{-1}}{3(1+q)}(1 + o(1)) = -2a^\varepsilon_j = -2a^\varepsilon_k,$$

$$A^\varepsilon_i = -a^\varepsilon_i \frac{\sqrt{q^\varepsilon_i}}{d^\varepsilon}, \ A^\varepsilon_j = -a^\varepsilon_j \frac{\sqrt{q^\varepsilon_j}}{d^\varepsilon}, \ A^\varepsilon_k = -a^\varepsilon_k \frac{\sqrt{q^\varepsilon_k}}{d^\varepsilon},$$

$$b^\varepsilon_i = 1 - a^\varepsilon_i \ln(r^\varepsilon_i/2), \ b^\varepsilon_j = -a^\varepsilon_j \ln(r^\varepsilon_j/2), \ b^\varepsilon_k = -a^\varepsilon_k \ln(r^\varepsilon_k/2),$$

$$B^\varepsilon_i = a^\varepsilon_i \ln d^\varepsilon + b^\varepsilon_i, \ B^\varepsilon_j = a^\varepsilon_j \ln d^\varepsilon + b^\varepsilon_j, \ B^\varepsilon_k = a^\varepsilon_k \ln d^\varepsilon + b^\varepsilon_k.$$

Direct calculations show that

$$\|\nabla^\varepsilon \widehat{v}^\varepsilon_{ijk}\|^2_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})} = \frac{4\pi}{3(1+q)}|\ln d^\varepsilon|^{-1}(1 + \bar{o}(1)) \to 0, \ \varepsilon \to 0, \tag{2.3.1}$$

$$(\nabla^\varepsilon \widehat{v}^\varepsilon_{ijk}, \nabla^\varepsilon \widehat{v}^\varepsilon_{jik})_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})} = -\frac{2\pi}{3(1+q)}|\ln d^\varepsilon|^{-1}(1 + \bar{o}(1)), \ \varepsilon \to 0. \tag{2.3.2}$$

Now we prove that w^ε_{ijk} gives vanishingly small contribution to W^ε_{ijk}. Since v^ε_{ijk} minimizes the functional $I^\varepsilon[v] = \|\nabla^\varepsilon v\|^2_{0\varepsilon}$ in the class of functions from $H^1(\widehat{\Gamma}^\varepsilon_{ijk})$ equal to 1 on $\widehat{S}^\varepsilon_i$ and equal to 0 on $\widehat{S}^\varepsilon_j \cup \widehat{S}^\varepsilon_k$, then $\|\nabla^\varepsilon v^\varepsilon_{ijk}\|^2_{L_2(M^\varepsilon)} \le \|\nabla^\varepsilon \widehat{v}^\varepsilon_{ijk}\|^2_{L_2(M^\varepsilon)}$ and threfore

$$\|\nabla^\varepsilon w^\varepsilon_{ijk}\|^2_{L_2(M^\varepsilon)} \le 2\left|(\nabla^\varepsilon w^\varepsilon_{ijk}, \nabla^\varepsilon \widehat{v}^\varepsilon_{ijk})_{L_2(M^\varepsilon)}\right|$$

---

[2] Here the normal derivatives are taken in arbitrary point of S^ε_i. It is easy to see that conditions 1)-3) guarantee that $\dfrac{\partial \widehat{v}^\varepsilon_{ijk}}{\partial \vec{n}}$ are constant on S^ε_i (and also on S^ε_j and S^ε_k). Condition 4) determines the constant M from the condition 3).

and using the properties of the function $\widehat{v}^\varepsilon_{ijk}$ we obtain

$$\|\nabla^\varepsilon w^\varepsilon_{ijk}\|^2_{0\varepsilon} \le 2\pi d^\varepsilon \left| \frac{A^\varepsilon_i}{\sqrt{q^\varepsilon_i}} w^\varepsilon_i + \frac{A^\varepsilon_j}{\sqrt{q^\varepsilon_j}} w^\varepsilon_j + \frac{A^\varepsilon_i}{\sqrt{q^\varepsilon_j}} w^\varepsilon_k \right| = 2\pi |a^\varepsilon_i w^\varepsilon_i + a^\varepsilon_j w^\varepsilon_j + a^\varepsilon_k w^\varepsilon_k| =$$
$$= 2\pi |a^\varepsilon_j (w^\varepsilon_j - w^\varepsilon_i) + a^\varepsilon_k (w^\varepsilon_k - w^\varepsilon_i)|, \qquad (2.3.3)$$

where $w^\varepsilon_i, w^\varepsilon_j, w^\varepsilon_k$ are the average values of w^ε_{ijk} in $S^\varepsilon_i, S^\varepsilon_j, S^\varepsilon_k$ correspondingly.

The following estimate is valid

$$|w^\varepsilon_i - w^\varepsilon_j| + |w^\varepsilon_i - w^\varepsilon_k| \le C\sqrt{|\ln \tan \frac{\theta^\varepsilon}{2}|} \cdot \|\nabla^\varepsilon v^\varepsilon_{ijk}\|_{0\varepsilon} \le$$
$$\le C\sqrt{|\ln \tan \frac{\theta^\varepsilon}{2}|} \cdot \|\nabla^\varepsilon \widehat{v}^\varepsilon_{ijk}\|_{0\varepsilon} \le C\sqrt{|\ln \tan \frac{\theta^\varepsilon}{2}|/|\ln d^\varepsilon|} \underset{\varepsilon \to 0}{\to} 0. \qquad (2.3.4)$$

The proof is similar to the proof of (1.2.21).

It follows from (2.3.1)-(2.3.3) and from the form of the coefficients $a^\varepsilon_i, a^\varepsilon_j, a^\varepsilon_k$ that

$$\|\nabla^\varepsilon w^\varepsilon\|^2_{0\varepsilon} = \bar{o}(|\ln d^\varepsilon|^{-1}). \qquad (2.3.5)$$

We have:

$$W^\varepsilon_{ijk} = -\big[(\widehat{v}^\varepsilon_{ijk}, \widehat{v}^\varepsilon_{jik})_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})} + (\widehat{v}^\varepsilon_{ijk}, w^\varepsilon_{jik})_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})} + (w^\varepsilon_{ijk}, \widehat{v}^\varepsilon_{jik})_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})} + (w^\varepsilon_{ijk}, w^\varepsilon_{jik})_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})}\big]. \qquad (2.3.6)$$

It follows from (2.3.1),(2.3.2),(2.3.5),(2.3.6) that

$$W^\varepsilon_{ij} \sim -(\widehat{v}^\varepsilon_{ijk}, \widehat{v}^\varepsilon_{jik})_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})} \sim \frac{2\pi}{3(1+q)} |\ln d^\varepsilon|^{-1}.$$

Let $w(x,y) \in C^\infty(\Omega)$. Then

$$\langle W^\varepsilon, w \rangle = \sum_{i,j,k: A_{ijk}=1} \frac{2\pi}{3(1+q)} w(x^\varepsilon_i, x^\varepsilon_j) |\ln d^\varepsilon|^{-1}.$$

By the construction of the manifold M^ε for any three squares $K^\varepsilon_\alpha, K^\varepsilon_\beta, K^\varepsilon_\gamma$ there are ones and only ones three holes $D^\varepsilon_{i_{\alpha\beta\gamma}}, D^\varepsilon_{j_{\alpha\beta\gamma}}, D^\varepsilon_{k_{\alpha\beta\gamma}}$ such that

$$D^\varepsilon_{i_{\alpha\beta\gamma}} \subset K^\varepsilon_\alpha,\ D^\varepsilon_{j_{\alpha\beta\gamma}} \subset K^\varepsilon_\beta,\ D^\varepsilon_{k_{\alpha\beta\gamma}} \subset K^\varepsilon_\gamma,\ A_{i_{\alpha\beta\gamma} j_{\alpha\beta\gamma} k_{\alpha\beta\gamma}} = 1.$$

Therefore it is easy to see that the sum above can be rewritten in the form:

$$\langle W^\varepsilon, w \rangle = \frac{2\pi}{3(1+q)} \sum_{\alpha,\beta,\gamma=1}^{n^2(\varepsilon)} w(x_{i_{\alpha\beta\gamma}}, x_{j_{\alpha\beta\gamma}}) |\ln d^\varepsilon|^{-1} =$$

$$= \frac{2\pi}{3(1+q)} \sum_{\alpha,\beta,\gamma=1}^{n^2(\varepsilon)} w(x_{i_{\alpha\beta\gamma}}, x_{j_{\alpha\beta\gamma}}) |K^\varepsilon_\alpha| \cdot |K^\varepsilon_\beta| \cdot |K^\varepsilon_\gamma| \underset{\varepsilon \to 0}{\to} \int\limits_K \int\limits_K \int\limits_K \frac{2\pi}{3(1+q)} w(x,y) dxdydz.$$

Thus

$$W(x,y) = \chi_K(x)\chi_K(y) \int\limits_K \frac{2\pi}{3(1+q)} dz = \frac{2\pi l^2}{3(1+q)} \chi_K(x)\chi_K(y),$$

where χ_K is the characteristic function of K.

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Про спектр риманових многовидүв з приэднаними тонкими ручками

Андрүй Храбустовський

Вивчаэться поведүнка спектру оператора Лапласа-Бельтрамү на риманових многовидах M^ε, що залежать вүд малого параметра ε. Вони складаються з фүксованого компактного многовиду з приэднаними ручками з радүусом, що прямуэ до нуля, коли $\varepsilon \to 0$. Ми розглянемо два випадки: коли күлькүсть ручок та »х довжини фүксованү, та коли күлькүсть ручок зростаэ, а »х довжини прямують до нуля, коли $\varepsilon \to 0$. Для цих двох випадкүв ми отримаэмо оператори, чий спектр притягуэ спектр Δ^ε, коли $\varepsilon \to 0$.