

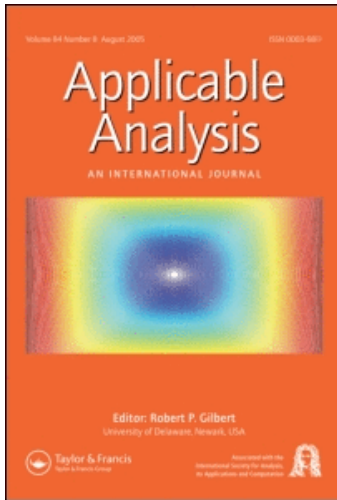
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Andrii Khrabustovskyi <sup>a</sup>

<sup>a</sup> Mathematical Division, B. Verkin Institute for Low Temperature Physics and Engineering, National Academy of Sciences of Ukraine, Kharkiv, Ukraine

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## RESEARCH ARTICLE

# Asymptotic behaviour of spectrum of Laplace–Beltrami operator on Riemannian manifolds with complex microstructure

Andrii Khrabustovskyi\*

Mathematical Division, B. Verkin Institute for Low Temperature Physics and Engineering,  
National Academy of Sciences of Ukraine, Kharkiv, Ukraine

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The article deals with a convergence of the spectrum of the Laplace–Beltrami operator  $\Delta^\varepsilon$  on a Riemannian manifold depending on a small parameter  $\varepsilon > 0$ . This manifold consists of a domain  $\Omega \subset \mathbb{R}^n$  with a large number of small ‘holes’ whose boundaries are glued to the boundaries of the  $n$ -dimensional spheres with small truncated segment. The number of the ‘holes’ increases, as  $\varepsilon \rightarrow 0$ , while their radii tend to zero. We prove that the spectrum converges to the spectrum of the homogenized operator having (in contrast to  $\Delta^\varepsilon$ ) a non-empty essential spectrum.

**Keywords:** homogenization; Laplace–Beltrami operator; spectrum; Riemannian manifold

**AMS Subject Classifications:** 35B27; 35P20; 58G25; 58G30

## 1. Introduction

The aim of this article is to study the asymptotic behaviour of the spectral problem

$$\begin{aligned} -\Delta^\varepsilon u^\varepsilon &= \lambda^\varepsilon u^\varepsilon, & \text{in } M^\varepsilon, \\ u^\varepsilon &= 0, & \text{on } \partial M^\varepsilon, \end{aligned} \tag{1.1}$$

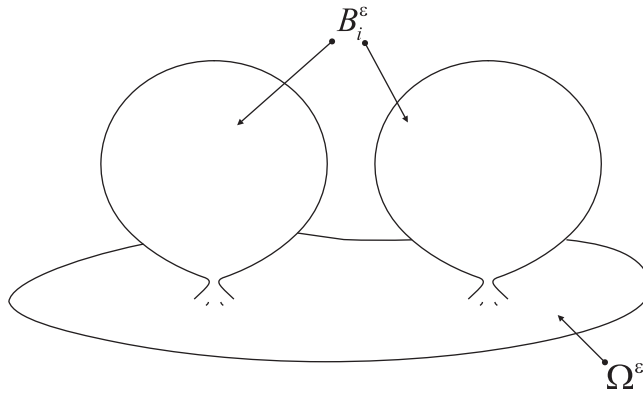
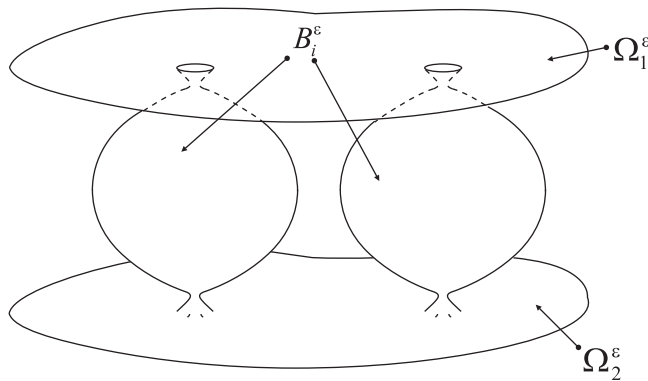
as  $\varepsilon \rightarrow 0$ . Here  $\Delta^\varepsilon$  is the Laplace–Beltrami operator,  $M^\varepsilon$  is a  $n$ -dimensional Riemannian manifold with complex microstructure depending on  $\varepsilon$ . It is constructed in the following way. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ , and  $\{D_i^\varepsilon : i = 1, \dots, N(\varepsilon)\}$  be a system of balls (‘holes’) in  $\Omega$  depending on  $\varepsilon$ ,  $\Omega^\varepsilon = \Omega \setminus \bigcup_{i=1}^{N(\varepsilon)} D_i^\varepsilon$ . Suppose that the boundary of each ‘hole’  $D_i^\varepsilon$  is glued to the boundary of  $B_i^\varepsilon$  (‘bubble’) – the  $n$ -dimensional sphere with small truncated segment. Thus, we obtain a manifold  $M^\varepsilon$  (Figure 1):

$$M^\varepsilon = \overline{\Omega}^\varepsilon \cup \left( \bigcup_{i=1}^{N(\varepsilon)} \overline{B}_i^\varepsilon \right).$$

More precise description of  $M^\varepsilon$  will be specified later in Section 2.

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\*Email: andry9@ukr.net

Figure 1. The manifold  $M^\varepsilon$ .Figure 2. The manifold  $M^\varepsilon$ .

On  $M^\varepsilon$  we introduce a Riemannian metric that coincides with the flat Euclidean metric on  $\Omega^\varepsilon$  and coincides with the spherical metric on the ‘bubbles’.

It is supposed that the centres of the ‘holes’ form a periodic lattice with the period  $\varepsilon$ , their radii are much smaller than the distance between them and are of order  $\exp(-1/\varepsilon^2)$  ( $n=2$ ) or  $\varepsilon^{n/(n-2)}$  ( $n > 2$ ). The radii of ‘bubbles’ are equivalent to  $\varepsilon$ . Clearly, the total volume of the ‘bubbles’ is bounded from above and from below uniformly in  $\varepsilon$ .

The following result is obtained. The spectrum  $\text{Sp}(-\Delta^\varepsilon)$  of the problem (1.1) converges to the spectrum of an operator  $\mathbf{A}$  in the Hausdorff sense. We give an explicit description of this operator later on. Now we only note that  $\mathbf{A}$  acts on a two-vector-functions from  $L_2(\Omega) \times L_2(\Omega)$ ,  $\mathbf{A}$  is self-adjoint, but in contrast to the operator  $-\Delta^\varepsilon$ , the spectrum of the operator  $\mathbf{A}$  is not purely discrete. Namely, if the radii of the ‘bubbles’ are identical, the essential spectrum contains one point. We also consider the manifold with different ‘bubbles’ – in this case the essential spectrum contains a segment.

We also generalize our result to a manifold consisting of two copies of the domain  $\Omega^\varepsilon$  and a system of the  $n$ -dimensional spheres with two small truncated segments (Figure 2). The topological type of  $M^\varepsilon$  increases as  $\varepsilon \rightarrow 0$ .

Spectral questions in homogenization have been studied by many authors, see, e.g. [1–7]. In particular, in [3,5] the spectrum of the homogenized operator is not purely discrete, but its structure differs from the spectrum of the operator  $\mathbf{A}$  in the present article.

Homogenization problems on a manifolds with complex microstructure have been studied in [2,8–11]. In particular, homogenization of semi-linear parabolic equations on a manifold described above (Figure 1) has been studied in [9] in the case of the identical radii of ‘bubbles’. The behaviour of the spectrum of the Laplace–Beltrami operator has been studied in [2]. The structure of the manifolds constructed in [2] differs from the structure of the manifold  $M^\varepsilon$  and in contrast to the present article, the homogenized operator have purely discrete spectrum.

We remark that various physical problems can be reduced to a study of homogenization problems on Riemannian manifolds depending on a small parameter. Namely, in paper [8], the authors apply their results for studying the asymptotic behaviour of coloured particles, which wander in the domain with many small obstacles reflecting the particles and changing their colour. The results in [10,11] are interpreted in terms of general relativity.

In the present article, we study proper oscillation of the membrane  $M^\varepsilon$  with complex ‘bubbles-like’ structure.

The article is organized as follows. In Section 2 we describe the structure of the manifold  $M^\varepsilon$  and formulate the main result of this article (Theorem 2.2). It is proved in Section 3. In Section 4 we study the case of the ‘bubbles’ with different radii (Theorem 4.1). In Section 5, we generalize our results for to a manifold with increasing topological type (Theorem 5.1).

## 2. The problem setting and the main result

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  ( $n \geq 2$ ) and  $\{D_i^\varepsilon : i = 1..N(\varepsilon)\}$  be the system of disjoint balls in  $\Omega$  depending on a small parameter  $\varepsilon$ . Let  $x_i^\varepsilon$  be the centre of the ball  $D_i^\varepsilon$  and let its radius be equal to  $d^\varepsilon$ . Suppose that points  $x_i^\varepsilon$  form a periodic lattice with period  $\varepsilon$ , i.e.  $x_i^\varepsilon = \varepsilon \sum_{j=1}^n e_j z_j^i$ , where  $\{e_i, i = 1, \dots, n\}$  is an orthonormal basis in  $\mathbb{R}^n$ ,  $z_j^i \in \mathbb{Z}$ .

We consider the following domain with ‘holes’

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{i=1}^{N(\varepsilon)} D_i^\varepsilon.$$

Suppose that each ‘hole’  $D_i^\varepsilon$  is glued to the truncated  $n$ -dimensional sphere (‘bubble’) with the radius  $b^\varepsilon$

$$B_i^\varepsilon = \left\{ \tilde{x} = (\theta_1, \theta_2, \dots, \theta_n) : \theta_1 \in [0, 2\pi], \theta_i \in [0, \pi], \quad i = 2, \dots, n-1, \theta_n \in \left[ \arcsin \frac{d^\varepsilon}{b^\varepsilon}, \pi \right] \right\},$$

where  $\theta_1, \dots, \theta_n$  are the spherical coordinates.

Namely, we identify  $\partial D_i^\varepsilon$  and the boundary of  $B_i^\varepsilon$ :

$$\partial B_i^\varepsilon = \left\{ x \in B_i^\varepsilon : \theta_n = \arcsin \frac{d^\varepsilon}{b^\varepsilon} \right\}.$$

As a result we obtain a manifold  $M^\varepsilon$  (the fragment is represented on the Figure 1)

$$M^\varepsilon = \overline{\Omega}^\varepsilon \cup \left( \bigcup_{i=1}^{N(\varepsilon)} \overline{B}_i^\varepsilon \right).$$

The boundary of  $M^\varepsilon$  coincides with  $\partial\Omega$ . We denote by  $\tilde{x}$  points of this manifold. If the point  $\tilde{x}$  belongs to  $\Omega^\varepsilon$ , sometimes we write  $x$  instead of  $\tilde{x}$  having in mind a corresponding point in  $\Omega$ .

Clearly,  $M^\varepsilon$  can be covered by a system of charts and compatible local coordinates  $\{x_1, \dots, x_n\} \rightarrow \tilde{x} \in M^\varepsilon$  can be introduced.

We suppose that  $M^\varepsilon$  is equipped by the metric  $g^\varepsilon$  that coincides with the flat Euclidean metric on  $\Omega^\varepsilon$  and coincides with the spherical metric on the ‘bubbles’.

We give more precise description of this metric at the points, where the boundary of the ‘holes’ is glued to the boundary of the ‘bubbles’. We introduce local coordinates  $(x_1, \dots, x_n)$  in a small neighbourhood of  $\partial D_i^\varepsilon$  as follows. Let  $(\theta_1, \dots, \theta_{n-1}, r)$  be the spherical coordinates in  $\Omega$  with the origin at  $x_i^\varepsilon$ . Here  $\theta_1, \dots, \theta_{n-1}$  are the angular coordinates,  $r$  is the distance to  $x_i^\varepsilon$  (in particular,  $r = d^\varepsilon$  for the points of  $\partial D_i^\varepsilon$ ). We set  $x_j = \theta_j$  ( $j = 1, \dots, n-1$ ),  $x_n = r - d^\varepsilon$  ( $x_n \geq 0$ ) for  $\tilde{x} \in \Omega^\varepsilon$  and  $x_n = -b^\varepsilon(\theta_n - \theta^\varepsilon)$  ( $x_n < 0$ ), where  $\theta^\varepsilon = \arcsin(d^\varepsilon/b^\varepsilon)$ , for  $\tilde{x} \in B_i^\varepsilon$ . Then the components of the corresponding metric tensor  $g_{\alpha\beta}^\varepsilon(x_1, \dots, x_n)$  have the following form:

$$g_{\alpha\beta}^\varepsilon = \begin{cases} g_{\alpha\beta}^{1\varepsilon} = \delta_{\alpha\beta}(x_n + d^\varepsilon)^2 \prod_{j=\alpha+1}^{n-1} \sin^2 x_j, & x_n \geq 0, \\ g_{\alpha\beta}^{2\varepsilon} = \delta_{\alpha\beta}(b^\varepsilon)^2 \sin^2\left(\frac{x_n}{b^\varepsilon} - \theta^\varepsilon\right) \prod_{j=\alpha+1}^{n-1} \sin^2 x_j, & x_n < 0, \end{cases} \quad \alpha = \overline{1, n-1}, g_{n\beta}^\varepsilon = \delta_{n\beta}$$

(for  $\alpha = n-1$  we set  $\prod_{j=\alpha+1}^{n-1} \sin^2 x_j := 1$ ). Here,  $\delta_{\alpha\beta}$  is the Kronecker’s delta.

*Remark 1* It is clear that the components of the metric tensor  $g_{\alpha\beta}^\varepsilon$  introduced above are continuous but not differentiable functions. In particular, they do not allow us to calculate the curvature at the points of  $\partial D_i^\varepsilon$ . Nevertheless, this tensor can be approximated by a smooth tensor  $g_{\alpha\beta}^{\varepsilon\delta}$  that differs from  $g_{\alpha\beta}^\varepsilon$  only in a small neighbourhood of  $\partial D_i^\varepsilon$ . Namely, we define  $g_{\alpha\beta}^{\varepsilon\delta}$  by the formula

$$g_{\alpha\beta}^{\varepsilon\delta}(x_1, \dots, x_n) = g_{\alpha\beta}^{1\varepsilon}(x_1, \dots, x_n) \cdot \varphi_\delta(x_n) + g_{\alpha\beta}^{2\varepsilon}(x_1, \dots, x_n) \cdot (1 - \varphi_\delta(x_n)),$$

where  $\varphi_\delta(x_n)$  is a smooth positive function equal to 1 for  $x_n \geq \delta$  and equal to 0 for  $x_n \leq -\delta$ ,  $\delta = \delta(\varepsilon) > 0$ . The obtained tensor is smooth, so the curvature can be calculated everywhere. Clearly, the curvature tends to infinity at the points of  $\partial D_i^\varepsilon$  as  $\delta \rightarrow 0$ .

In order to simplify our calculations, we will further consider the piecewise smooth tensor  $g_{\alpha\beta}^\varepsilon$ . However, all results (in particular, the formula (2.5) below) are still valid for the smooth tensor  $g_{\alpha\beta}^{\varepsilon\delta}$  if  $\delta(\varepsilon)$  converges to 0 sufficiently fast as  $\varepsilon \rightarrow 0$  (see, also, [8, subsection 3.2], [11, section 2], for similar assertions).

Our main assumption is about the radii  $d^\varepsilon$  of the ‘holes’:

$$d^\varepsilon = \begin{cases} \exp\left(-\frac{1}{a\varepsilon^2}\right), & n = 2, \\ a\varepsilon^{n/n-2}, & n > 2, \end{cases} \quad (2.1)$$

where  $a$  is a positive constant.

We suppose that the ‘bubbles’ radii  $b^\varepsilon$  are the following

$$b^\varepsilon = b \cdot \varepsilon, \quad b \text{ is a positive constant.} \tag{2.2}$$

It is easy to see that the total volume of the ‘bubbles’ is bounded from above and from below, namely  $\lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{N(\varepsilon)} |B_i^\varepsilon| = b^n \cdot \omega_n \cdot |\Omega|$ , where  $\omega_n$  is the volume of the  $n$ -dimensional unit sphere.

We consider the following eigenvalue problem

$$-\Delta^\varepsilon u^\varepsilon = \lambda u^\varepsilon, \quad \tilde{x} \in M^\varepsilon, \quad u^\varepsilon = 0, \quad \tilde{x} \in \partial M^\varepsilon, \tag{2.3}$$

where  $\Delta^\varepsilon$  is the Laplace–Beltrami operator on  $M^\varepsilon$ , which have the following form on the local coordinates

$$\Delta^\varepsilon = \frac{1}{\sqrt{G^\varepsilon}} \sum_{\alpha, \beta=1}^n \frac{\partial}{\partial x_\alpha} \left( \sqrt{G^\varepsilon} g_\varepsilon^{\alpha\beta} \frac{\partial}{\partial x_\beta} \right).$$

Here,  $G^\varepsilon = \det g_\varepsilon^{\alpha\beta}$ ,  $g_\varepsilon^{\alpha\beta}$  are the components of the tensor inverse to  $g_\varepsilon^{\alpha\beta}$ .

Let  $0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \lambda_3^\varepsilon \leq \dots \leq \lambda_k^\varepsilon \leq \dots \rightarrow \infty$  be the eigenvalues of the problem (2.3), written with account of their multiplicity and  $u_1^\varepsilon, u_2^\varepsilon, \dots, u_k^\varepsilon, \dots$  be the corresponding eigenfunctions such that  $(u_k^\varepsilon, u_l^\varepsilon)_{0\varepsilon} = \delta_{kl}$ . We denote by  $\text{Sp}(-\Delta^\varepsilon)$  the spectrum of the problem (2.3):  $\text{Sp}(-\Delta^\varepsilon) = \{\lambda_i^\varepsilon, i = 1, \dots, \infty\}$ .

The goal of this article is a description of asymptotic behaviour of  $\text{Sp}(-\Delta^\varepsilon)$  as  $\varepsilon \rightarrow 0$ . Notice that the Dirichlet boundary condition on  $\partial M^\varepsilon$  is irrelevant; it can be replaced by any other one.

We use the following notations:

$$\begin{aligned} R_i^\varepsilon &= \left\{ \tilde{x} \in \Omega^\varepsilon : d^\varepsilon \leq |x - x_i^\varepsilon| < \frac{\varepsilon}{2} \right\}, \\ G_i^\varepsilon &= R_i^\varepsilon \cup B_i^\varepsilon, \\ S_i^\varepsilon &= \left\{ \tilde{x} \in \Omega^\varepsilon : |x - x_i^\varepsilon| = \frac{\varepsilon}{2} \right\} \equiv \partial G_i^\varepsilon. \end{aligned}$$

We denote by  $\sigma^\varepsilon$  the first eigenvalue of the Dirichlet problem

$$-\Delta^\varepsilon v^\varepsilon = \lambda^\varepsilon v^\varepsilon, \quad \tilde{x} \in G_i^\varepsilon, \quad v^\varepsilon = 0, \quad \tilde{x} \in S_i^\varepsilon. \tag{2.4}$$

and let  $\sigma = \lim_{\varepsilon \rightarrow 0} \sigma^\varepsilon$ . It is possible to show (see, e.g. [8,9] for close assertions) that  $\sigma$  is finite and equal to

$$\sigma = \begin{cases} \frac{a}{4b^2}, & n = 2, \\ \frac{n-2}{2} \cdot \frac{a^{n-2} \omega_{n-1}}{b^n \omega_n}, & n > 2. \end{cases} \tag{2.5}$$

Note, that in spite of the fact that the volume of  $G_i^\varepsilon$  converges to zero as  $\varepsilon \rightarrow 0$ ,  $\sigma^\varepsilon$  does not blow up because of a weak connection between  $B_i^\varepsilon$  and  $R_i^\varepsilon$ .

We introduce the following functional spaces:

$L_2(M^\varepsilon)$  is the Hilbert space of real-valued functions on  $M^\varepsilon$  with the norm

$$\|u^\varepsilon\|_{0\varepsilon} = \left\{ \int_{M^\varepsilon} (u^\varepsilon)^2 d\tilde{x} \right\}^{1/2},$$

where  $d\tilde{x} = \sqrt{G^\varepsilon} dx_1, \dots, dx_n$  is the volume form on  $M^\varepsilon$ ;

$\mathbf{H}$  is the Hilbert space of real-valued two-vector-functions from  $L_2(\Omega) \times L_2(\Omega)$  with the norm

$$\|U\|_0 = \left\{ \int_{\Omega} \left[ (u_1(x))^2 + \rho(u_2(x))^2 \right] dx \right\}^{1/2}, \quad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where the weight  $\rho = b^n \omega_n$ .

*Definition 2.1* Let  $\mathcal{A}^\varepsilon \subset \mathbb{R}$  be a set depending on a positive parameter  $\varepsilon$ . We say that  $\mathcal{A}^\varepsilon$  converges as  $\varepsilon \rightarrow 0$  in the Hausdorff sense to the set  $\mathcal{A}$  if the following conditions hold:

- (A) if  $\lambda^\varepsilon \in \mathcal{A}^\varepsilon$  and  $\lim_{\varepsilon \rightarrow 0} \lambda^\varepsilon = \lambda$ , then  $\lambda \in \mathcal{A}$ ,
- (B) for any  $\lambda \in \mathcal{A}$  there is  $\lambda^\varepsilon \in \mathcal{A}^\varepsilon$  such that  $\lim_{\varepsilon \rightarrow 0} \lambda^\varepsilon = \lambda$ .

Now, we formulate the main theorem.

**THEOREM 2.2** *The spectrum  $\text{Sp}(-\Delta^\varepsilon)$  converges to the spectrum  $\text{Sp}\mathbf{A}$  of the self-adjoint operator  $\mathbf{A}: \mathbf{H} \rightarrow \mathbf{H}$  in the Hausdorff sense.  $\mathbf{A}$  is defined by the operation*

$$\mathbf{A}U = \begin{pmatrix} -\Delta u_1 + \sigma\rho(u_1 - u_2) \\ \sigma(u_2 - u_1) \end{pmatrix}, \quad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (2.6)$$

and the boundary condition

$$u_1 = 0, \quad x \in \partial\Omega. \quad (2.7)$$

Before investigating the limit  $\lim_{\varepsilon \rightarrow 0} \text{Sp}(-\Delta^\varepsilon)$  at first, we describe the spectrum of the operator  $\mathbf{A}$ . Namely, the following simple proposition is valid:

**PROPOSITION 2.3** *Let  $0 < \mu_1^\varepsilon \leq \mu_2^\varepsilon \leq \mu_3^\varepsilon \leq \dots \leq \mu_k^\varepsilon \leq \dots \rightarrow \infty$  be the eigenvalues of the Laplace operator in  $L_2(\Omega)$  with Dirichlet boundary condition,  $f_1, f_2, \dots, f_k \dots$  be the corresponding eigenfunctions. Then the spectrum of the operator  $\mathbf{A}$  (2.6) and (2.7) has the following structure:*

$$\text{Sp}(\mathbf{A}) = \{\lambda_0\} \cup \{\lambda_k^-, k = 1, 2, 3, \dots\} \cup \{\lambda_k^+, k = 1, 2, 3, \dots\}, \quad (2.8)$$

where

$$\lambda_0 = \sigma, \quad \lambda_k^\pm = \frac{1}{2} \left( \mu_k + \sigma\rho + \sigma \pm \sqrt{(\mu_k + \sigma\rho + \sigma)^2 - 4\sigma\mu_k} \right).$$

The points  $\lambda_k^\pm$ ,  $k = 1, 2, 3, \dots$ , belong to the discrete spectrum,  $\lambda_0$  is a point of the essential spectrum:

$$0 < \lambda_1^- \leq \lambda_2^- \leq \dots \leq \lambda_k^- \rightarrow \lambda_0 = \sigma < \sigma\rho + \sigma < \lambda_1^+ \leq \lambda_2^+ \leq \dots \leq \lambda_k^+ \rightarrow \infty.$$

The corresponding eigenfunctions have the form

$$U_k^\pm = f_k \begin{pmatrix} 1 \\ \sigma(\sigma - \lambda_k^\pm)^{-1} \end{pmatrix}.$$

**3. Proof of the Theorem 1.2**

**3.1. Verification of the condition (A)**

Let  $\lambda^\varepsilon \in \text{Sp}(-\Delta^\varepsilon)$  converge to some  $\lambda \in \mathbb{R}$ . We prove that  $\lambda$  belongs to the spectrum of the operator  $\mathbf{A}$  (2.6) and (2.7).

Let  $u^\varepsilon$  be the eigenfunction corresponding to  $\lambda^\varepsilon$  such that  $\|u^\varepsilon\|_{0\varepsilon}^2 = 1$  (and so  $\|\nabla^\varepsilon u^\varepsilon\|_{0\varepsilon}^2 = \lambda^\varepsilon$ ). In order to describe the behaviour of  $u^\varepsilon$  in  $\Omega^\varepsilon$  as  $\varepsilon \rightarrow 0$ , we introduce an extension operator  $\Pi_1^\varepsilon : H^1(M^\varepsilon) \rightarrow H^1(\Omega)$  with the following properties:  $\forall u^\varepsilon \in H^1(M^\varepsilon)$

1.  $\Pi_1^\varepsilon u^\varepsilon(x) = u^\varepsilon(\tilde{x}), \quad \forall x \in \Omega^\varepsilon,$
2.  $\|\Pi_1^\varepsilon u^\varepsilon\|_{H^1(\Omega)} \leq \gamma_1 \|u^\varepsilon\|_{H^1(\Omega^\varepsilon)},$  where  $\gamma_1$  does not depend on  $\varepsilon$ .

Such operator exists, see [8].

Also, we introduce the operator  $\Pi_2^\varepsilon : L_2(M^\varepsilon) \rightarrow L_2(\Omega)$

$$\Pi_2^\varepsilon u(x) = \begin{cases} u_i^\varepsilon, & x \in \square^{\varepsilon i}, \\ 0, & x \in \Omega \setminus \bigcup_i \square^{\varepsilon i}, \end{cases}$$

where  $\square_i^\varepsilon$  is the cube with the centre in  $x_i^\varepsilon$  and the side-length  $\varepsilon$ ,  $u_i^\varepsilon = (1/|B_i^\varepsilon|) \int_{B_i^\varepsilon} u^\varepsilon d\tilde{x}$  is the average value of the function  $u^\varepsilon$  over the domain  $B_i^\varepsilon$ .  $\Pi_2^\varepsilon u^\varepsilon$  describes the behaviour of  $u^\varepsilon$  in  $\bigcup_i B_i^\varepsilon$  as  $\varepsilon \rightarrow 0$ . It is easy to see that the following inequality holds:

$$\|\Pi_2^\varepsilon u^\varepsilon\|_{L_2(\Omega)} \leq \gamma_2 \|u^\varepsilon\|_{L_2(M^\varepsilon)}, \text{ where } \gamma_2 \text{ does not depend on } \varepsilon. \tag{3.2}$$

Using (3.1), (3.2) and the imbedding theorem, we obtain the subsequence (still denoted by  $\varepsilon$ ) such that

$$\begin{aligned} \Pi_1^\varepsilon u^\varepsilon &\rightarrow u_1 \in H_0^1(\Omega) \text{ strongly in } L_2(\Omega), \\ \Pi_2^\varepsilon u^\varepsilon &\rightharpoonup u_2 \in L_2(\Omega) \text{ weakly in } L_2(\Omega). \end{aligned}$$

We have two cases of the limit function  $u_1$ .

*Case I*  $u_1 \neq 0$ . One has the following integral equality for any  $w^\varepsilon$  from the definitional domain of the operator  $\Delta^\varepsilon$  with Dirichlet boundary condition:

$$-\int_{M^\varepsilon} \Delta w^\varepsilon u^\varepsilon d\tilde{x} = \lambda^\varepsilon \int_{M^\varepsilon} w^\varepsilon u^\varepsilon d\tilde{x}. \tag{3.3}$$

We choose the following test-function  $w^\varepsilon$ :

$$w^\varepsilon(\tilde{x}) = \begin{cases} w_1(x), & \tilde{x} \in \Omega^\varepsilon \setminus \bigcup_i R_i^\varepsilon, \\ w_1(x) + (w_1(x_i^\varepsilon) - w_1(x)) \Phi\left(\frac{|x - x_i^\varepsilon|}{8d^\varepsilon}\right) \\ + (w_2(x_i^\varepsilon) - w_1(x_i^\varepsilon)) v_i^\varepsilon(\tilde{x}) \Phi\left(\frac{|x - x_i^\varepsilon|}{\varepsilon}\right), & \tilde{x} \in R_i^\varepsilon, \\ w_2(x_i^\varepsilon) + (w_1(x_i^\varepsilon) - w_2(x_i^\varepsilon))(1 - v_i^\varepsilon(\tilde{x})), & \tilde{x} \in B_i^\varepsilon. \end{cases} \tag{3.4}$$

Here  $w_1(x) \in C_0^\infty(\Omega)$ ,  $w_2(x) \in C^\infty(\Omega)$  are arbitrary functions,  $\Phi(r)$  is a smooth function equal to 1 as  $0 \leq r \leq 1/4$  and equal to 0 as  $r \geq 1/2$ ,  $v_i^\varepsilon$  is the eigenfunction of the problem (2.4) corresponding to  $\sigma^\varepsilon$  and normalized by the condition

$$\int_{B_i^\varepsilon} v_i^\varepsilon(\tilde{x}) d\tilde{x} = |B_i^\varepsilon|.$$

At first we investigate the integrals in (3.3) over the domain  $\Omega^\varepsilon$ . One has:

$$\begin{aligned} & \left| \sum_{i=1}^{N(\varepsilon)} \int_{R_i^\varepsilon} \Delta \left\{ (w_1(x_i^\varepsilon) - w_1(x)) \Phi \left( \frac{|x - x_i^\varepsilon|}{8d^\varepsilon} \right) \right\} u^\varepsilon(x) dx \right| \\ & \leq \sum_{i=1}^{N(\varepsilon)} \int_{R_i^\varepsilon \cup D_i^\varepsilon} \left| \nabla^\varepsilon \left\{ (w_1(x_i^\varepsilon) - w_1(x)) \Phi \left( \frac{|x - x_i^\varepsilon|}{8d^\varepsilon} \right) \right\} \cdot \nabla^\varepsilon \Pi_1^\varepsilon u^\varepsilon(x) \right| dx \\ & \quad + C \sum_{i=1}^{N(\varepsilon)} |D_i^\varepsilon| \leq C_1 \sum_{i=1}^{N(\varepsilon)} |D_i^\varepsilon \cup \text{supp} \nabla^\varepsilon \Phi| \rightarrow 0, \quad \varepsilon \rightarrow 0. \end{aligned} \quad (3.5)$$

We have for an arbitrary smooth function  $f(x)$

$$\sum_{i=1}^{N(\varepsilon)} \int_{R_i^\varepsilon} \left| \Delta \left\{ (w_2(x_i^\varepsilon) - w_1(x_i^\varepsilon)) v_i^\varepsilon(\tilde{x}) \Phi \left( \frac{|x - x_i^\varepsilon|}{\varepsilon} \right) \right\} \right|^2 dx < C, \quad (3.6)$$

$$\begin{aligned} & \sum_{i=1}^{N(\varepsilon)} \int_{R_i^\varepsilon} \Delta \left\{ (w_2(x_i^\varepsilon) - w_1(x_i^\varepsilon)) v_i^\varepsilon(\tilde{x}) \Phi \left( \frac{|x - x_i^\varepsilon|}{\varepsilon} \right) \right\} f(x) dx \\ & = \sum_{i=1}^{N(\varepsilon)} f(x_i^\varepsilon) (w_2(x_i^\varepsilon) - w_1(x_i^\varepsilon)) \int_{\partial D_i^\varepsilon} \frac{\partial v_i^\varepsilon}{\partial \bar{n}} ds_x + \bar{o}(1) \\ & = \sum_{i=1}^{N(\varepsilon)} \sigma^\varepsilon \rho \varepsilon^n f(x_i^\varepsilon) (w_2(x_i^\varepsilon) - w_1(x_i^\varepsilon)) + \bar{o}(1) \\ & \rightarrow \sigma \rho \int_{\Omega} f(x) (w_2(x) - w_1(x)) dx, \quad \varepsilon \rightarrow 0. \end{aligned} \quad (3.7)$$

( $ds_x$  is an area form on  $\partial\Omega$ .) We have used the following estimates proved in [9]

$$|D^\alpha v_i^\varepsilon(x)| \leq \begin{cases} C\varepsilon^2 (\ln|x - x_i^\varepsilon|)^{(\alpha)}, & n = 2, \\ \frac{C\varepsilon^n}{|x - x_i^\varepsilon|^{n-2+|\alpha|}}, & n > 2, \end{cases} \quad \tilde{x} \in R_i^\varepsilon \quad \text{and} \quad |x - x_i^\varepsilon| \geq \frac{\varepsilon}{4}, \quad (3.8)$$

$$\int_{R_i^\varepsilon} |v_i^\varepsilon(x)|^2 dx \leq C\varepsilon^{n+2}. \quad (3.9)$$

It follows from (3.6) and (3.7) that  $-\sum_{i=1}^{N(\varepsilon)} \Delta \{ (w_2(x_i^\varepsilon) - w_1(x_i^\varepsilon)) v_i^\varepsilon(x) \Phi(|x - x_i^\varepsilon|/\varepsilon) \} dx$  weakly converges to  $\sigma\rho(w_1 - w_2)$ . Using this fact, (3.1) and (3.5), we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega^\varepsilon} -\Delta w^\varepsilon u^\varepsilon dx = \int_{\Omega} [-\Delta w_1 u_1 + \sigma\rho(w_1 - w_2) u_1] dx. \quad (3.10)$$

In the same way, we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega^\varepsilon} \lambda^\varepsilon w^\varepsilon u^\varepsilon \, dx = \int_{\Omega} \lambda w_1 u_1 \, dx. \tag{3.11}$$

Now, we investigate the behaviour of the integrals in (3.3) over the union of the ‘bubbles’. We have:

$$\begin{aligned} & \sum_{i=1}^{N(\varepsilon)} \int_{B_i^\varepsilon} \Delta^\varepsilon \left[ (w_1(x_i^\varepsilon) - w_2(x_i^\varepsilon))(1 - v_i^\varepsilon(\tilde{x})) \right] u^\varepsilon(\tilde{x}) \, d\tilde{x} \\ &= - \sum_{i=1}^{N(\varepsilon)} u_i^\varepsilon (w_1(x_i^\varepsilon) - w_2(x_i^\varepsilon)) \int_{\partial B_i^\varepsilon} \frac{\partial v^\varepsilon}{\partial \bar{n}} \, ds_x + \bar{o}(1) \\ &= \sigma^\varepsilon \rho \sum_{i=1}^{N(\varepsilon)} u_i^\varepsilon (w_1(x_i^\varepsilon) - w_2(x_i^\varepsilon)) \varepsilon^n + \bar{o}(1) \\ &= \sigma^\varepsilon \rho \int_{\Omega} Q^\varepsilon [w_1 - w_2](x) \cdot \Pi_2 u^\varepsilon(x) \, dx + \bar{o}(1), \quad \varepsilon \rightarrow 0, \end{aligned}$$

where the operator  $Q^\varepsilon : L_2(\Omega) \rightarrow L_2(\Omega)$  is defined by the formula

$$Q^\varepsilon w(x) = \begin{cases} w(x_i^\varepsilon), & x \in \square^{\varepsilon i}, \\ 0, & x \in \Omega \setminus \bigcup_i \square^{\varepsilon i}. \end{cases}$$

Here we use the following estimate proved in [9]:

$$\int_{B_i^\varepsilon} |v_i^\varepsilon(x)|^2 \, dx = |B_i^\varepsilon| + O(\varepsilon^{n+1}). \tag{3.12}$$

It is clear that  $Q^\varepsilon [w_1 - w_2](x)$  strongly converges to  $w_1(x) - w_2(x)$ . Therefore,

$$\lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{N(\varepsilon)} \int_{B_i^\varepsilon} -\Delta \left\{ (w_1^\varepsilon(\tilde{x}) - w_2(x_i^\varepsilon))(1 - v_i^\varepsilon(\tilde{x})) \right\} u^\varepsilon \, d\tilde{x} = \sigma \rho \int_{\Omega} u_2(x)(w_2(x) - w_1(x)) \, dx. \tag{3.13}$$

In a similar manner, we obtain

$$\lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{N(\varepsilon)} \int_{B_i^\varepsilon} \lambda^\varepsilon w^\varepsilon u^\varepsilon \, d\tilde{x} = \rho \int_{\Omega} \lambda w_2 u_2 \, dx. \tag{3.14}$$

Thus, from (3.10), (3.11), (3.13) and (3.14) we conclude that  $u_1$  and  $u_2$  satisfy to the equality:

$$\int_{\Omega} \left[ -\Delta w_1 u_1 + \sigma \rho u_1 (w_1 - w_2) + \sigma \rho u_2 (w_2 - w_1) \right] \, dx = \lambda \int_{\Omega} \left[ u_1 w_1 + \rho u_2 w_2 \right] \, dx \tag{3.15}$$

for any functions  $w_1 \in C_0^\infty(\Omega)$ ,  $w_2 \in C^\infty(\Omega)$ .

It follows easily from (3.15) that  $\lambda$  is an eigenvalue of  $\mathbf{A}$ .

*Case II*  $u_1 = 0$ . We prove that  $\lim_{\varepsilon \rightarrow 0} \lambda^\varepsilon = \sigma$ .

In order to prove this, we construct a suitable approximation  $v^\varepsilon$  for the eigenvalue  $u^\varepsilon$ . We represent the eigenvalue  $u^\varepsilon$  in the form  $u^\varepsilon = v^\varepsilon - g^\varepsilon + w^\varepsilon$ . Here

$$v^\varepsilon = \begin{cases} 0, & \tilde{x} \in \Omega^\varepsilon \setminus \bigcup_i R_i^\varepsilon, \\ u_i^\varepsilon v_i^\varepsilon(x) \Phi\left(\frac{|x - x_i^\varepsilon|}{\varepsilon}\right), & \tilde{x} \in R_i^\varepsilon, \\ u_i^\varepsilon v_i^\varepsilon(\tilde{x}), & \tilde{x} \in B_i^\varepsilon. \end{cases} \quad (3.16)$$

Recall, that  $u_i^\varepsilon$  is the average value of the function  $u^\varepsilon$  over the domain  $B_i^\varepsilon$ ;  $v_i^\varepsilon$  and  $\Phi$  are introduced in the Case I.  $g^\varepsilon$  is the projection of  $v^\varepsilon$  onto the linear span of the eigenfunctions  $u_1, \dots, u_{n^\varepsilon-1}$ , where  $n^\varepsilon$  is a number of the eigenvalue  $\lambda^\varepsilon$ :

$$g^\varepsilon = \sum_{k=1}^{n^\varepsilon-1} u_k^\varepsilon (v^\varepsilon, u_k^\varepsilon)_{0\varepsilon}.$$

From Poincare inequality for the ‘bubble’  $B_i^\varepsilon$ , one has

$$1 = \lim_{\varepsilon \rightarrow 0} \|u^\varepsilon\|_{0\varepsilon}^2 = \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{N(\varepsilon)} (u_i^\varepsilon)^2 |B_i^\varepsilon|. \quad (3.17)$$

Using (3.17) and the properties of  $v_i^\varepsilon$  (3.8), (3.9) and (3.12), we have

$$\|\nabla^\varepsilon v^\varepsilon\|_{0\varepsilon}^2 = \sigma \sum_{i=1}^{N(\varepsilon)} (u_i^\varepsilon)^2 |B_i^\varepsilon| + \bar{o}(1) \xrightarrow{\varepsilon \rightarrow 0} \sigma, \quad (3.18)$$

$$\|v^\varepsilon\|_{0\varepsilon}^2 = \sum_{i=1}^{N(\varepsilon)} (u_i^\varepsilon)^2 |B_i^\varepsilon| + \bar{o}(1) \xrightarrow{\varepsilon \rightarrow 0} 1, \quad (3.19)$$

$$\|\Delta^\varepsilon v^\varepsilon\|_{0\varepsilon}^2 \leq C, \quad (3.20)$$

$$\frac{1}{2} \|u^\varepsilon - v^\varepsilon\|_{0\varepsilon}^2 \leq \|u^\varepsilon\|_{0,\Omega^\varepsilon}^2 + \|v^\varepsilon\|_{0,\Omega^\varepsilon}^2 + \sum_{i=1}^{N(\varepsilon)} \left\{ \|u^\varepsilon - u_i^\varepsilon\|_{0,B_i^\varepsilon}^2 + \|v^\varepsilon - u_i^\varepsilon\|_{0,B_i^\varepsilon}^2 \right\} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (3.21)$$

Using the Bessel’s inequality and (3.21), one has

$$\|g^\varepsilon\|_{0\varepsilon}^2 = \sum_{k=1}^{n^\varepsilon-1} |(v^\varepsilon, u_k^\varepsilon)_{0\varepsilon}|^2 = \sum_{k=1}^{n^\varepsilon-1} |(v^\varepsilon - u^\varepsilon, u_k^\varepsilon)_{0\varepsilon}|^2 \leq \|u^\varepsilon - v^\varepsilon\|_{0\varepsilon}^2 \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (3.22)$$

$$\|\nabla^\varepsilon g^\varepsilon\|_{0\varepsilon}^2 = \sum_{k=1}^{n^\varepsilon-1} \lambda_k^\varepsilon |(v^\varepsilon - u^\varepsilon, u_k^\varepsilon)_{0\varepsilon}|^2 \leq \lambda^\varepsilon \|u^\varepsilon - v^\varepsilon\|_{0\varepsilon}^2 \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (3.23)$$

Now, we estimate the remainder  $w^\varepsilon$ . It follows from (3.21) and (3.22) that

$$\|w^\varepsilon\|_{0\varepsilon} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

It is clear that  $\hat{v}^\varepsilon = v^\varepsilon - g^\varepsilon$  is orthogonal to the eigenfunctions  $u_1^\varepsilon, \dots, u_{n^\varepsilon-1}^\varepsilon$ . Therefore, we have

$$\lambda^\varepsilon \equiv \|\nabla^\varepsilon u^\varepsilon\|_{0\varepsilon}^2 \leq \frac{\|\nabla^\varepsilon \hat{v}^\varepsilon\|_{0\varepsilon}^2}{\|\hat{v}^\varepsilon\|_{0\varepsilon}^2}$$

and so

$$\begin{aligned} \|\nabla^\varepsilon w^\varepsilon\|^2 &\leq 2|(\nabla^\varepsilon \hat{v}^\varepsilon, \nabla^\varepsilon w^\varepsilon)_{0\varepsilon}| + \left| \frac{\|\nabla^\varepsilon \hat{v}^\varepsilon\|_{0\varepsilon}^2}{\|\hat{v}^\varepsilon\|_{0\varepsilon}^2} - \|\nabla^\varepsilon \hat{v}^\varepsilon\|_{0\varepsilon}^2 \right| \\ &= |(\Delta^\varepsilon v^\varepsilon, w^\varepsilon)_{0\varepsilon}| + |(\nabla^\varepsilon g^\varepsilon, \nabla^\varepsilon w^\varepsilon)_{0\varepsilon}| + \left| \frac{\|\nabla^\varepsilon \hat{v}^\varepsilon\|_{0\varepsilon}^2}{\|\hat{v}^\varepsilon\|_{0\varepsilon}^2} - \|\nabla^\varepsilon \hat{v}^\varepsilon\|_{0\varepsilon}^2 \right|. \end{aligned}$$

Using (3.18)–(3.23), we obtain that

$$\lim_{\varepsilon \rightarrow 0} \|\nabla^\varepsilon w^\varepsilon\|_{0\varepsilon} = 0. \tag{3.24}$$

Therefore, from (3.18), (3.23) and (3.24) we have

$$\lambda^\varepsilon \equiv \|\nabla^\varepsilon u^\varepsilon\|_{0\varepsilon}^2 = \|\nabla^\varepsilon v^\varepsilon\|_{0\varepsilon}^2 + \bar{o}(1) \rightarrow \sigma, \quad \varepsilon \rightarrow 0.$$

The fulfilment of the condition (A) is proved.

### 3.2. Verification of the condition (B)

Let  $\lambda$  belong to the spectrum of the operator  $\mathbf{A}$ . We prove that there exists  $\lambda^\varepsilon \in \text{Sp}(-\Delta^\varepsilon)$  such that  $\lim_{\varepsilon \rightarrow 0} \lambda^\varepsilon = \lambda$ .

Proving this indirectly we assume the opposite. Then the subsequence (still denoted by  $\varepsilon$ ) exists and a positive number  $\delta$  exists such that

$$\text{dist}(\lambda, \text{Sp}(-\Delta^\varepsilon)) > \delta. \tag{3.25}$$

Since  $\lambda \in \text{Sp}(\mathbf{A})$  there exists the function  $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \in \mathbf{H}$ , such that  $f \notin \text{Im}(\mathbf{A} - \lambda \mathbf{I})$ , where  $\mathbf{I}$  is the identical operator.

We consider the following problem

$$-\Delta^\varepsilon u^\varepsilon - \lambda u^\varepsilon = f^\varepsilon, \quad \tilde{x} \in M^\varepsilon, \quad u^\varepsilon = 0, \quad \tilde{x} \in \partial M^\varepsilon, \tag{3.26}$$

where  $f^\varepsilon(\tilde{x}) \in L_2(M^\varepsilon)$ :

$$f^\varepsilon(\tilde{x}) = \begin{cases} f_1(x), & \tilde{x} \in \Omega^\varepsilon, \\ \frac{1}{|\square_i^\varepsilon|} \int_{\square_i^\varepsilon} f_2(x) dx, & \tilde{x} \in B_i^\varepsilon. \end{cases}$$

In consequence of (3.25), the problem (3.26) has the unique solution  $u^\varepsilon(\tilde{x}) \in H_0^1(M^\varepsilon)$  such that

$$\|u^\varepsilon\|_{0\varepsilon}^2 \leq \frac{\|f^\varepsilon\|_{0\varepsilon}^2}{\delta} \leq C_1,$$

where  $C_1$  does not depend on  $\varepsilon$ . Moreover, it is easy to see that

$$\|\nabla u^\varepsilon\|_{0\varepsilon}^2 \leq 2\left(\|f^\varepsilon\|_{0\varepsilon} \cdot \|u^\varepsilon\|_{0\varepsilon} + \lambda \|u^\varepsilon\|_{0\varepsilon}^2\right) \leq C_2,$$

where  $C_2$  does not depend on  $\varepsilon$ .

Then there exists a subsequence (still denoted by  $\varepsilon$ ) such that

$$\begin{aligned}\Pi_1^\varepsilon u^\varepsilon &\rightarrow u_1 \in H_0^1(\Omega) \text{ strongly in } L_2(\Omega), \\ \Pi_2^\varepsilon u^\varepsilon &\rightharpoonup u_2 \in L_2(\Omega) \text{ weakly in } L_2(\Omega).\end{aligned}$$

Using the same calculations as in Case I of the previous section, we obtain

$$\mathbf{A}U - \lambda U = f, \quad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Therefore,  $f \in \text{Im}(\mathbf{A} - \lambda I)$  and we obtain a contradiction. The fulfilment of condition (B) and so Theorem 2.2 are proved.

#### 4. Case of the ‘bubbles’ with different radii

We consider the manifold  $M^\varepsilon$  constructed in the same way as in Section 2. We again suppose that the radii of the ‘holes’  $D_i^\varepsilon$  satisfy (2.1), while the radii of the ‘bubbles’ are the following

$$b_i^\varepsilon = b(x_i^\varepsilon)\varepsilon,$$

where  $b(x)$  be a positive function from  $C^\infty(\overline{\Omega})$ .

We denote

$$\rho(x) = (b(x))^n \cdot \omega_n, \quad \sigma(x) = \begin{cases} \frac{a}{4(b(x))^2}, & n = 2, \\ \frac{n-2}{2} \cdot \frac{a^{n-2} \omega_{n-1}}{(b(x))^n \omega_n}, & n > 2. \end{cases}$$

Let  $\mathbf{H}$  be the Hilbert space of real-valued two-vector-function from  $L_2(\Omega) \times L_2(\Omega)$  with the scalar product

$$(u, v)_0 = \int_{\Omega} [u_1(x)v_1(x) + \rho(x)u_2(x)v_2(x)] dx.$$

We have the analogous to the Theorem 2.2 result.

**THEOREM 4.1** *The spectrum  $\text{Sp}(-\Delta^\varepsilon)$  of the problem (2.3) converges in the Hausdorff sense to the spectrum  $\text{Sp } \mathbf{A}$  of the self-adjoint operator  $\mathbf{A} : \mathbf{H} \rightarrow \mathbf{H}$  defined by the operation*

$$U = \begin{pmatrix} -\Delta u_1 + \sigma(x)\rho(x)(u_1 - u_2) \\ \sigma(x)(u_2 - u_1) \end{pmatrix}, \quad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad (4.1)$$

and the boundary condition (2.7).

The proof is based on the following

**LEMMA 4.2** *The essential spectrum  $\sigma_{\text{ess}}(\mathbf{A})$  of the operator  $\mathbf{A}$  contains the segment  $[\min_{x \in \overline{\Omega}} \sigma(x), \max_{x \in \overline{\Omega}} \sigma(x)]$ .*

*Proof* Let  $\sigma_0$  is a point of the segment  $[\min_{x \in \overline{\Omega}} \sigma(x), \max_{x \in \overline{\Omega}} \sigma(x)]$ . Then there exists the point  $x_0 \in \overline{\Omega}$  such that  $\sigma(x_0) = \sigma_0$ . We denote  $\rho_0 = \rho(x_0)$ . First, we suppose that  $x_0 \in \text{int}\Omega$ .

In order to prove that  $\sigma_0 \in \sigma_{\text{ess}}(\mathbf{A})$ , we construct the sequence  $U^m \in \mathbf{H}$  with the following properties: (1)  $U^m$  is bounded, (2)  $U^m$  is noncompact, (3)  $(\mathbf{A} - \sigma_0 \mathbf{I})U^m \xrightarrow[n \rightarrow \infty]{} 0$  strongly in  $\mathbf{H}$ .

We denote by  $\mathbf{A}_0$  the operation

$$\mathbf{A}_0 U = \begin{pmatrix} -\Delta u_1 + \sigma_0 \rho_0 (u_1 - u_2) \\ \sigma_0 (u_2 - u_1) \end{pmatrix}, \quad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{4.2}$$

Let  $B^n$  be the ball with the centre in the point  $x_0$  and the radius  $1/n$ . Let  $\mu_k^n$  be the  $k$ th eigenvalue of the Laplace operator on the domain  $B^n$  with Dirichlet boundary conditions. Let  $e_k^n$  be the corresponding eigenfunction, such that  $(e_k^n, e_l^n)_{L_2(B^n)} = \delta_{kl}$ . We redefine  $e_k^n$  by zero in  $\Omega \setminus B^n$ .

We denote

$$\lambda_k^n = \frac{1}{2} \left( \mu_k^n + \sigma_0 \rho_0 + \sigma_0 - \sqrt{(\mu_k^n + \sigma_0 \rho_0 + \sigma_0)^2 - 4\sigma_0 \mu_k^n} \right).$$

Analogous to Proposition 2.3, we have that  $\lambda_k^n$  is the eigenvalue of the operator defined by the operation  $\mathbf{A}_0$  and Dirichlet boundary condition on  $\partial B^n$ . The corresponding eigenfunction has the form:

$$\hat{U}_k^n = c_n^k e_k^n \begin{pmatrix} 1 \\ \sigma_0 (\sigma_0 - \lambda_k^n)^{-1} \end{pmatrix}, \quad c_n^k \text{ is a constant.}$$

We choose the constant factor  $c_n^k$  in such a way as to have  $\|\hat{U}_k^n\|_0 = 1$ . Then one has

$$(c_n^k)^2 = \left[ \|e_k^n\|_{L_2(\Omega)}^2 + \|e_k^n\|_{L_{2\rho}(\Omega)}^2 \left( \frac{\sigma_0}{\sigma_0 - \lambda_k^n} \right)^2 \right]^{-1}.$$

It is easy to obtain the inequalities

$$\min_{x \in \Omega} \rho(x) \leq \|e_k^n\|_{L_{2\rho}}^2 \leq \max_{x \in \Omega} \rho(x), \tag{4.3}$$

$$|\lambda_k^n - \sigma_0| \leq \frac{M_1}{\mu_k^n}, \quad M_1 \text{ does not depend on } n \text{ and } k, \tag{4.4}$$

$$c_n^k \leq \frac{M_2}{\mu_k^n}, \quad M_2 \text{ does not depend on } n \text{ and } k. \tag{4.5}$$

Now we redefine  $\hat{U}_k^n$  by zero in  $\Omega \setminus B^n$  and set

$$U^n = \hat{U}_k^n \Phi_n(|x - x_0|), \quad k = k(n),$$

where  $\Phi_n(r) = \Phi(nr)$ ,  $\Phi(r)$  is a smooth function equal to 1 as  $r \leq (1/2)$  and equal to 0 as  $r \geq 1$ . The number  $k = k(n)$  we choose later. It is clear that  $U^n$  is from the definitional domain of the operator  $\mathbf{A}$ .

It is easy to see that  $\|U^n\|_0 \leq \|\hat{U}_k^n\|_0 = 1$  and exists constant  $C > 0$  such that  $C \leq \|U^n\|_0$ . On the other hand,  $U^n$  converges nearly everywhere to zero (since the support of  $U^n$  tightens to the point  $x_0$ ). Therefore,  $U^n$  is non-compact.

We have

$$\|(\mathbf{A} - \sigma_0 \mathbf{I})U^n\|_0 \leq \|(\mathbf{A}_0 - \sigma_0 \mathbf{I})U^n\|_0 + \|(\mathbf{A} - \mathbf{A}_0)U^n\|_0. \tag{4.6}$$

It is obvious that the second term in (4.6) converges to zero as  $n \rightarrow \infty$

$$\|(\mathbf{A} - \mathbf{A}_0)U^n\|_0 \leq M \max_{x \in B^n} (|\sigma(x)\rho(x) - \sigma_0\rho_0| + |\sigma(x) - \sigma_0|) \leq \frac{M}{n}.$$

Now we estimate the first term. Using the inequalities (4.3)–(4.5), we have

$$\begin{aligned} \|(\mathbf{A}_0 - \sigma_0\mathbf{I})U^n\|_0 &\leq \|((A_0 - \sigma_0\mathbf{I})\hat{U}_k^n) \cdot \Phi_n\|_0 + \|2c_n^k \nabla^\varepsilon e_k^n \cdot \nabla^\varepsilon \Phi_n + c_n^k e_k^n \cdot \Delta^\varepsilon \Phi_n\|_{L_2(\Omega)} \\ &\leq |\lambda_k^n - \sigma_0| + M \left( \frac{n}{\sqrt{\mu_k^n}} + \frac{n^2}{\mu_k^n} \right). \end{aligned}$$

We choose  $k=k(n)$  such that  $\mu_k^n \geq n^3$ . Then the first term also converges to zero. We obtain that  $(\mathbf{A} - \sigma_0\mathbf{I})U^n \rightarrow 0$ ,  $n \rightarrow \infty$  and so  $\sigma_0$  is the point of  $\sigma_{\text{ess}}(\mathbf{A})$ .

We have proved that any  $\sigma_0$  such that  $\sigma_0 = \sigma(x_0)$  with  $x_0 \in \text{int}\Omega$  belongs to the essential spectrum. The set of such  $\sigma_0$  is dense in the segment  $[\min_{x \in \overline{\Omega}} \sigma(x), \max_{x \in \overline{\Omega}} \sigma(x)]$ .

Since the essential spectrum is a closed set we conclude that Lemma is valid for any point of this segment.

*Proof of Theorem 4.1* The fulfilment of the condition (B) is proved in the similar way as it proved in Theorem 2.2. As for the property (A), similar to the proof of the Theorem 2.2 computations show that if  $\lambda^\varepsilon \in \text{Sp}(-\Delta^\varepsilon)$  converges to some  $\lambda$ , then  $\lambda$  is an eigenvalue of the operator  $\mathbf{A}$  (Case 1) or  $\lambda \in [\min_{x \in \overline{\Omega}} \sigma(x), \max_{x \in \overline{\Omega}} \sigma(x)]$  (Case 2) and therefore, by Lemma 4.2  $\lambda$  belongs to the essential spectrum of the operator  $\mathbf{A}$ .

## 5. One generalization

We consider the manifold  $M^\varepsilon$  whose structure differs from the manifold in previous sections. Namely, let  $\Omega_1^\varepsilon$  and  $\Omega_2^\varepsilon$  be the two copies ('sheets') of the domain with 'holes'  $\Omega^\varepsilon$  constructed in Section 2. We denote by  $\partial D_{ki}^\varepsilon$ ,  $k=1,2$  the boundary of the  $i$ th 'hole' on the  $k$ th 'sheet'. Let  $B_i^\varepsilon$  be the  $n$ -dimensional sphere of radius  $b^\varepsilon$  with two truncated small segments ('bubble'):

$$\begin{aligned} B_i^\varepsilon = \left\{ x = (\theta_1, \theta_2, \dots, \theta_n) : \theta_1 \in [0, 2\pi], \quad \theta_i \in [0, \pi], \quad i = 2, \dots, n-1, \right. \\ \left. \theta_n \in \left[ \arcsin \frac{d^\varepsilon}{b^\varepsilon}, \pi - \arcsin \frac{d^\varepsilon}{b^\varepsilon} \right] \right\}. \end{aligned}$$

Recall, that  $d^\varepsilon$  is the radius of the 'holes'  $D_{ki}^\varepsilon$ . Suppose that the boundaries of the 'holes'  $\partial D_{1i}^\varepsilon$  and  $\partial D_{2i}^\varepsilon$  are connected by means of the 'bubble'  $B_i^\varepsilon$ . Namely, we identify  $\partial D_{1i}^\varepsilon$  and the first component of the boundary of  $B_i^\varepsilon$

$$[\partial B_i^\varepsilon]_1 = \left\{ x \in B_i^\varepsilon : \theta_n = \arcsin \frac{d^\varepsilon}{b^\varepsilon} \right\}.$$

and identify  $\partial D_{2i}^\varepsilon$  and the second component of the boundary of  $B_i^\varepsilon$

$$[\partial B_i^\varepsilon]_2 = \left\{ x \in B_i^\varepsilon : \theta_n = \pi - \arcsin \frac{d^\varepsilon}{b^\varepsilon} \right\}.$$

As a result, we obtain a manifold  $M^\varepsilon$  (the fragment is represented on Figure 2)

$$M^\varepsilon = \overline{\Omega}_1^\varepsilon \cup \left( \bigcup_{i=1}^{N(\varepsilon)} B_i^\varepsilon \right) \cup \overline{\Omega}_2^\varepsilon.$$

We denote by  $\tilde{x}$  points of this manifold. If the point  $\tilde{x}$  belongs to  $\Omega_k^\varepsilon$ , we assign the pair  $(x, k)$  to  $\tilde{x}$  where  $x$  is the corresponding point in  $\Omega$ .

We introduce on  $M^\varepsilon$  a metric tensor  $g_{\alpha\beta}^\varepsilon(\tilde{x})$  such that the metric is the flat Euclidean metric on  $\Omega_1^\varepsilon \cup \Omega_2^\varepsilon$  and the spherical metric on  $B_i^\varepsilon$  outside some small neighbourhood of the boundary of  $B_i^\varepsilon$ .

We suppose that the radii  $d^\varepsilon$  of the ‘holes’ satisfy (2.1) and the ‘bubbles’ radii have the form (2.2).

We consider the eigenvalue problem (2.3) on this manifold. An asymptotic behaviour of the spectrum  $\text{Sp}(-\Delta^\varepsilon)$  of this problem as  $\varepsilon \rightarrow 0$ , is described by the following theorem.

**THEOREM 5.1** *The set  $\text{Sp}(-\Delta^\varepsilon)$  converges in the Hausdorff sense to the spectrum  $\text{Sp } \mathbf{A}$  of the operator  $\mathbf{A}$  defined by the operation*

$$U = \begin{pmatrix} -\Delta u_1 + \sigma\rho(u_1 - v) \\ \sigma(2v - u_1 - u_2) \\ -\Delta u_2 + \sigma\rho(u_2 - v) \end{pmatrix}, \quad U = \begin{pmatrix} u_1 \\ v \\ u_2 \end{pmatrix}$$

and the boundary conditions  $u_1 = u_2 = 0$ ,  $x \in \partial\Omega$ , where  $\sigma$  is defined by the formula (2.5),  $\rho = b^n \omega_n$ .

The spectrum of the operator  $\text{Sp}(\mathbf{A})$  has the structure (2.8), where  $\lambda_0 = 2\sigma$ ,  $\lambda_k^\pm = 1/2(\mu_k + \sigma\rho + 2\sigma \pm \sqrt{(\mu_k + \sigma\rho + 2\sigma)^2 - 8\sigma\mu_k})$ .

*Proof* The proof is similar to that of Theorem 2.2. We only underline the main difference – a choice of a test-function  $w^\varepsilon$  in the integral equality (3.3). Namely,

$$w^\varepsilon(\tilde{x}) = \begin{cases} w_k(x), & \tilde{x} \in \Omega_k^\varepsilon \setminus \bigcup_i R_{ki}^\varepsilon, k = 1, 2, \\ w_k(x) + (w_k(x_i^\varepsilon) - w_k(x))\Phi\left(\frac{|x - x_i^\varepsilon|}{8d^\varepsilon}\right) \\ \quad + (v_{ki}^\varepsilon(\tilde{x}) - w_k(x_i^\varepsilon))\Phi\left(\frac{|x - x_i^\varepsilon|}{\varepsilon}\right), & \tilde{x} \in R_{ki}^\varepsilon, \quad k = 1, 2, \\ W(x_i^\varepsilon) + (v_{1i}(\tilde{x}) - W(x_i^\varepsilon))\Psi(\theta_n) \\ \quad + (v_{2i}(\tilde{x}) - W(x_i^\varepsilon))\Psi(\pi - \theta_n), & \tilde{x} = (\theta_1, \dots, \theta_n) \in B_i^\varepsilon. \end{cases}$$

Here  $R_{ki}^\varepsilon = \{\tilde{x} = (x, k) \in \Omega_k^\varepsilon : d^\varepsilon \leq |x - x_i^\varepsilon| < \varepsilon/2\}$  ( $k = 1, 2$ ),  $w_1, w_2(x) \in C_0^\infty(\Omega)$ ,  $W(x) \in C^\infty(\Omega)$ ,  $\Phi(r)$  is a smooth function equal to 1 as  $0 \leq r \leq (1/4)$  and equal to 0 as  $r \geq (1/2)$ ,  $\Psi(\theta)$  is a smooth function equal to 1 as  $0 \leq \theta \leq (\pi/6)$  and equal to 0 as  $\theta \geq (\pi/3)$ ,  $v_{1i}^\varepsilon$  and  $v_{2i}^\varepsilon$  are the following functions:

$$\begin{cases} \Delta^\varepsilon v_{1i}^\varepsilon = 0, & \tilde{x} \in R_{1i}^\varepsilon \cup \widehat{B}_{1i}, \\ v_{1i}^\varepsilon = w_1(x_i^\varepsilon), & \tilde{x} \in S_{1i}^\varepsilon, \\ v_{1i}^\varepsilon = W(x_i^\varepsilon), & \tilde{x} \in \Gamma_i^\varepsilon \cup \widehat{B}_{2i}; \end{cases} \quad \begin{cases} \Delta^\varepsilon v_{2i}^\varepsilon = 0, & \tilde{x} \in R_{2i}^\varepsilon \cup \widehat{B}_{2i}, \\ v_{2i}^\varepsilon = w_2(x_i^\varepsilon), & \tilde{x} \in S_{2i}^\varepsilon, \\ v_{2i}^\varepsilon = W(x_i^\varepsilon), & \tilde{x} \in \Gamma_i^\varepsilon \cup \widehat{B}_{1i}, \end{cases}$$

$$\begin{aligned}
S_{ki}^\varepsilon &= \left\{ \tilde{x} = (x, k) \in \Omega_k^\varepsilon: |x - x_i^\varepsilon| = \frac{\varepsilon}{2} \right\}, \quad k = 1, 2, \\
\widehat{B}_{1i}^\varepsilon &= \left\{ \tilde{x} = (\theta_1, \dots, \theta_n) \in B_i^\varepsilon: \arcsin \frac{d^\varepsilon}{b^\varepsilon} \leq \theta_n < \frac{\pi}{2} \right\}, \\
\widehat{B}_{2i}^\varepsilon &= \left\{ \tilde{x} = (\theta_1, \dots, \theta_n) \in B_i^\varepsilon: \frac{\pi}{2} < \theta_n \leq \pi - \arcsin \frac{d^\varepsilon}{b^\varepsilon} \right\}, \\
\Gamma_i^\varepsilon &= \left\{ \tilde{x} = (\theta_1, \dots, \theta_n) \in B_i^\varepsilon: \theta_n = \frac{\pi}{2} \right\}.
\end{aligned}$$

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