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Positivity and time behavior of a linear reaction–diffusion system, non-local in space and time

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SUMMARY

We consider a general linear reaction–diffusion system in three dimensions and time, containing diffusion (local interaction), jumps (nonlocal interaction) and memory effects. We prove a maximum principle and positivity of the solution and investigate its asymptotic behavior. Moreover, we give an explicit expression of the limit of the solution for large times. In order to obtain these results, we use the following method: We construct a Riemannian manifold with complicated microstructure depending on a small parameter. We study the asymptotic behavior of the solution to a simple diffusion equation on this manifold as the small parameter tends to zero. It turns out that the homogenized system coincides with the original reaction–diffusion system. Using this and the facts that the diffusion equation on manifolds satisfies the maximum principle and its solution converges to a easily calculated constant, we can obtain analogous properties for the original system. Copyright © 2008 John Wiley & Sons, Ltd.

KEY WORDS: homogenization; diffusion–reaction systems; Riemannian manifold; maximum principle; positive solutions

1. INTRODUCTION

Linear reaction–diffusion systems play an important role in applied mathematics. They describe, for instance, the transport of particles of various species in a random medium and the transformation of the particles into each other, which means linear reactions. The transport can be forced by

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local (diffusion) and nonlocal interaction (jumps) of the particles with the medium. Moreover, the transport and the reactions can be nonlocal in time (memory effects).

Let Ω be a bounded domain in $\mathbb{R}^3$ and $[0,\infty)$ the time interval. We consider m kinds of species with concentrations $u_k = u_k(x,t)$, for $k=1,\dots,m$, $x\in\Omega$ and $t\in[0,\infty)$. In the following, we give every species a special color and call transformations of the particles into each other 'changing of color'.

We consider a general linear reaction–diffusion system in $\Omega\times[0,\infty)$, describing the mentioned reactions and transport effects:

$$\begin{aligned} 0 = {} & \frac{\partial u_k}{\partial t} - \Delta u_k + \sum_{l=1|l\neq k}^{m} A_{kl}(x)(u_k(x,t)-u_l(x,t)) \\ & + \sum_{l=1}^{m}\int_\Omega B_{kl}(x,y)(u_k(x,t)-u_l(y,t))\,\mathrm{d}y + \frac{\partial}{\partial t}\int_0^t C_k(x)\mathrm{e}^{-D_k(x)(t-\tau)}u_k(x,\tau)\,\mathrm{d}\tau \\ & + \sum_{l=1|l\neq k}^{m}\frac{\partial}{\partial t}\int_0^t E_{kl}(x)\mathrm{e}^{-F_{kl}(x)(t-\tau)}(u_k(x,\tau)+u_l(x,\tau))\,\mathrm{d}\tau \\ & + \sum_{l=1}^{m}\int_\Omega\frac{\partial}{\partial t}\int_0^t G_{kl}(x,y)\mathrm{e}^{-H_{kl}(x,y)(t-\tau)}(u_k(x,\tau)+u_l(y,\tau))\,\mathrm{d}\tau\,\mathrm{d}y \end{aligned} \tag{1}$$

$$u_k(x,0) = f_k(x) \tag{2}$$

$$0 = \frac{\partial u_k}{\partial \mathbf{n}} + \sum_{l=1}^{m} U_{kl}\frac{\partial u_l}{\partial \mathbf{n}}, \quad x\in\partial\Omega \tag{3}$$

with $k=1,\dots,m$, a smooth function $f_k(x)$, strictly positive smooth functions $A_{kl}(x)$, $B_{kl}(x,y),\dots,$ $H_{kl}(x,y)$ satisfying the following conditions

$$\begin{aligned} A_{kl}(x) &= A_{lk}(x), \quad B_{kl}(x,y) = B_{lk}(y,x) \\ E_{kl}(x) &= E_{lk}(x), \quad F_{kl}(x) = F_{lk}(x) \\ G_{kl}(x,y) &= G_{lk}(y,x), \quad H_{kl}(x,y) = H_{lk}(y,x) \\ A_{kl}(x) &> E_{kl}(x), \quad B_{kl}(x,y) > G_{kl}(x,y) \end{aligned} \tag{4}$$

and a symmetric matrix $U=\{U_{kl},\ k,l=1,m\}$ consisting of zeros and unities with one and only one unity in every line.

From a physical point of view, our system can be understood in the following way. System (1) describes the diffusion of particles of m colors with concentrations u_k, which can change their coordinates and colors, in the following way in each point of Ω:

- Change their color (this is described by the terms with A_{kl}).
- Jump from one point of Ω to another (described by the B_{kk} terms).
- Jump and change its color simultaneously (described by the B_{kl} terms with $k\neq l$).
- Disappear and appear after some period of time in the same place without change in color (this is described by the terms with C_k and D_k).

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- Disappear and appear after some period of time in the same point of Ω, but with another color (described by the terms with E_{kl} and F_{kl}). This process also makes a contribution to the terms with A_{kl}.
- Disappear and appear after some period of time without change in color, but in another point of Ω (described by the terms with G_{kk} and H_{kk}). This process also makes a contribution to the terms with B_{kk}.
- Disappear and appear after some period of time with another color and in another point (described by the terms with G_{kl} and H_{kl} for $k \neq l$). This process also makes a contribution to the terms with B_{kl} for $k \neq l$.

When a particle with the kth color reaches the boundary of Ω, it is reflected and if $U_{kl} = 1$ for some $l \neq k$, it changes to the lth color.

Important for the positivity of the solution to our system is the absence of differential operators on the off-diagonal of the main part. This problem was investigated in [1] for general linear drift–diffusion systems without memory effects.

We analyze the system (1)–(3), transforming the analytical difficulties into geometric ones but for a much simpler equation. The idea of this method comes from the article [2], where the authors consider the diffusion equation on a Riemannian manifold with a complicated microstructure. After homogenization they obtain a system of equations, which describes nonlocal spatial and time interactions of a system of various species. In some sense we solve the inverse to this problem such that we construct a special Riemannian manifold M^ε and homogenize the diffusion equation (Theorem 3). As a result we obtain the desired system (1)–(3) and are able to prove some of its important properties (Theorems 1 and 2).

The homogenization of the parabolic equation was studied by many authors (see, for example, the monographs [3–8] and the references therein). The effect of the appearance of memory terms in the homogenized equation was also investigated in many articles, see, in particular, [9–14]. Homogenization problems on manifolds with complicated microstructure were studied, except [2], in [15–19].

This paper is organized as follows. In Section 2 we formulate our main results and give an idea about how to construct the manifold M^ε. Sections 3–5 are devoted to the homogenization of the diffusion equation on this manifold: in Section 3 we give an explicit construction of M^ε, in Section 4 we formulate the homogenization theorem and prove it in Section 5. In Section 6 we prove the main results. The proof is based on the previously obtained homogenization result and on a uniform (with respect to ε) Poincare inequality.

2. THE IDEAS AND MAIN RESULTS

It is well known that the solution $u(x,t)$ to the initial-boundary value problem for the simple diffusion equation

$$\frac{\partial u}{\partial t} - \Delta u = 0 \tag{5}$$

$$u(x,0) = f(x) \tag{6}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega \tag{7}$$

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satisfies the following properties:

(I) $\max_{x\in\Omega}\max_{t>0} u(x,t)=\max_{x\in\Omega} f(x)$ (maximum principle).
(II) If the function $f(x)$ is nonnegative, then $u(x,t)$ is nonnegative for all $t>0$ (conservation of positivity).
(III) $u(x,t)$ converges to the constant $M=1/|\Omega|\int_\Omega f(x)\,\mathrm{d}x$ as $t\to\infty$, where $|\Omega|$ is the volume of the domain Ω. M is the solution to the stationary equation.
(IV) $\int_\Omega u(x,t)\,\mathrm{d}x=\int_\Omega f(x)\,\mathrm{d}x$ for $t>0$ (conservation of mass).

These properties are valid for general linear evolution problems, conserving positivity (see, e.g. [20, 21]).

The goal of the present paper is to prove analogous statements I, II and the first part of III for problem (1)–(3). It is not possible to prove all properties I–IV because of the memory effects contained in the system: A constant function equal to the time limit of the solution, $u(x)\equiv M$, is not to be a solution to the system, i.e. the time limit M is not necessarily a solution to the stationary equation (second part of property III). Moreover, the conservation of mass can be invalid (property IV).

Theorem 1
System (1)–(3) has a unique solution $u(x,t)=(u_1(x,t),\ldots,u_m(x,t))$ with the following properties:

(1) If $M:=\max_k\max_{x\in\Omega} f_k(x)\geqslant 0$, we have

$$u_k(x,t)\leqslant M \text{ for almost all } (x,t)\in\Omega\times[0,\infty)\quad\forall k$$

(2) If $m:=\min_k\min_{x\in\Omega} f_k(x)\leqslant 0$, we have

$$m\leqslant u_k(x,t) \text{ for almost all } (x,t)\in\Omega\times[0,\infty)\quad\forall k$$

Corollary
Let $f_k(x)\geqslant 0$, $k=1,\ldots,m$. Then $u_k(x,t)\geqslant 0$ for almost all $(x,t)\in\Omega\times[0,\infty)$, $\forall k$.

Theorem 2
Let $u(x,t)=(u_1(x,t),\ldots,u_m(x,t))$ be the solution to (1)–(3). Then, $\forall k$ $u_k(x,t)$ converges in $L_2(\Omega)$ as $t\to\infty$ to the constant

$$L=\frac{\sum_{k=1}^m\int_\Omega f_k(x)\,\mathrm{d}x}{m\cdot|\Omega|+\sum_{k=1}^m\int_\Omega(C_k/D_k)\,\mathrm{d}x+2\sum_{k,l=1|k\neq l}^m\int_\Omega(E_{kl}/F_{kl})\,\mathrm{d}x+2\sum_{k,l=1}^m\int_\Omega\int_\Omega(G_{kl}/H_{kl})\,\mathrm{d}x\,\mathrm{d}y}$$

In order to prove these theorems, we use the following method. We construct a special Riemannian manifold $\widetilde{M}^\varepsilon$, called the main manifold, depending on a small parameter ε. On $\widetilde{M}^\varepsilon$ we consider the initial-boundary problem for the usual diffusion equation

$$\frac{\partial u^\varepsilon}{\partial t}-\Delta^\varepsilon u^\varepsilon=0,\quad (\tilde{x},t)\in\widetilde{M}^\varepsilon\times[0,T] \tag{8}$$

$$u^\varepsilon(\tilde{x},0)=f^\varepsilon(\tilde{x}) \tag{9}$$

$$\frac{\partial u^\varepsilon}{\partial \mathbf{n}}=0,\quad \tilde{x}\in\partial\widetilde{M}^\varepsilon \tag{10}$$

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where f^{ε} is a smooth function, and Δ^{ε} is the Laplace–Beltrami operator. We prove that it is possible to choose such manifolds $\widetilde{M}^{\varepsilon}$ and initial functions f^{ε} such that the solution to (8)–(10) $u^{\varepsilon}(\tilde{x},t)$ converges (in a certain sense) to the solution to (1)–(3) $u(x,t)=(u_1(x,t),\ldots,u_m(x,t))$ as $\varepsilon\to 0$. It is well known that the statements I–III are still true for problem (8)–(10). Using the convergence of $u^{\varepsilon}(\tilde{x},t)$ to $u(x,t)$, we will extend the statements I–III to and problem (1)–(3).

It seems to be possible to prove Theorems 1 and 2 directly analyzing (1)–(3). This is done for some particular cases. Our method gives a microscopic interpretation of the terms of the system as diffusing particles in different domains and allows us to calculate the constant L explicitly.

At first, we give an idea about how to choose the manifold $\widetilde{M}^{\varepsilon}$ (see Figures 1 and 2). Note that all objects in the following are three dimensional. Because we cannot draw them, we will use two-dimensional figures and two-dimensional notations for the objects such as sheets, holes, tubes and bubbles.

Instead of particles of m colors moving in the domain Ω, we consider particles with *one* color moving on m copies (sheets) of the domain Ω which are connected between each other in a special manner. On the sheets are distributed holes $D_k^{\varepsilon i}$. All holes on all sheets are connected by special manifolds consisting of tubes, or bubbles and tubes.

All kinds of interactions between the particles and the medium and between different kinds of particles can be realized by a simple diffusion on explicitly constructed manifolds. We call

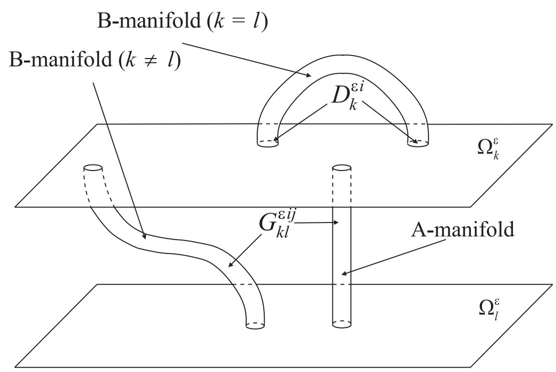

Figure 1. Manifolds without bubbles.

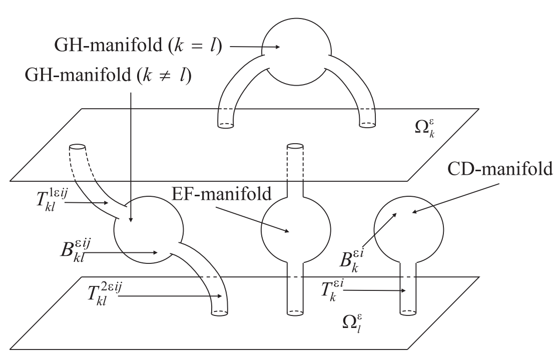

Figure 2. Manifolds with bubbles.

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these manifolds A-, B-, CD-, EF- and GH-manifolds to show the underlying connection with the term in system (1)–(3), containing the functions A_{kl}, B_{kl}, C_k and D_k, E_{kl} and F_{kl}, G_{kl} and H_{kl}, respectively. Note that the EF-manifolds give a contribution to the terms with A_{kl} and the GH-manifolds to the terms with B_{kl}, too.

- Color change: diffusion through a thin tube connecting two points with the same coordinate in Ω but on different sheets (see Figure 1: A-manifold).
- Jump from one point to another: diffusion through a thin tube connecting two points on the same sheet, but with different coordinates in Ω (see Figure 1: B-manifold ($k=l$)).
- Simultaneous change in color and jump from one point to another: diffusion through a thin tube connecting two points with different coordinates in Ω and on different sheets (see Figure 1: B-manifold ($k\neq l$)).
- Disappearance of a particle and appearance after some period of time: diffusion in a bubble which is joined to the sheet by a thin tube (see Figure 2: CD-manifold).
- Disappearance of a particle and appearance after some period of time with another color and/or in another place: diffusion through a manifold connecting two points with different coordinates in Ω and/or lying on the different sheets. This manifold consists of bubble and two thin tubes (see Figure 2: EF-/GH-manifold).
- The behavior of particles on the boundary of the domain Ω can be realized by connecting the external boundaries of the kth and lth sheets if $U_{kl}=1, k\neq l$.

3. CONSTRUCTION OF THE MAIN MANIFOLD

Let Ω be a bounded domain in $\mathbb{R}^3$, $\{D^{\varepsilon i}\subset\Omega, i=1,\ldots,N(\varepsilon)\}$ be a system of balls (holes) of radius d_i^ε and centers x_i^ε and

$$\Omega^\varepsilon=\Omega\Bigg\backslash\bigcup_{i=1}^{N(\varepsilon)}D^{\varepsilon i}$$

We consider m copies of the domain Ω^ε. We denote by Ω_k^ε the kth copy and call it the kth sheet. By $D_k^{\varepsilon i}$ we denote the copy of the ith ball on the kth sheet.

We associate with each hole $D_k^{\varepsilon i}$ at most one hole $D_l^{\varepsilon j}$ (possibly, $k=l; i=j$) and connect them via a manifold $G_{kl}^{\varepsilon ij}$ with boundary ${}^\alpha\Gamma_{kl}^{\varepsilon ij}$, where α counts the components of the boundary.

- If $k=l$ and $i=j$, we glue to $\partial D_k^{\varepsilon i}$ a three-dimensional manifold $G_{kk}^{\varepsilon ii}$ with a boundary consisting of one component ${}^0\Gamma_k^{\varepsilon i}$. More exactly, we suppose that ${}^0\Gamma_k^{\varepsilon i}$ is diffeomorphic to $\partial D_k^{\varepsilon i}$; according to these diffeomorphisms, we glue the manifold $G_{kk}^{\varepsilon ii}$ to the sheet Ω_k^ε identifying ${}^0\Gamma_k^{\varepsilon i}$ and $\partial D_k^{\varepsilon i}$ (see Figure 2: CD-manifold).
- If $k\neq l$ or $i\neq j$, we connect $\partial D_k^{\varepsilon i}$ and $\partial D_l^{\varepsilon j}$ by a three-dimensional manifold $G_{kl}^{\varepsilon ij}$ with boundary consisting of components ${}^1\Gamma_{kl}^{\varepsilon ij}$ and ${}^2\Gamma_{kl}^{\varepsilon ij}$. More exactly, we suppose that ${}^1\Gamma_{kl}^{\varepsilon ij}$ is diffeomorphic to $\partial D_k^{\varepsilon i}$, and ${}^2\Gamma_{kl}^{\varepsilon ij}$ is diffeomorphic to $\partial D_l^{\varepsilon j}$; according to these diffeomorphisms, we glue manifold $G_{kl}^{\varepsilon ij}$ to sheets Ω_k^ε and Ω_l^ε identifying ${}^1\Gamma_{kl}^{\varepsilon ij}$ and $\partial D_k^{\varepsilon i}$, and ${}^2\Gamma_{kl}^{\varepsilon ij}$ and $\partial D_l^{\varepsilon j}$ (see Figures 1 and 2: A-, B-, EF-, GH-manifolds).

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As a result we obtain a differentiable manifold M^ε

$$M^\varepsilon=\left(\bigcup_k \overline{\Omega_k^\varepsilon}\right)\cup\left(\bigcup_{k,l,i,j} G_{kl}^{\varepsilon ij}\right)$$

Let $U=\{U_{kl},\ k,l=1,m\}$ be the symmetric matrix described in the previous section. If $U_{kl}=1$ we identify the boundaries of the kth and lth sheets. We denote the obtained manifold by $\widetilde{M}^\varepsilon$.

The boundary of $\widetilde{M}^\varepsilon$ consists of $\bigcup_{k:U_{kk}=1}\partial\Omega_k$ and the boundaries of holes $D_k^{\varepsilon i}$ which do not have an associated hole $D_l^{\varepsilon j}$.

We denote the points of $\widetilde{M}^\varepsilon$ by $\tilde{x}$. If $\tilde{x}\in\Omega_k^\varepsilon$, then we assign the pair (x,k) to $\tilde{x}$, where x is the corresponding point in Ω.

We supposed that $\widetilde{M}^\varepsilon$ is equipped by the Riemannian metrics $g_{\alpha\beta}^\varepsilon(\tilde{x})$, which coincides with the Euclidian metrics on $\bigcup_k\Omega_k^\varepsilon$.

Now, we specify the size of the holes $D^{\varepsilon i}$ and the form of the manifold $G_{kl}^{\varepsilon ij}$. We consider two holes $D_k^{\varepsilon i}$ and $D_l^{\varepsilon j}$ associated with each other. We set

$$d_i^\varepsilon=\begin{cases} a\varepsilon^3, & i=j,\\ a\varepsilon^6, & i\neq j,\end{cases}\quad a>0 \tag{11}$$

Moreover, we suppose that

$$\exists c_1,\quad c_2>0,\quad \forall i: c_1\cdot r_i^{\varepsilon 3}<d_i^\varepsilon<c_2\cdot r_i^{\varepsilon 3},\quad \varepsilon<\varepsilon_0 \tag{12}$$

where $r_i^\varepsilon=\min_j(\operatorname{dist}(x_i^\varepsilon,x_j^\varepsilon))$.

We introduce a set of smooth positive functions

$$q_{kl}^A(x),\quad q_{kl}^B(x,y),\quad q_k^C(x),\quad b_k^D(x),\quad q_{kl}^E(x),\quad b_{kl}^F(x),\quad q_{kl}^G(x,y),\quad b_{kl}^H(x,y)$$

such that

$$\begin{aligned} &q_{kl}^A(x)=q_{lk}^A(x),\quad q_{kl}^B(x,y)=q_{lk}^B(y,x),\quad q_{kl}^E(x)=q_{lk}^E(x),\quad b_{kl}^F(x)=b_{lk}^F(x)\\ &q_{kl}^G(x,y)=q_{lk}^G(y,x),\quad b_{kl}^H(x,y)=b_{lk}^H(y,x)\end{aligned} \tag{13}$$

They will describe the metrics on the manifolds $G_{kl}^{\varepsilon ij}$ and the coefficients of (1) will depend on these functions.

We describe the form of the manifolds $G_{kl}^{\varepsilon ij}$ and the metrics on them.

- If $k=l$ and $i=j$, then

$$G_{kl}^{\varepsilon ij}=B_k^{\varepsilon i}\cup T_k^{\varepsilon i}\quad \text{(CD-manifold)} \tag{14}$$

where

$$B_k^{\varepsilon i}=\{(\varphi,\psi,\theta):\varphi\in[0,2\pi],\psi\in[0,\pi],\theta\in[\theta_k^{\varepsilon i},\pi]\}\quad \text{(bubble)}$$

$$T_k^{\varepsilon i}=\{(\varphi,\psi,z):\varphi\in[0,2\pi],\psi\in[0,\pi],z\in[0,1]\}\quad \text{(tube)}$$

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so that ${}^{0}\Gamma_k^i = \{\tilde{x} \in T_k^{\varepsilon i} | z=0\}$, $B_k^{\varepsilon i}$ and $T_k^{\varepsilon i}$ are connected in the points $\theta = \theta_k^{\varepsilon i}$ and $z=1$, correspondingly. The metrics is defined by the formula for the element of length:

$$\mathrm{d}s^2 = \begin{cases} (q_k^{\varepsilon i})^2 \,\mathrm{d}z^2 + (d_i^{\varepsilon})^2(\sin^2\psi \,\mathrm{d}\varphi^2 + d\psi^2), & \tilde{x} \in T_k^{\varepsilon i} \\ (b_k^{\varepsilon i})^2(\sin^2\theta \sin^2\psi \,\mathrm{d}\varphi^2 + \sin^2\theta \,\mathrm{d}\psi^2 + \mathrm{d}\theta^2), & \tilde{x} \in B_k^{\varepsilon i} \end{cases}$$

where $b_k^{\varepsilon i} = b_k^D(x_i^{\varepsilon}) \cdot \sqrt[3]{d_i^{\varepsilon}}$, $q_k^{\varepsilon i} = q_k^C(x_i^{\varepsilon}) \cdot d_i^{\varepsilon}$, $\sin\theta_k^{\varepsilon i} = d_i^{\varepsilon}/b_k^{\varepsilon i}$.[‡]

- If $k \neq l$ or $i \neq j$, then two situations are possible
  - $$G_{kl}^{\varepsilon ij} = \{(\varphi, \psi, z) : \varphi \in [0, 2\pi], \psi \in [0, \pi], z \in [0, 1]\} \tag{15}$$

    so that ${}^{1}\Gamma_{kl}^{ij} = \{\tilde{x} \in G_{kl}^{\varepsilon ij} | z=0\}$ and ${}^{2}\Gamma_{kl}^{ij} = \{\tilde{x} \in G_{kl}^{\varepsilon ij} | z=1\}$. The metrics is defined by the formula

    $$\mathrm{d}s^2 = (q_{kl}^{\varepsilon ij})^2 \,\mathrm{d}z^2 + (d_i^{\varepsilon})^2(\sin^2\psi \,\mathrm{d}\varphi^2 + \mathrm{d}\psi^2)$$

    where

    $$q_{kl}^{\varepsilon ij} = \begin{cases} q_{kl}^A(x_i^{\varepsilon}) \cdot d_i^{\varepsilon}, & i=j \quad \text{(A-manifold)} \\ q_{kl}^B(x_i^{\varepsilon}, x_j^{\varepsilon}) \cdot d_i^{\varepsilon}, & i \neq j \quad \text{(B-manifold)} \end{cases}$$

  - $$G_{kl}^{\varepsilon ij} = T_{kl}^{1\varepsilon ij} \cup B_{kl}^{\varepsilon ij} \cup T_{kl}^{2\varepsilon ij} \tag{16}$$

    where

    $$B_{kl}^{\varepsilon ij} = \{(\varphi, \psi, \theta) : \varphi \in [0, 2\pi], \psi \in [0, \pi], \theta \in [\theta_{kl}^{\varepsilon ij}, \pi - \theta_{kl}^{\varepsilon ij}]\} \tag{17}$$

    $$T_{kl}^{1\varepsilon ij} = T_{kl}^{2\varepsilon ij} = \{(\varphi, \psi, z) : \varphi \in [0, 2\pi], \psi \in [0, \pi], z \in [0, 1]\} \tag{18}$$

    so that ${}^{1}\Gamma_{kl}^{ij} = \{\tilde{x} \in T_{kl}^{1\varepsilon ij} | z=0\}$, ${}^{2}\Gamma_{kl}^{ij} = \{\tilde{x} \in T_{kl}^{2\varepsilon ij} | z=1\}$, $B_{kl}^{\varepsilon ij}$ and $T_{kl}^{1\varepsilon ij}$ are joined in the points with $\theta = \theta_{kl}^{\varepsilon ij}$ and $z=1$, correspondingly, and $B_{kl}^{\varepsilon ij}$ and $T_{kl}^{2\varepsilon ij}$ are joined in the points with $\theta = \pi - \theta_{kl}^{\varepsilon ij}$ and $z=0$, correspondingly.

    The metrics is defined by the formula

    $$\mathrm{d}s^2 = \begin{cases} (q_{kl}^{\varepsilon ij})^2 \,\mathrm{d}z^2 + (d_i^{\varepsilon})^2(\sin^2\psi \,\mathrm{d}\varphi^2 + \mathrm{d}\psi^2), & \tilde{x} \in T_{kl}^{1\varepsilon ij} \cup T_{kl}^{2\varepsilon ij} \\ (b_{kl}^{\varepsilon ij})^2(\sin^2\theta \sin^2\psi \,\mathrm{d}\varphi^2 + \sin^2\theta \,\mathrm{d}\psi^2 + \mathrm{d}\theta^2), & \tilde{x} \in B_{kl}^{\varepsilon ij} \end{cases} \tag{19}$$

    where

    $$b_{kl}^{\varepsilon ij} = \begin{cases} b_{kl}^F(x_i^{\varepsilon}) \cdot \sqrt[3]{d_i^{\varepsilon}}, & i=j \\ b_{kl}^H(x_i^{\varepsilon}, x_j^{\varepsilon}) \cdot \sqrt[3]{d_i^{\varepsilon}}, & i \neq j \end{cases}$$

    $$q_{kl}^{\varepsilon ij} = \begin{cases} q_{kl}^E(x_i^{\varepsilon}) \cdot d_i^{\varepsilon}, & i=j \quad \text{(EF-manifold)} \\ q_{kl}^G(x_i^{\varepsilon}, x_j^{\varepsilon}) \cdot d_i^{\varepsilon}, & i \neq j \quad \text{(GH-manifold)} \end{cases} \qquad \sin\theta_{kl}^{\varepsilon ij} = \frac{d_i^{\varepsilon}}{b_{kl}^{\varepsilon ij}}$$

[‡]The metrics on $B_k^{\varepsilon i}$ is the usual metrics on the sphere $S^3 \subset \mathbb{R}^4$ with radius $b_k^{\varepsilon i}$. The metrics on $T_k^{\varepsilon i}$ is the usual metrics on the cylinder $S^2 \times [0, 1]$ with radius d_i^{ε} and length $q_k^{\varepsilon i}$.

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4. HOMOGENIZATION OF THE DIFFUSION EQUATION

We consider the following Cauchy problem on $\widetilde{M}^\varepsilon$:

$$\frac{\partial u^\varepsilon}{\partial t} - \Delta^\varepsilon u^\varepsilon = 0, \quad (\tilde{x}, t) \in \widetilde{M}^\varepsilon \times [0, T] \tag{20}$$

$$u^\varepsilon(\tilde{x}, 0) = f^\varepsilon(\tilde{x}) \tag{21}$$

$$\frac{\partial u^\varepsilon}{\partial \mathbf{n}} = 0, \quad \tilde{x} \in \partial \widetilde{M}^\varepsilon \tag{22}$$

where Δ^ε is the Laplace–Beltrami operator which has the following form in local coordinates

$$\Delta^\varepsilon = \frac{1}{\sqrt{G^\varepsilon}} \sum_{\alpha,\beta=1}^{3} \frac{\partial}{\partial x_\alpha} \left(\sqrt{G^\varepsilon} g_\varepsilon^{\alpha\beta} \frac{\partial}{\partial x_\beta} \right)$$

where $G^\varepsilon = \det g_{\alpha\beta}^\varepsilon$, $g_\varepsilon^{\alpha\beta}$ are the components of the tensor inverse to $g_{\alpha\beta}^\varepsilon$, and f^ε is a smooth function, which coincides with $f_k(x)$ on the sheets and is equal to zero on $G_{kl}^{\varepsilon ij}$ (outside some small neighborhood of $\partial G_{kl}^{\varepsilon ij}$). More exactly

$$f^\varepsilon(\tilde{x}) = \begin{cases} f_k(x), & \tilde{x} = (x, k) \in \Omega_k^\varepsilon \\ 0, & \tilde{x} \in G_{kl}^{\varepsilon ij} \setminus U_{kl}^{\varepsilon ij}(\delta) \end{cases}$$

where

$$U_{kl}^{\varepsilon ij}(\delta) = \begin{cases} \{\tilde{x} \in G_{kl}^{\varepsilon ij} | \tilde{x} = (\varphi, \psi, z) \in T_k^{\varepsilon i} : |z| < \delta\} \quad \text{if } G_{kl}^{\varepsilon ij} \text{ is of type (14)} \\ \{\tilde{x} \in G_{kl}^{\varepsilon ij} | \tilde{x} = (\varphi, \psi, z) \in G_{kl}^{\varepsilon ij} : |z| < \delta \vee |1 - z| < \delta\} \\ \quad \text{if } G_{kl}^{\varepsilon ij} \text{ is of type (15)} \\ \{\tilde{x} \in G_{kl}^{\varepsilon ij} | \tilde{x} = (\varphi, \psi, z) \in T_{kl}^{1\varepsilon ij} : |z| < \delta \vee \\ \quad \vee \tilde{x} = (\varphi, \psi, z) \in T_{kl}^{2\varepsilon ij} : |1 - z| < \delta\} \quad \text{if } G_{kl}^{\varepsilon ij} \text{ is of type (16)} \end{cases}$$

is a δ-neighborhood of $\partial G_{kl}^{\varepsilon ij}$. We also require $0 \leqslant f^\varepsilon(\tilde{x}) \leqslant \max_{k=1,\ldots,m} \max_{x \in \Omega} f_k(x), \tilde{x} \in \bigcup_{k,l,i,j} U_{kl}^{\varepsilon ij}(\delta)$. We set $\delta < 1$.

Let $L_2(\widetilde{M}^\varepsilon)$ be the Hilbert space of real-valued functions with the norm

$$\|u^\varepsilon\|_{0\varepsilon} = \left\{ \int_{\widetilde{M}^\varepsilon} (u^\varepsilon)^2 \, \mathrm{d}\tilde{x} \right\}^{1/2}$$

where $\mathrm{d}\tilde{x} = \sqrt{G^\varepsilon} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3$ is a volume element on $\widetilde{M}^\varepsilon$; let $H^1(\widetilde{M}^\varepsilon)$ be the Hilbert space of real-valued functions with the norm

$$\|u^\varepsilon\|_{H^1(M^\varepsilon)}^2 = \|u^\varepsilon\|_{0\varepsilon}^2 + \|\nabla^\varepsilon u^\varepsilon\|_{0\varepsilon}^2$$

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Let $L_2(\Omega)^m$ be the Hilbert space of the real-valued m vector functions with the norm

$$\|u\|_0=\left\{\int_\Omega\sum_{k=1}^m(u_k)^2\,\mathrm{d}x\right\}^{1/2}$$

It is well known that system (20)–(22) has a unique generalized solution in $L_2(0,T;H^1(M^\varepsilon))$ (see, e.g. [22, 23]).

We say that $f^\varepsilon\in L_2(\widetilde{M}^\varepsilon)$ converges to the function $f\in L_2(\Omega)^m$ if

$$\lim_{\varepsilon\to 0}\|Q^\varepsilon f^\varepsilon-f\|_{L_2(\Omega)^m}=0 \tag{23}$$

where the operator $Q^\varepsilon: L_2(\widetilde{M}^\varepsilon)\to L_2(\Omega)^m$ is defined by the equality

$$(Q^\varepsilon u^\varepsilon)_k(x)=\begin{cases}u^\varepsilon(\tilde{x}),\tilde{x}=(x,k) & \text{if } x\in\Omega^\varepsilon\\ 0 & \text{if } x\in\bigcup_i D^{\varepsilon i}\end{cases}$$

Similarly, we say that $u^\varepsilon\in L_2(\widetilde{M}^\varepsilon\times[0,T])$ converges to the function $u\in L_2(\Omega\times[0,T])^m$ if

$$\lim_{\varepsilon\to 0}\int_0^T\|Q^\varepsilon u^\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L_2(\Omega)^m}\,\mathrm{d}t=0 \tag{24}$$

Theorem 3

Let $A_{kl}(x), B_{kl}(x,y),\ldots,H_{kl}(x,y)$ be an arbitrary set of smooth positive functions that satisfy conditions (4). Then there exist a number $a>0$ (see, (11)), a distribution of the points x_i^ε, which satisfy (12), and a set of functions $q_{kl}^A(x),\ldots,b_{kl}^H(x,y)$, which satisfies condition (13), such that the solution $u^\varepsilon(\tilde{x},t)$ to (20)–(22) converges in the sense (24) to the solution $u(x,t)$ to the initial-boundary value problem (1)–(3). Moreover, the following equalities are valid:

$$\begin{aligned}A_{kl}(x)&=E_{kl}(x)+\frac{4a\pi}{q_{kl}^A(x)+2}, \quad B_{kl}(x)=G_{kl}+\frac{4a\pi}{q_{kl}^B(x,y)+2}\\ C_k(x)&=\frac{4a\pi}{q_k^C(x)+2}, \quad D_k(x)=\frac{2}{\pi(b_k^D(x))^3(q_k^C(x)+2)}\\ E_{kl}(x)&=\frac{2a\pi}{q_{kl}^E(x)+2}, \quad F_{kl}(x)=\frac{4}{\pi(b_{kl}^F(x))^3(q_{kl}^E(x)+2)}\\ G_{kl}(x,y)&=\frac{2a\pi}{q_{kl}^G(x,y)+2}, \quad F_{kl}(x,y)=\frac{4}{\pi(b_{kl}^H(x,y))^3(q_{kl}^G(x,y)+2)}\end{aligned} \tag{25}$$

5. PROOF OF THEOREM 3

The asymptotic behavior of the solution to the diffusion equation on Riemannian manifolds with the same form as in Section 3 was investigated in [2]. We will use the results obtained there.

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Let

$$R_k^{\varepsilon i}=\{\tilde{x}=(x,k)\in\Omega_k^\varepsilon : d_i^\varepsilon \leqslant |x-x_i^\varepsilon|\leqslant r_i^\varepsilon/2\}$$

$$S_k^{\varepsilon i}=\{\tilde{x}=(x,k)\in\Omega_k^\varepsilon : |x-x_i^\varepsilon| = r_i^\varepsilon/2\}$$

$$\widetilde{G_{kl}^{\varepsilon ij}}=\begin{cases} R_k^{\varepsilon i}\cup G_{kk}^{\varepsilon ii}, & k=l\wedge i=j \\ R_k^{\varepsilon i}\cup G_{kl}^{\varepsilon ij}\cup R_l^{\varepsilon j}, & k\neq l\vee i\neq j \end{cases}$$

We consider the following boundary value problem in the domain $\widetilde{G_{kl}^{\varepsilon ij}}$:

$$-\Delta^\varepsilon v+\lambda\chi_{kl}^{\varepsilon ij} v=0, \quad \tilde{x}\in\widetilde{G_{kl}^{\varepsilon ij}}, \quad \lambda>0 \tag{26}$$

$$v=1, \quad \tilde{x}\in S_k^{\varepsilon i} \tag{27}$$

$$v=0, \quad \tilde{x}\in S_l^{\varepsilon j} \text{ (if } k\neq l\vee i\neq j) \tag{28}$$

where $\chi_{kl}^{\varepsilon ij}$ is the characteristic function of $G_{kl}^{\varepsilon ij}$.

Let $v_{kl}^{\varepsilon ij}$ be the solution to problem (26)–(28). We set

$$V_{kl}^{\varepsilon ij}=\int_{\widetilde{G_{kl}^{\varepsilon ij}}}\left\{\sum_{\alpha,\beta=1}^{3} g_\varepsilon^{\alpha\beta}\frac{\partial v_{kl}^{\varepsilon ij}}{\partial x_\alpha}\frac{\partial v_{kl}^{\varepsilon ij}}{\partial x_\beta}+\lambda\chi_{kl}^{\varepsilon ij}(v_{kl}^{\varepsilon ij})^2\right\}\mathrm{d}\tilde{x}$$

$$W_{kl}^{\varepsilon ij}=\int_{\widetilde{G_{kl}^{\varepsilon ij}}}\left\{\sum_{\alpha,\beta=1}^{3} g_\varepsilon^{\alpha\beta}\frac{\partial v_{kl}^{\varepsilon ij}}{\partial x_\alpha}\frac{\partial v_{lk}^{\varepsilon ji}}{\partial x_\beta}+\lambda\chi_{kl}^{\varepsilon ij} v_{kl}^{\varepsilon ij} v_{lk}^{\varepsilon ji}\right\}\mathrm{d}\tilde{x}$$

and introduce the following $m\times m$ matrix-valued generalized functions

$$V^\varepsilon(x,\lambda)=\left\{\sum_{i=1}^{N(\varepsilon)} W_{kl}^{\varepsilon ii}\delta(x-x_i^\varepsilon); k,l=1,\dots,m,\ k\neq l\right\}$$

$$+\mathrm{diag}\left\{\sum_{i,j=1}^{N(\varepsilon)} V_{kl}^{\varepsilon ij}\delta(x-x_i^\varepsilon); k=1,\dots,m\right\}$$

$$W^\varepsilon(x,\lambda)=\left\{\sum_{i,j=1,i\neq j}^{N(\varepsilon)} W_{kl}^{\varepsilon ij}\delta(x-x_i^\varepsilon)\delta(y-x_j^\varepsilon); k,l=1,\dots,m\right\}$$

We suppose that it $\forall\lambda>0$, the following limits exist (in $\mathscr{D}'(\mathbb{R}^n)$ and $\mathscr{D}'(\mathbb{R}^n\times\mathbb{R}^n)$, respectively)

$$\lim_{\varepsilon\to 0} V^\varepsilon(x,\lambda)=V(x,\lambda), \quad \lim_{\varepsilon\to 0} W^\varepsilon(x,y,\lambda)=W(x,y,\lambda) \tag{29}$$

where $V(x,\lambda)$ and $W(x,y,\lambda)$ are continuous matrix-valued functions in Ω and $\Omega\times\Omega$, respectively.

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It is possible to show that $V(x,\lambda)$ and $W(x,\lambda)$ have an analytic continuation with respect to parameter λ to the domain $\mathbb{C}\backslash\{\arg\lambda=\pi\}$, where the matrix-valued functions $\lambda^{-1}V(x,\lambda)$ and $\lambda^{-1}W(x,\lambda)$ are the Laplace transforms of the matrix-valued functions $V(x,t)$ and $W(x,t)$:

$$V(x,t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{V(x,\lambda)}{\lambda}\mathrm{e}^{\lambda t}\,\mathrm{d}\lambda$$

$$W(x,t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{W(x,y,\lambda)}{\lambda}\mathrm{e}^{\lambda t}\,\mathrm{d}\lambda,\quad \sigma>0$$

Now, we formulate the main result of [2].

Theorem 4
Let

(i) condition (12) be fulfilled;
(ii) the limits (29) exist;
(iii) the function $f^{\varepsilon}(\tilde{x})$ converges in the sense (23) to the vector function $f(x)=(f_1(x),\ldots,f_m(x))$;
(iv) $\lim_{\varepsilon\to 0}\sum_{k,l=1}^{m}\sum_{i,j=1}^{N(\varepsilon)}\int_{G_{kl}^{\varepsilon ij}}(f^{\varepsilon}(\tilde{x}))^2\,\mathrm{d}\tilde{x}=0$

Then the solution $u^{\varepsilon}(\tilde{x},t)$ to problem (20)–(22) converges in the sense (24) to the solution to the following problem:

$$\frac{\partial u_k}{\partial t}-\Delta u_k+\sum_{l=1}^{m}\frac{\partial}{\partial t}\int_0^t V_{kl}(x,t-\tau)u_l(x,\tau)\,\mathrm{d}\tau$$
$$+\sum_{l=1}^{m}\int_{\Omega}\frac{\partial}{\partial t}\int_0^t W_{kl}(x,y,t-\tau)u_l(y,\tau)\,\mathrm{d}\tau\,\mathrm{d}y=0,\quad k=1,\ldots,m \tag{30}$$

$$u_k(x,0)=f_k(x) \tag{31}$$

$$\frac{\partial u_k}{\partial \mathbf{n}}+\sum_{l=1}^{m}U_{kl}\frac{\partial u_l}{\partial \mathbf{n}}=0,\quad x\in\partial\Omega \tag{32}$$

In the case when $G_{kl}^{\varepsilon ij}$ has a structure such as (14), (15) or (16), then (with a suitable distribution of the points x_i^{ε}) the limits (29) exist and it is possible to find the functions $V(x,t)$ and $W(x,t)$ explicitly.

At first we consider some typical cases of the manifold $\widetilde{M}^{\varepsilon}$ with different types of $G_{kl}^{\varepsilon ij}$. We restrict ourself to the case of a manifold $\widetilde{M}^{\varepsilon}$, which consist of $m=2$ sheets. For the case $m>2$, the theorem is proved in a similar way.

We do not replay all details in every case and explain in more detail the most complicated Case 4 (for another cases see [24]).

Case 1: We divide the domain Ω into cubes $K^{\varepsilon i}$ in such a way that they form a periodic cubic lattice with side length ε. The number i counts the cubes, and x_i^{ε} are the centers of the cubes. Further, in the center of each cube $K^{\varepsilon i}$, fully lying in Ω, we cut out a ball $D^{\varepsilon i}$ with radius $d_i^{\varepsilon}=a\varepsilon^3$ and center x_i^{ε} (Figure 3). As before, $\Omega^{\varepsilon}=\Omega\backslash\bigcup_{i=1}^{N(\varepsilon)}D^{\varepsilon i}$, Ω_1^{ε} and Ω_2^{ε} are the two copies of the

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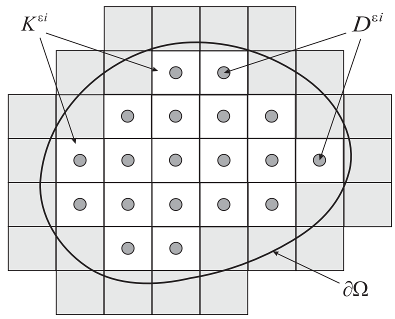

Figure 3. Case 1.

domain Ω, and $D_k^{\varepsilon i}$ is the copy of the ith ball on the kth sheet, $k=1,2$. We connect the boundaries of $D_1^{\varepsilon i}$ and $D_2^{\varepsilon i}$ by the manifold $G_{12}^{\varepsilon ij}$ which has the form (15). Finally, we obtain the manifold $\widetilde{M}^{\varepsilon}=\Omega_1^{\varepsilon}\cup\Omega_2^{\varepsilon}\cup(\bigcup_{i=1}^{N(\varepsilon)}G_{12}^{ii})$. It is easy to see that the conditions (i), (iii) and (iv) of Theorem 4 hold.

We obtain

$$V_{12}^{\varepsilon ii}=V_{21}^{\varepsilon ii}=-W_{12}^{\varepsilon ii}=-W_{21}^{\varepsilon ii}=\frac{4a\pi\varepsilon^3}{q_{12}^A(x_i^{\varepsilon})+2}(1+\bar{o}(1))$$

Let $\varphi(x)\in C^{\infty}(\Omega)$, then

$$\begin{aligned}\left\langle\sum_i V_{12}^{\varepsilon ii}\delta(x-x_i^{\varepsilon});\varphi(x)\right\rangle&=\sum_i V_{12}^{\varepsilon ii}\varphi(x_i^{\varepsilon})(1+\bar{o}(1))\\&=\sum_i\frac{4a\pi\varphi(x_i^{\varepsilon})}{q_{12}^A(x_i^{\varepsilon})+2}|K^{\varepsilon i}|(1+\bar{o}(1))\underset{\varepsilon\to 0}{\longrightarrow}\int_{\Omega}\frac{4a\pi\varphi(x)\,\mathrm{d}x}{q_{12}^A(x)+2}\end{aligned}$$

i.e.

$$\begin{aligned}V_{12}(x,\lambda)&=V_{21}(x,\lambda)=-W_{12}(x,\lambda)=-W_{21}(x,\lambda)=\frac{4a\pi}{q_{12}^A(x)+2}\\&\Longrightarrow V_{12}(x,t)=V_{21}(x,t)=-W_{12}(x,t)=-W_{21}(x,t)=\frac{4a\pi}{q_{12}^A(x)+2}\end{aligned}$$

Thus, the homogenized system has the form

$$\begin{cases}\dfrac{\partial u_1}{\partial t}-\Delta u_1+\dfrac{4a\pi}{q_{12}^A(x)+2}(u_1(x,t)-u_2(x,t))=0\\[2ex]\dfrac{\partial u_2}{\partial t}-\Delta u_2+\dfrac{4a\pi}{q_{12}^A(x)+2}(u_2(x,t)-u_1(x,t))=0\end{cases}$$

This is a two species diffusion–reaction system.

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Case 2: We divide the domain Ω into cubes $K^{\varepsilon i}$ in such a way that they form a periodic cubic lattice with side length ε. Let $n(\varepsilon)$ be the number of cubes that fully lie in Ω. In each of such cubes, we cut out $n(\varepsilon)$ balls $D^{\varepsilon i}$ with radii $d_i^\varepsilon = a\varepsilon^6$ and centers x_i^ε. Thus, the total number of holes is equal to $n^2(\varepsilon)$. Moreover, we distribute the balls in such a way that condition (12) holds (e.g. choosing the centers of the balls in the knots of the periodic lattice with period $\sim c\varepsilon^2$).

Again $\Omega^\varepsilon = \Omega \setminus \bigcup_{i=1}^{N(\varepsilon)} D^{\varepsilon i}$, Ω_1^ε and Ω_2^ε are the two copies of Ω, and $D_k^{\varepsilon i}$ is the copy of the ball $D^{\varepsilon i}$ on the kth sheet, $k = 1, 2$.

Within each cube $K^{\varepsilon\alpha}$, we renumber the out-cut balls from 1 to $n(\varepsilon)$. For each ball $D^{\varepsilon i}$, we denote by $\alpha(i)$ the number of the cube containing the ball and by $\beta(i)$ the number of the balls inside this cube. If $\alpha(i) = \beta(j), a(j) = \beta(i)$ (and only in this case), we join the boundaries of the balls $D_1^{\varepsilon i}$ and $D_2^{\varepsilon j}$ by manifolds $G_{12}^{\varepsilon ij}$ having form (15). Figure 4 shows an example of two copies of Ω where some holes are connected by tubes.

We obtain the manifold $\widetilde{M}^\varepsilon = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup (\bigcup_{i,j} G_{12}^{\varepsilon ij})$. Conditions (i), (iii) and (iv) of Theorem 4 hold.

In this case we have

$$V_{12}^{\varepsilon ij} = V_{21}^{\varepsilon ji} = -W_{12}^{\varepsilon ij} = -W_{21}^{\varepsilon ji} = \frac{4a\pi\varepsilon^6}{q_{12}^B(x_i^\varepsilon, x_j^\varepsilon) + 2}(1 + \bar{o}(1))$$

Let $\varphi(x, y) \in C^\infty(\Omega \times \Omega)$, then

$$\left\langle \sum_{i,j} W_{12}^{\varepsilon ij} \delta(x - x_i^\varepsilon)\delta(x - x_j^\varepsilon); \varphi(x) \right\rangle = \sum_{i,j} W_{12}^{\varepsilon ij} \varphi(x_i^\varepsilon, x_j^\varepsilon)(1 + \bar{o}(1))$$

$$= -\sum_{i,j} \frac{4a\pi\varphi(x_i^\varepsilon, x_j^\varepsilon)}{q_{12}^B(x_i^\varepsilon, x_j^\varepsilon) + 2} |K^{\varepsilon\alpha(i)}| \cdot |K^{\varepsilon\alpha(j)}| (1 + \bar{o}(1)) \quad (33)$$

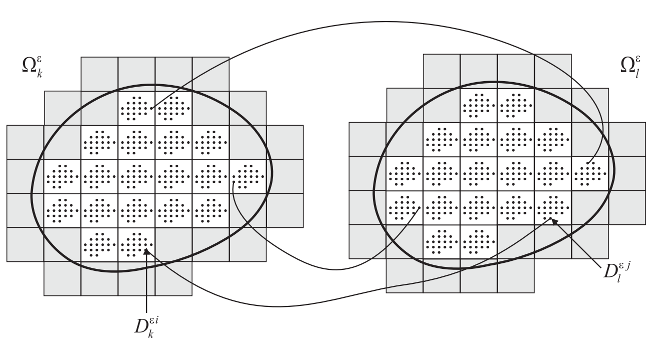

Figure 4. Case 2.

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where the sum contains only the terms with pairs (i, j), which are connected by the tube $G_{kl}^{\varepsilon ij}$. Since $\forall K^{\varepsilon\alpha_1}, K^{\varepsilon\alpha_2}$ there exists a joining pair of the holes $D_1^{\varepsilon i}$ and $D_2^{\varepsilon j}$ such that $D_1^{\varepsilon i} \subset K^{\varepsilon\alpha_1}$, $D_2^{\varepsilon j} \subset K^{\varepsilon\alpha_2}$ (by the construction of the $\widetilde{M}^{\varepsilon}$), the sum (33) is an integral sum for the function $4\pi\varphi(x, y)/(q_{12}^B(x, y)+2)$, i.e.

$$W_{12}(x, y, \lambda) = -\frac{4a\pi}{q_{12}^B(x, y)+2}, \quad W_{21}(x, y, \lambda) = -\frac{4a\pi}{q_{21}^B(x, y)+2}$$

$$V_{12}(x, \lambda) = -\int_\Omega W_{12}(x, y)\,\mathrm{d}y, \quad V_{21}(x, \lambda) = -\int_\Omega W_{21}(x, y)\,\mathrm{d}y$$

Finally, the homogenized system has the form

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \Delta u_1 + \displaystyle\int_\Omega \frac{4a\pi}{q_{12}^B(x, y)+2}(u_1(x, t) - u_2(y, t))\,\mathrm{d}y = 0 \\[2ex] \dfrac{\partial u_2}{\partial t} - \Delta u_2 + \displaystyle\int_\Omega \frac{4a\pi}{q_{21}^B(x, y)+2}(u_2(x, t) - u_1(y, t))\,\mathrm{d}y = 0 \end{cases}$$

This is a two-species diffusion–reaction system with nonlocal spatial interaction.

Case 3: Let Ω^{ε} be the domain constructed in Case 1. To the boundary of the ith hole, we glue a manifold G_{11}^{ii} of the form (14). Hence, we obtain the manifold $\widetilde{M}^{\varepsilon} = \Omega^{\varepsilon} \cup (\bigcup_i G_{11}^{ii})$.

In this case we have

$$V_{11}(x, \lambda) = \frac{4a\pi^2(b_1^D(x))^3\lambda}{2+\lambda\pi(b_1^D(x))^3(q_1^C(x)+2))}$$

Hence

$$V_{11}(x, t) = \frac{4a\pi}{q_1^C(x)+2}\exp\left(\frac{-2t}{\pi(b_1^D(x))^3(q_1^C(x)+2)}\right)$$

The homogenized equation has the form

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\partial}{\partial t}\int_0^t \frac{4a\pi}{q_1^C(x)+2}\exp\left(\frac{-2(t-\tau)}{\pi(b_1^D(x))^3(q_1^C(x)+2)}\right)u(x, \tau)\,\mathrm{d}\tau = 0$$

This is a one-species diffusion equation with memory.

Case 4: We construct the manifold $\widetilde{M}^{\varepsilon}$ in the same way as in Case 1, but $G_{12}^{\varepsilon ii}$ we choose in the form (16). The metrics on $G_{12}^{\varepsilon ii}$ is defined by (19).

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Now, we show how to calculate the function $V_{12}(x,\lambda)$. Let $v^\varepsilon \equiv v_{12}^{\varepsilon ii}(\tilde{x},\lambda)$ be the solution to (26)–(28). In order to find a suitable approximation for v^ε, we consider the function

$$\widehat{v}^{\,\varepsilon} = \begin{cases} 1+\alpha_{1i}^\varepsilon \dfrac{d_i^\varepsilon}{|x-x_i^\varepsilon|}\Phi(|x-x_i^\varepsilon|\cdot\varepsilon^{-1}), & \tilde{x}=(x,1)\in\Omega_1^\varepsilon \\ A_{1i}^\varepsilon z + B_{1i}^\varepsilon, & \tilde{x}=(\varphi,\psi,z)\in T_{12}^{1\varepsilon ii} \\ C_i^\varepsilon + \beta_{1i}^\varepsilon \dfrac{\cot\theta}{\cot\theta_{12}^{\varepsilon ii}}\Psi(\theta) + \beta_{2i}^\varepsilon \dfrac{\cot\theta}{\cot\theta_{12}^{\varepsilon ii}}\Psi(\pi-\theta), & \tilde{x}=(\varphi,\psi,\theta)\in B_{12}^{\varepsilon ii} \\ A_{2i}^\varepsilon z + B_{2i}^\varepsilon, & \tilde{x}=(\varphi,\psi,z)\in T_{12}^{2\varepsilon ii} \\ \alpha_{2i}^\varepsilon \dfrac{d_i^\varepsilon}{|x-x_i^\varepsilon|}\Phi(|x-x_i^\varepsilon|\cdot\varepsilon^{-1}), & \tilde{x}=(x,2)\in\Omega_2^\varepsilon \end{cases} \tag{34}$$

where $\Phi(r)$ is a smooth function equal to 1 when $r<\frac{1}{4}$ and equal to 0 when $r\geqslant\frac{1}{2}$ and $\Psi(\theta)$ is a smooth function equal to 1 when $\theta<\pi/4$ and equal to 0 when $\theta\geqslant\pi/2$.

We choose the constants $\alpha_{1i}^\varepsilon, \beta_{1i}^\varepsilon, A_{1i}^\varepsilon, B_{1i}^\varepsilon, C_i^\varepsilon, \alpha_{2i}^\varepsilon, \beta_{2i}^\varepsilon, A_{2i}^\varepsilon, B_{2i}^\varepsilon$ in such a way that the limiting values and normal derivatives of the function $\widehat{v}^{\,\varepsilon}$ coincide on the places of gluing between $\Omega_1^\varepsilon, T_{12}^{1\varepsilon ii}, B_{12}^{\varepsilon ii}, T_{12}^{2\varepsilon ii}, \Omega_2^\varepsilon$. Moreover, we require that the following equality

$$\int_{G_{12}^{\varepsilon ii}} (-\Delta^\varepsilon \widehat{v}^{\,\varepsilon} + \lambda\widehat{v}^{\,\varepsilon})\,\mathrm{d}\tilde{x} = 0 \tag{35}$$

holds. As a result we obtain for $\varepsilon\to 0$ the asymptotics

$$\begin{cases} \beta_{1i}^\varepsilon = \dfrac{2a+\lambda\pi(b_{12}^F(x_i^\varepsilon))^3(q_{12}^E(x_i^\varepsilon)+2)}{(q_{12}^E(x_i^\varepsilon)+2)(4a+\lambda\pi(b_{12}^F(x_i^\varepsilon))^3(q_{12}^E(x_i^\varepsilon)+2))}(1+\bar{o}(1)) = -\alpha_{1i}^\varepsilon\cos\theta_{12}^{\varepsilon ii} \\ \beta_{2i}^\varepsilon = \dfrac{2a}{(q_{12}^E(x_i^\varepsilon)+2)(4a+\lambda\pi(b_{12}^F(x_i^\varepsilon))^3(q_{12}^E(x_i^\varepsilon)+2))}(1+\bar{o}(1)) = \alpha_{2i}^\varepsilon\cos\theta_{12}^{\varepsilon ii} \\ C_i^\varepsilon = \dfrac{2a}{4a+\lambda\pi(b_{12}^F(x_i^\varepsilon))^3(q_{12}^E(x_i^\varepsilon)+2)}(1+\bar{o}(1)) \\ A_{1i}^\varepsilon = \alpha_{1i}^\varepsilon q_{12}^E(x_i^\varepsilon), \quad A_{2i}^\varepsilon = -\alpha_{2i}^\varepsilon q_{12}^E(x_i^\varepsilon), \quad B_{1i}^\varepsilon = 1+\alpha_{1i}^\varepsilon, \quad B_{2i}^\varepsilon = \alpha_{2i}^\varepsilon - A_{2i}^\varepsilon \end{cases} \tag{36}$$

We represent v^ε in the form $v^\varepsilon = \widehat{v}^{\,\varepsilon} + w^\varepsilon$.

Estimating w^ε, we set

$$I^\varepsilon[v^\varepsilon] = \int_{\widetilde{G_{12}^{\varepsilon ii}}} \left\{ \sum_{\alpha,\beta=1}^{3} g_\varepsilon^{\alpha\beta} \frac{\partial v^\varepsilon}{\partial x_\alpha}\frac{\partial v^\varepsilon}{\partial x_\beta} + \lambda\chi^\varepsilon\cdot\left(v^\varepsilon\right)^2 \right\} \mathrm{d}\tilde{x}$$

where χ^ε is the characteristic function of $G_{12}^{\varepsilon ii}$.

Since v^ε minimizes the functional I^ε in the class of functions in $H^1(\widetilde{G_{12}^{\varepsilon ii}})$ equal to 1 on $S_1^{\varepsilon i}$ and equal to 0 on $S_2^{\varepsilon i}$, w^ε minimizes the functional

$$J^\varepsilon[w^\varepsilon] = I^\varepsilon[w^\varepsilon] - 2\int_{\widetilde{G_{12}^{\varepsilon ii}}} (\Delta^\varepsilon \widehat{v}^{\,\varepsilon} - \lambda\chi^\varepsilon\widehat{v}^{\,\varepsilon}) w^\varepsilon\,\mathrm{d}\tilde{x}$$

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in the class $H_0^1(\widetilde{G_{12}^{\varepsilon ii}})$. Therefore, $J[w^\varepsilon]\leqslant J[0]\equiv 0$ and using (35) we have

$$I^\varepsilon[w^\varepsilon]\leqslant 2\left|\int_{R_1^{\varepsilon i}\cup R_2^{\varepsilon i}}\Delta\widehat{v}^\varepsilon w^\varepsilon\,\mathrm{d}x\right|+2\left|\int_{G_{12}^{\varepsilon ii}}(\Delta\widehat{v}^\varepsilon-\lambda\widehat{v}^\varepsilon)\cdot(w^\varepsilon-\bar{w}^\varepsilon)\,\mathrm{d}\tilde{x}\right| \tag{37}$$

where $\bar{w}^\varepsilon$ is the average value of w^ε in $G_{12}^{\varepsilon ii}$.

We have the following fact (see [2]): operator $\Pi^\varepsilon: H^1(\widetilde{M}^\varepsilon)\to H^1(\Omega)^m$ exists such that $\forall u^\varepsilon\in H^1(\widetilde{M}^\varepsilon)$:

(1) $(\Pi^\varepsilon u^\varepsilon)_k(x)=u^\varepsilon(\tilde{x}),\ \forall k=1,\dots,m,\ \forall x\in\Omega^\varepsilon,\ \tilde{x}=(x,k)$;
(2) $\|(\Pi^\varepsilon u^\varepsilon)_k\|_{H^1(\Omega)}\leqslant\gamma\|u^\varepsilon\|_{H^1(\Omega_k^\varepsilon)}$, where γ does not depend on ε.

It follows from this fact and Friedrich's and Poincare's inequalities that

$$\begin{aligned}\|w^\varepsilon\|^2_{L_2(R_k^{\varepsilon i})}&\leqslant C\varepsilon^2\|(\Pi^\varepsilon w^\varepsilon)_1\|^2_{H^1(R_k^{\varepsilon i}\cup D_k^{\varepsilon i})}\leqslant\widehat{C}\varepsilon^2 I^\varepsilon[w^\varepsilon],\quad k=1,2\\ \|w^\varepsilon-\bar{w}^\varepsilon\|^2_{L_2(G_{12}^{\varepsilon ii})}&\leqslant C\varepsilon^2 I^\varepsilon[w^\varepsilon]\end{aligned} \tag{38}$$

Moreover, from (34) and (36) we have

$$\int_{R_1^{\varepsilon i}\cup R_2^{\varepsilon i}}|\Delta^\varepsilon\widehat{v}^\varepsilon|^2\,\mathrm{d}\tilde{x}+\int_{G_{12}^{\varepsilon ii}}(|\Delta^\varepsilon\widehat{v}^\varepsilon|^2+\lambda|\widehat{v}^\varepsilon|^2)\,\mathrm{d}\tilde{x}\leqslant C\cdot\varepsilon^3 \tag{39}$$

Taking into account the inequalities (38), (39) and using Cauchy's inequality, from (37) we obtain the estimate

$$I^\varepsilon[w^\varepsilon]\leqslant C\cdot\varepsilon^5 \tag{40}$$

On the other hand, from (34) and (36) we obtain

$$I^\varepsilon[\widehat{v}^\varepsilon]=4a\pi\frac{2+\lambda\pi(b_{12}^F(x_i^\varepsilon))^3(q_{12}^E(x_i^\varepsilon)+2)}{(q_{12}^E(x_i^\varepsilon)+2)\cdot(4+\lambda\pi(b_{12}^F(x_i^\varepsilon))^3(q_{12}^E(x_i^\varepsilon)+2))}\varepsilon^3\cdot(1+\bar{o}(1)) \tag{41}$$

Therefore, from (40)–(41) we have $V_{12}^{\varepsilon ii}\sim I^\varepsilon[\widehat{v}^\varepsilon],\varepsilon\to 0$.

In the same way as in Case 1, we obtain

$$V_{12}(x,\lambda)=V_{21}(x,\lambda)=4a\pi\frac{2+\lambda\pi(b_{12}^F(x))^3(q_{12}^E(x)+2)}{(q_{12}^E(x)+2)\cdot(4+\lambda\pi(b_{12}^F(x))^3(q_{12}^E(x)+2))}$$

Using the same methods, it is easy to obtain

$$W_{12}(x,\lambda)=W_{21}(x,\lambda)=-4a\pi\frac{2}{(q_{12}^E(x)+2)\cdot(4+\lambda\pi(b_{12}^F(x))^3(q_{12}^E(x)+2))}$$

Hence,

$$V_{12}(x,t)=V_{21}(x,t)=\frac{2a\pi}{q_{12}^E(x)+2}\left\{1+\exp\left(\frac{-4t}{\lambda\pi(b_{12}^F(x))^3(q_{12}^E(x)+2)}\right)\right\}$$

$$W_{12}(x,t)=W_{21}(x,t)=-\frac{2a\pi}{q_{12}^E(x)+2}\left\{1-\exp\left(\frac{-4t}{\lambda\pi(b_{12}^F(x))^3(q_{12}^E(x)+2)}\right)\right\}$$

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The homogenized system has the form

$$\frac{\partial u_1}{\partial t}-\Delta u_1+\frac{2a\pi}{q_{12}^E(x)+2}(u_1(x,t)-u_2(x,t))$$

$$+\frac{\partial}{\partial t}\int_0^t \frac{2a\pi}{q_{12}^E(x)+2}\exp\left(\frac{-4(t-\tau)}{\pi(b_{12}^F(x))^3(q_{12}^E(x)+2)}\right)(u_1(x,\tau)+u_2(x,\tau))\,\mathrm{d}\tau=0$$

$$\frac{\partial u_2}{\partial t}-\Delta u_2+\frac{2a\pi}{q_{12}^E(x)+2}(u_2(x,t)-u_1(x,t))$$

$$+\frac{\partial}{\partial t}\int_0^t \frac{2a\pi}{q_{12}^E(x)+2}\exp\left(\frac{-4(t-\tau)}{\pi(b_{12}^F(x))^3(q_{12}^E(x)+2)}\right)(u_2(x,\tau)+u_1(x,\tau))\,\mathrm{d}\tau=0$$

Case 5: Finally, we construct the manifold $\widetilde{M}^\varepsilon$ in the same way as in Case 2, but $G_{12}^{\varepsilon ij}$ we choose in the form (16). In this case the homogenized system has the form

$$\frac{\partial u_1}{\partial t}-\Delta u_1+\int_\Omega \frac{2a\pi}{q_{12}^G(x,y)+2}(u_1(x,t)-u_2(y,t))\,\mathrm{d}y$$

$$+\int_\Omega \frac{\partial}{\partial t}\int_0^t \frac{2a\pi}{q_{12}^G(x,y)+2}\exp\left(\frac{-4(t-\tau)}{\pi(b_{12}^H(x,y))^3(q_{12}^G(x,y)+2)}\right)$$

$$\times(u_1(x,\tau)+u_2(y,\tau))\,\mathrm{d}\tau\,\mathrm{d}y=0$$

$$\frac{\partial u_2}{\partial t}-\Delta u_2+\int_\Omega \frac{2a\pi}{q_{21}^G(x,y)+2}(u_2(x,t)-u_1(y,t))\,\mathrm{d}y$$

$$+\int_\Omega \frac{\partial}{\partial t}\int_0^t \frac{2a\pi}{q_{21}^G(x,y)+2}\exp\left(\frac{-4(t-\tau)}{\pi(b_{21}^H(x,y))^3(q_{21}^G(x,y)+2)}\right)$$

$$\times(u_2(x,\tau)+u_1(y,\tau))\,\mathrm{d}\tau\,\mathrm{d}y=0$$

Let us combine the results of Cases 1–5. We divide the domain Ω into cubes $K^{\varepsilon i}$ in such a way that they form a periodic cubic lattice with side length ε. In each cube we pick out seven disjoint cubes $K^{s\varepsilon\alpha}, s=1,\dots,7$, such that $\operatorname{diam} K^{s\varepsilon\alpha}\sim c\varepsilon$. We call them sub-cubes. In the sub-cubes $K^{1\varepsilon}, K^{2\varepsilon}, K^{3\varepsilon}$ we cut out a single hole—a ball with radius $a\varepsilon^3$, whereas in the sub-cubes $K^{4\varepsilon}, K^{5\varepsilon}, K^{6\varepsilon}, K^{7\varepsilon}$ we cut out $n(\varepsilon)$ holes—balls with radius $a\varepsilon^6$ (we require that condition (12) holds—see Case 2). We obtain a system of balls $D^{\varepsilon i}, i=1,\dots,N(\varepsilon)=3n(\varepsilon)+4n^2(\varepsilon)$. As before, $\Omega^\varepsilon=\Omega\setminus\bigcup_{i=1}^{N(\varepsilon)} D^{\varepsilon i}$. Now, we consider two copies (sheets) of the domain Ω^ε—Ω_1^ε and Ω_2^ε. We denote by $D_k^{\varepsilon i}$ the copy of the ith ball on the kth sheet $(k=1,2)$. We can express the index i in the form $i=i_{\alpha,s,\beta}$, where α is the number of the cube containing the ball, s is the number of the sub-cube, and index β appears only in the case $s=4,5,6,7$ and denote the number of the ball within the sub-cube.

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Now, we connect the manifolds $G_{kl}^{\varepsilon ij}$ with the sheets by the following rules:

(1) Via the manifold $G_{12}^{\varepsilon i_{\alpha,1} i_{\alpha,1}}$ of the form (15) (see Case 1) we join the boundaries of the holes $D_1^{\varepsilon i_{\alpha,1}}$ and $D_2^{\varepsilon i_{\alpha,1}}$.

(2) Via the manifold $G_{12}^{\varepsilon i_{\alpha,2} i_{\alpha,2}}$ of the form (16) (see Case 4) we join the boundaries of the holes $D_1^{\varepsilon i_{\alpha,2}}$ and $D_2^{\varepsilon i_{\alpha,2}}$.

(3) We glue the manifold $G_{11}^{\varepsilon i_{\alpha,3} i_{\alpha,3}}$ of the form (14) to the boundary of the hole $D_1^{\varepsilon i_{\alpha,3}}$ and glue the manifold $G_{22}^{\varepsilon i_{\alpha,3} i_{\alpha,3}}$ of the form (14) to the boundary of the hole $D_2^{\varepsilon i_{\alpha,3}}$ (see Case 3).

(4) Via the manifold $G_{12}^{\varepsilon i_{\alpha,4,\beta} i_{\beta,4,\alpha}}$ of the form (15) (see Case 2) we join the boundaries of the holes $D_1^{\varepsilon i_{\alpha,4,\beta}}$ and $D_2^{\varepsilon i_{\beta,4,\alpha}}$.

(5) Via the manifold $G_{12}^{\varepsilon i_{\alpha,5,\beta} i_{\beta,5,\alpha}}$ of the form (16) (see Case 5) we join the boundaries of the holes $D_1^{\varepsilon i_{\alpha,5,\beta}}$ and $D_2^{\varepsilon i_{\beta,5,\alpha}}$.

(6) Via the manifold $G_{11}^{\varepsilon i_{\alpha,6,\beta} i_{\beta,6,\alpha}}$ of the form (15) we join the boundaries of the holes $D_1^{\varepsilon i_{\alpha,6,\beta}}$ and $D_1^{\varepsilon i_{\beta,6,\alpha}}$, and via the manifold $G_{22}^{\varepsilon i_{\alpha,6,\beta} i_{\beta,6,\alpha}}$ of the form (15) we join the boundaries of the holes $D_2^{\varepsilon i_{\alpha,6,\beta}}$ and $D_2^{\varepsilon i_{\beta,6,\alpha}}$. This is analogous to Case 2, but here the tube starts and ends on the same sheet.

(7) Via the manifold $G_{11}^{\varepsilon i_{\alpha,7,\beta} i_{\beta,7,\alpha}}$ of the form (16) we join the boundaries of the holes $D_1^{\varepsilon i_{\alpha,7,\beta}}$ and $D_1^{\varepsilon i_{\beta,7,\alpha}}$, and via the manifold $G_{22}^{\varepsilon i_{\alpha,7,\beta} i_{\beta,7,\alpha}}$ of the form (16) we join the boundaries of the holes $D_2^{\varepsilon i_{\alpha,7,\beta}}$ and $D_2^{\varepsilon i_{\beta,7,\alpha}}$. This is analogous to Case 5, but as before the tube starts and ends on the same sheet.

As a result we obtain the manifold $\widetilde{M}^{\varepsilon}$ as a combination of Cases 1–5. In this case the homogenized system has the form

$$
\begin{aligned}
&\frac{\partial u_k}{\partial t} - \Delta u_k + \sum_{l\neq k}\left(\frac{4a\pi}{q_{kl}^A(x)+2} + \frac{2a\pi}{q_{kl}^E(x)+2}\right)(u_k(x,t)-u_l(x,t))\\
&\quad + \sum_{l=1}^{2}\int_\Omega \left(\frac{4a\pi}{q_{kl}^B(x,y)+2} + \frac{2a\pi}{q_{kl}^G(x,y)+2}\right)(u_k(x,t)-u_l(y,t))\,\mathrm{d}y\\
&\quad + \frac{\partial}{\partial t}\int_0^t \frac{4a\pi}{q_k^C(x)+2}\exp\left(\frac{-2(t-\tau)}{\pi(b_k^D(x))^3(q_k^C(x)+2)}\right)u_k(x,\tau)\,\mathrm{d}\tau\\
&\quad + \sum_{l\neq k}\frac{\partial}{\partial t}\int_0^t \frac{2a\pi}{q_{kl}^E(x)+2}\exp\left(\frac{-4(t-\tau)}{\pi(b_{kl}^F(x))^3(q_{kl}^E(x)+2)}\right)(u_k(x,\tau)+u_l(x,\tau))\,\mathrm{d}\tau\\
&\quad + \sum_{l=1}^{2}\int_\Omega \frac{\partial}{\partial t}\int_0^t \frac{2a\pi}{q_{kl}^G(x,y)+2}\exp\left(\frac{-4(t-\tau)}{\pi(b_{kl}^H(x,y))^3(q_{kl}^G(x,y)+2)}\right)\\
&\quad \times (u_k(x,\tau)+u_l(y,\tau))\,\mathrm{d}\tau\,\mathrm{d}y = 0, \quad k=1,2
\end{aligned}
\tag{42}
$$

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We set

$$
\begin{aligned}
q_{kl}^{A}(x) &= \frac{4a\pi}{A_{kl}(x)-E_{kl}(x)}-2, \quad q_{kl}^{B}(x)=\frac{4a\pi}{B_{kl}(x,y)-G_{kl}(x,y)}-2\\
q_{k}^{C}(x) &= \frac{4a\pi}{C_{k}(x)}-2, \quad b_{k}^{D}(x)=\sqrt[3]{\frac{C_{k}(x)}{2a\pi^{2}D_{k}(x)}}\\
q_{kl}^{E}(x) &= \frac{2a\pi}{E_{kl}(x)}-2, \quad b_{kl}^{F}(x)=\sqrt[3]{\frac{2E_{kl}(x)}{a\pi^{2}F_{kl}(x)}}\\
q_{kl}^{G}(x,y) &= \frac{2a\pi}{G_{kl}(x,y)}-2, \quad b_{kl}^{H}(x,y)=\sqrt[3]{\frac{2G_{kl}(x,y)}{a\pi^{2}H_{kl}(x,y)}}
\end{aligned}
\tag{43}
$$

Functions (43) satisfy conditions (13) and are positive, if a is sufficiently large. Then system (42) has the form (1). In the same way, the proof can be done for $m>2$ sheets . Theorem 3 is proved.

6. PROOFS OF THE MAIN THEOREMS

Proof of Theorem 1
Existence and uniqueness of the generalized solution to (1)–(3) $u(x,t)\in L_2(0,T;L_2(\Omega)^m)$ follow from the results in [2] (see also [14]).

We construct the manifold $\widetilde{M}^{\varepsilon}$ in the same way as in Theorem 3. Moreover, we require that the point $x_{\max}$, providing the maximum to $\max_k \max_{x\in\Omega} f_k(x)$, does not lie in any out-cut ball $D^{\varepsilon i}$. This can be done, because of the construction of the manifold $\widetilde{M}^{\varepsilon}$. Let the function $f^{\varepsilon}(\tilde{x})$ be the same as in the proof of Theorem 3, i.e. it coincides with $f_k(x)$ if $\tilde{x}=(x,k)\in\Omega_k^{\varepsilon}$ and is equal to zero in $G_{kl}^{\varepsilon ij}$, except for small neighborhoods of ${}^{0}\Gamma_{k}^{\varepsilon i}$, ${}^{1}\Gamma_{kl}^{\varepsilon ij}$ and ${}^{2}\Gamma_{kl}^{\varepsilon ij}$. In these neighborhoods we construct f^{ε} in such a way that $0\leqslant f^{\varepsilon}(\tilde{x})\leqslant \max_{k=1,\ldots,m}\max_{x\in\Omega} f_k(x)$. Then, $\max_{\widetilde{M}^{\varepsilon}} f^{\varepsilon}(\tilde{x})$ is reached on some sheet and, therefore, $\max_{\widetilde{M}^{\varepsilon}} f^{\varepsilon}(\tilde{x})=\max_l \max_{x\in\Omega} f_l(x)\equiv M$.

In view of the maximum principle (see, e.g. [22, 23]), we have

$$u^{\varepsilon}(\tilde{x},t)\leqslant \max_{\widetilde{M}^{\varepsilon}} f^{\varepsilon}(\tilde{x})=M \quad \text{for almost all } x\in\Omega,\ t>0$$

where u^{ε} is a solution to problem (20)–(22). We obtain

$$(Q^{\varepsilon}u^{\varepsilon})_k(x,t)\leqslant M \quad \text{for almost all } x\in\Omega,\ \ t>0\ \ \forall k$$

By Theorem 3, $(Q^{\varepsilon}u^{\varepsilon})_k$ converges to u_k in $L_2(\Omega\times[0,T])$, $\forall T>0$. Therefore, there exists a sequence $\varepsilon=\varepsilon_n$ such that for almost all $x\in\Omega,\ t\in[0,T]$:

$$(Q^{\varepsilon}u^{\varepsilon})_k(x,t)\to u_k(x,t), \quad \varepsilon=\varepsilon_n\to 0$$

Then, for almost all $x\in\Omega,\ t>0, \forall k: u_k(x,t)\leqslant M$.

In the same way, the minimum principle and Corollary can be proved. □

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Remark
We give an example which shows that the condition $\max_k \max_{x\in\Omega} f_k(x)\geqslant 0$ is essential for the maximum principle.

We consider a particular case of problem (1)–(3):

$$\frac{\partial u}{\partial t}-\Delta u+\int_\Omega B(x,y)(u(x)-u(y))\,\mathrm{d}y+\frac{\partial}{\partial t}\int_0^t C\mathrm{e}^{-D(t-\tau)}u(x,\tau)\,\mathrm{d}\tau$$
$$+\int_\Omega \frac{\partial}{\partial t}\int_0^t G\mathrm{e}^{-H(t-\tau)}(u(x,\tau)+u(y,\tau))\,\mathrm{d}\tau=0 \tag{44}$$

$$u(x,0)=f,\quad \frac{\partial u}{\partial \mathbf{n}}=0 \tag{45}$$

where C, D and f are constants, $D=H$.

It is easy to see that the function

$$u(x,t)=\frac{f}{C+D+2G\cdot|\Omega|}\{D+(C+2G\cdot|\Omega|)\cdot \mathrm{e}^{-t(C+D+2G\cdot|\Omega|)}\}$$

is a solution to (44)–(45). If $f<0$, then, obviously, the maximum principle is not fulfilled.

Proof of Theorem 2
We construct the manifold $\widetilde{M}^\varepsilon$ in the same way as in Theorem 3. Let $u^\varepsilon(\tilde{x},t)$ be the solution to problem (20)–(22). In order to estimate $u^\varepsilon(\tilde{x},t)$, we prove the following uniform Poincare inequality.

Lemma
For all $u^\varepsilon\in H^1(\widetilde{M}^\varepsilon)$ such that $\bar{u}^\varepsilon\equiv\frac{1}{|\widetilde{M}^\varepsilon|}\int_{\widetilde{M}^\varepsilon}u^\varepsilon(\tilde{x})\,\mathrm{d}\tilde{x}=0$, the following inequality holds:

$$\int_{\widetilde{M}^\varepsilon}(u^\varepsilon(\tilde{x}))^2\,\mathrm{d}\tilde{x}\leqslant c_p\int_{\widetilde{M}^\varepsilon}\sum_{\alpha,\beta=1}^{3} g_\varepsilon^{\alpha\beta}\frac{\partial u^\varepsilon}{\partial x_\alpha}\frac{\partial u^\varepsilon}{\partial x_\beta}\,\mathrm{d}\tilde{x} \tag{46}$$

where the constant c_p does not depend on ε.

Proof
We prove the Lemma for one special case of the manifold $\widetilde{M}^\varepsilon$. For the general case, it can be proved in a similar way (see the proof for another case in [24]).

Suppose that our manifold $\widetilde{M}^\varepsilon$ has the same form as in Case 4 in the proof of Theorem 3:

$$\widetilde{M}^\varepsilon=\Omega_1^\varepsilon\cup\left(\bigcup_{i=1}^{N(\varepsilon)}G_{12}^{\varepsilon ii}\right)\cup\Omega_2^\varepsilon$$

where $G_{12}^{\varepsilon ii}=T_{12}^{1\varepsilon ii}\cup B_{12}^{\varepsilon ii}\cup T_{12}^{2\varepsilon ii}$, $T_{12}^{1\varepsilon ii}, B_{12}^{\varepsilon ii}, T_{12}^{2\varepsilon ii}$ are defined by formulas (17), (18).

Proving the lemma indirectly, we assume the opposite, i.e. (46) does not hold. Then a sequence (still denoted by ε) exists and functions $u^\varepsilon(\tilde{x}) \in H^1(\widetilde{M}^\varepsilon)$ exist such that

$$\begin{aligned} &\int_{\widetilde{M}^\varepsilon} (u^\varepsilon(\tilde{x}))^2 \,\mathrm{d}\tilde{x} = 1 \\ &\bar{u}^\varepsilon \equiv \frac{1}{|\widetilde{M}^\varepsilon|} \int_{\widetilde{M}^\varepsilon} u^\varepsilon(\tilde{x}) \,\mathrm{d}\tilde{x} = 0 \\ &\int_{\widetilde{M}^\varepsilon} \sum_{\alpha,\beta=1}^{3} g_\varepsilon^{\alpha\beta} \frac{\partial u^\varepsilon}{\partial x_\alpha} \frac{\partial u^\varepsilon}{\partial x_\beta} \,\mathrm{d}\tilde{x} \to 0, \quad \varepsilon \to 0 \end{aligned} \tag{47}$$

It follows from (47) that a sequence (still denoted by ε) exists such that $(\Pi^\varepsilon u^\varepsilon)_1$ converges in $L_2(\Omega)$ to some constant C_1 and $(\Pi^\varepsilon u^\varepsilon)_2$ converges in $L_2(\Omega)$ to some constant C_2 as $\varepsilon \to 0$. The operator $\Pi^\varepsilon : H^1(M^\varepsilon) \to H^1(\Omega)^m$ was introduced in the proof of Theorem 3 (Case 4).

Denote by C_i^ε the average value of u^ε in the domain $B_{12}^{\varepsilon ii}$, i.e.

$$C_i^\varepsilon = \frac{1}{|B_{12}^{\varepsilon ii}|} \int_{B_{12}^{\varepsilon ii}} u^\varepsilon(\tilde{x}) \,\mathrm{d}\tilde{x}$$

We represent u^ε in the form $u^\varepsilon = v^\varepsilon + w^\varepsilon$, where

$$v^\varepsilon = \begin{cases} C_1 + \sum_{i=1}^{N(\varepsilon)} \alpha_{1i}^\varepsilon \dfrac{d_i^\varepsilon}{|x - x_i^\varepsilon|} \Phi(|x - x_i^\varepsilon| \cdot \varepsilon^{-1}), & \tilde{x} = (x, 1) \in \Omega_1^\varepsilon \\ A_{1i}^\varepsilon z + B_{1i}^\varepsilon, & \tilde{x} = (\varphi, \psi, z) \in T_{12}^{1\varepsilon ii} \\ C_i^\varepsilon + \beta_{1i}^\varepsilon \dfrac{\cot\theta}{\cot\theta_{12}^{\varepsilon ii}} \Psi(\theta) + \beta_{2i}^\varepsilon \dfrac{\cot\theta}{\cot\theta_{12}^{\varepsilon ii}} \Psi(\pi - \theta), & \tilde{x} = (\varphi, \psi, \theta) \in B_{12}^{\varepsilon ii} \\ A_{2i}^\varepsilon z + B_{2i}^\varepsilon, & \tilde{x} = (\varphi, \psi, z) \in T_{12}^{2\varepsilon ii} \\ C_2 + \sum_{i=1}^{N(\varepsilon)} \alpha_{2i}^\varepsilon \dfrac{d_i^\varepsilon}{|x - x_i^\varepsilon|} \Phi(|x - x_i^\varepsilon| \cdot \varepsilon^{-1}), & \tilde{x} = (x, 2) \in \Omega_2^\varepsilon \end{cases}$$

with

$$\begin{aligned} &\alpha_{1i}^\varepsilon = \frac{C_i^\varepsilon - C_1}{1 + \cos\theta_{12}^{\varepsilon ii} + q_{12}^E(x_i^\varepsilon)}, \quad \beta_{1i}^\varepsilon = -\alpha_{1i}^\varepsilon \cdot \cos\theta_{12}^{\varepsilon ii} \\ &A_{1i}^\varepsilon = \alpha_{1i}^\varepsilon \cdot q_{12}^E(x_i^\varepsilon), \quad B_{1i}^\varepsilon = C_1 + \alpha_{1i}^\varepsilon \\ &\alpha_{2i}^\varepsilon = \frac{C_i^\varepsilon - C_2}{1 + \cos\theta_{12}^{\varepsilon ii} + q_{12}^E(x_i^\varepsilon)}, \quad \beta_{2i}^\varepsilon = \alpha_{2i}^\varepsilon \cdot \cos\theta_{12}^{\varepsilon ii} \\ &A_{2i}^\varepsilon = -\alpha_{2i}^\varepsilon \cdot q_{12}^E(x_i^\varepsilon), \quad B_{2i}^\varepsilon = C_2 + \alpha_{2i}^\varepsilon - A_{2i}^\varepsilon \end{aligned}$$

$\Phi(r)$ is a smooth function equal to 1 when $r < \frac{1}{4}$ and equal to 0 when $r \geqslant \frac{1}{2}$; $\Psi(\theta)$ is a smooth function equal to 1 when $\theta < \pi/4$ and equal to 0 when $\theta \geqslant \pi/2$. The coefficients $\alpha_{1i}^\varepsilon, \beta_{1i}^\varepsilon, A_{1i}^\varepsilon, B_{1i}^\varepsilon, \alpha_{2i}^\varepsilon, \beta_{2i}^\varepsilon, A_{2i}^\varepsilon, B_{2i}^\varepsilon$ are taken in such a way that the limiting values and normal derivatives of the function v^ε coincide on the places of gluing between $\Omega_1^\varepsilon, T_{12}^{1\varepsilon ii}, B_{12}^{\varepsilon ii}, T_{12}^{2\varepsilon ii}, \Omega_2^\varepsilon$.

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We have

$$\|\nabla^{\varepsilon}u^{\varepsilon}\|_{0\varepsilon}^{2}\geqslant\|\nabla^{\varepsilon}v^{\varepsilon}\|_{0\varepsilon}^{2}+2(\nabla^{\varepsilon}v^{\varepsilon},\nabla^{\varepsilon}w^{\varepsilon})_{0\varepsilon}=\|\nabla^{\varepsilon}v^{\varepsilon}\|_{0\varepsilon}^{2}-2(\Delta^{\varepsilon}v^{\varepsilon},w^{\varepsilon})_{0\varepsilon} \tag{48}$$

From the explicit form of the function v^{ε} and using Poincare inequality for the domain $B_{12}^{\varepsilon ii}$, we obtain the following inequalities:

$$\begin{aligned}
M_1\cdot\sigma^{\varepsilon}\leqslant\|\nabla^{\varepsilon}v^{\varepsilon}\|_{0\varepsilon}^{2}&\leqslant M_2\cdot\sigma^{\varepsilon}\\
\|\Delta^{\varepsilon}v^{\varepsilon}\|_{0\varepsilon}^{2}&\leqslant M_3\cdot\sigma^{\varepsilon}\\
\|w^{\varepsilon}\|_{L_2(\Omega_1^{\varepsilon}\cup\Omega_2^{\varepsilon})}^{2}&\leqslant 2\|u^{\varepsilon}-C_1\|_{L_2(\Omega_1^{\varepsilon})}^{2}+2\|u^{\varepsilon}-C_2\|_{L_2(\Omega_2^{\varepsilon})}^{2}+M_5\varepsilon^4\cdot\sigma^{\varepsilon}\\
\sum_{i=1}^{N(\varepsilon)}\|w^{\varepsilon}\|_{L_2(B_{12}^{\varepsilon ii})}^{2}&\leqslant\varepsilon^2M_6\sum_{i=1}^{N(\varepsilon)}\|\nabla^{\varepsilon}w^{\varepsilon}\|_{L_2(B_{12}^{\varepsilon ii})}^{2}+\sum_{i=1}^{N(\varepsilon)}\frac{1}{|B_{12}^{\varepsilon ii}|}\left(\int_{B_1^{i\varepsilon}}(C_i^{\varepsilon}-v^{\varepsilon})\,\mathrm{d}\tilde{x}\right)^2\\
&\leqslant 2\varepsilon^2M_6(\|\nabla^{\varepsilon}v^{\varepsilon}\|_{0\varepsilon}^{2}+\|\nabla^{\varepsilon}u^{\varepsilon}\|_{0\varepsilon}^{2})+\varepsilon^4M_7\cdot\sigma^{\varepsilon}
\end{aligned} \tag{49}$$

where $\sigma^{\varepsilon}=\sum_{i=1}^{N(\varepsilon)}[(C_1-C_i^{\varepsilon})^2+(C_2-C_i^{\varepsilon})^2]\cdot|B_{12}^{\varepsilon ii}|$ and $M_i, i=1,\dots,7$, are positive constants.

Further we prove that

$$\exists c_1,c_2,>0: c_1<\sigma^{\varepsilon}<c_2 \tag{50}$$

From the inequalities (49), (50), using Cauchy inequality, we have

$$\begin{aligned}
\|\nabla^{\varepsilon}v^{\varepsilon}\|_{0\varepsilon}^{2}&\geqslant c_1\cdot M_1>0\\
\left|(\Delta^{\varepsilon}v^{\varepsilon},w^{\varepsilon})_{0\varepsilon}\right|&\leqslant\|\Delta^{\varepsilon}v^{\varepsilon}\|_{L_2(\Omega_1^{\varepsilon}\cup\Omega_2^{\varepsilon})}\cdot\|w^{\varepsilon}\|_{L_2(\Omega_1^{\varepsilon}\cup\Omega_2^{\varepsilon})}\\
&\quad+\left[\sum_{i=1}^{N(\varepsilon)}\|\Delta^{\varepsilon}v^{\varepsilon}\|_{L_2(B_{12}^{\varepsilon ii})}^{2}\right]^{1/2}\cdot\left[\sum_{i=1}^{N(\varepsilon)}\|w^{\varepsilon}\|_{L_2(B_{12}^{\varepsilon ii})}^{2}\right]^{1/2}\underset{\varepsilon\to0}{\longrightarrow}0
\end{aligned}$$

Then, from (48) we have $\lim_{\varepsilon\to0}\|\nabla u^{\varepsilon}\|_{0\varepsilon}>0$—a contradiction.

Now, we prove inequality (50). The right-hand side follows from the equality $\|u^{\varepsilon}\|_{0\varepsilon}=1$. Suppose that the left-hand side inequality does not hold. Then there exists a sequence (again denoted by ε) such that

$$\lim_{\varepsilon\to0}\sum_{i=1}^{N(\varepsilon)}(C_1-C_i^{\varepsilon})^2|B_{12}^{\varepsilon ii}|=0,\quad \lim_{\varepsilon\to0}\sum_{i=1}^{N(\varepsilon)}(C_2-C_i^{\varepsilon})^2|B_{12}^{\varepsilon ii}|=0 \tag{51}$$

From (51) follow the inequalities

$$\begin{aligned}
\left(\sum_{i=1}^{N(\varepsilon)}(C_\alpha-C_i^{\varepsilon})|B_{12}^{\varepsilon ii}|\right)^2&\leqslant N(\varepsilon)\sum_{i=1}^{N(\varepsilon)}(C_\alpha-C_i^{\varepsilon})^2|B_{12}^{\varepsilon ii}|^2\\
&\leqslant c\sum_{i=1}^{N(\varepsilon)}(C_\alpha-C_i^{\varepsilon})^2|B_{12}^{\varepsilon ii}|,\quad c>0,\ \alpha=1,2
\end{aligned} \tag{52}$$

and so the left-hand side of (52) also converges to zero.

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Using Poincare inequality for the domain $B_{12}^{\varepsilon ii}$, we have

$$0 \leqslant \sum_{i=1}^{N(\varepsilon)} \left\{ \int_{B_{12}^{\varepsilon ii}} (u^{\varepsilon}(\tilde{x}))^2 \,\mathrm{d}\tilde{x} - (C_i^{\varepsilon})^2 |B_{12}^{\varepsilon ii}| \right\} \leqslant c \cdot \varepsilon^2 \sum_{i=1}^{N(\varepsilon)} \|\nabla^{\varepsilon} u^{\varepsilon}\|^2_{L_2(B_{12}^{\varepsilon ii})} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

Moreover, it is easy to show that $\lim_{\varepsilon\to 0} \sum_{i=1}^{N(\varepsilon)} \|u^{\varepsilon}\|^2_{L_2(T_{12}^{1\varepsilon ij} \cup T_{12}^{2\varepsilon ij})} = 0$.

Hence,

$$1 = \lim_{\varepsilon\to 0} \|u^{\varepsilon}\|^2_{0\varepsilon} = (C_1^2 + C_2^2) \cdot |\Omega| + \lim_{\varepsilon\to 0} \sum_{i=1}^{N(\varepsilon)} (C_i^{\varepsilon})^2 |B_{12}^{\varepsilon ii}| \tag{53}$$

Further we obtain

$$0 = \lim_{\varepsilon\to 0} \int_{\widetilde{M}^{\varepsilon}} u^{\varepsilon}(\tilde{x}) \,\mathrm{d}\tilde{x} = (C_1 + C_2)|\Omega| + \lim_{\varepsilon\to 0} \sum_{i=1}^{N(\varepsilon)} C_i^{\varepsilon} |B_{12}^{\varepsilon ii}|$$

$$= \left[\begin{array}{l} C_1 \left(\lim\limits_{\varepsilon\to 0} \sum\limits_{i=1}^{N(\varepsilon)} |B_{12}^{\varepsilon ii}| + |\Omega| \right) + C_2|\Omega| + \lim\limits_{\varepsilon\to 0} \sum\limits_{i=1}^{N(\varepsilon)} (C_i^{\varepsilon} - C_1)|B_{12}^{\varepsilon ii}| \\ C_2 \left(\lim\limits_{\varepsilon\to 0} \sum\limits_{i=1}^{N(\varepsilon)} |B_{12}^{\varepsilon ii}| + |\Omega| \right) + C_1|\Omega| + \lim\limits_{\varepsilon\to 0} \sum\limits_{i=1}^{N(\varepsilon)} (C_i^{\varepsilon} - C_2)|B_{12}^{\varepsilon ii}| \end{array} \right. \tag{54}$$

It follows from (52), (54) that $C_1 = C_2 = 0$. However, this together with (51) contradicts (53). Then the right-hand side of (50) is true.

The lemma is proved. □

We continue the proof of Theorem 2.

From Gronwall's lemma, we obtain that the solution to (20)–(22) satisfies the inequality

$$\|u^{\varepsilon} - L^{\varepsilon}\|^2_{0\varepsilon} \leqslant \|f^{\varepsilon} - L^{\varepsilon}\|^2_{0\varepsilon} \cdot \exp\left[\frac{-2t}{c_p}\right], \quad L^{\varepsilon} = \frac{1}{|\widetilde{M}^{\varepsilon}|} \int_{\widetilde{M}^{\varepsilon}} f^{\varepsilon}(\tilde{x}) \,\mathrm{d}\tilde{x} \tag{55}$$

It is clear that the norms $\|f^{\varepsilon} - L^{\varepsilon}\|_{0\varepsilon}$ are uniformly bounded with respect to ε. Therefore, $\exists\, c_1 > 0$:

$$\|u^{\varepsilon} - L^{\varepsilon}\|^2_{0\varepsilon} \leqslant c_1 \cdot \exp\left[\frac{-2t}{c_p}\right] \tag{56}$$

Hence,

$$\|Q^{\varepsilon} u^{\varepsilon} - L^{\varepsilon}\|^2_0 \leqslant c_1 \cdot \exp\left[\frac{-2t}{c_p}\right] + (L^{\varepsilon})^2 \sum_{k=1}^{m} \sum_{i=1}^{N(\varepsilon)} |D_i^{\varepsilon}| \tag{57}$$

By the construction of $\widetilde{M}^{\varepsilon}$, we have

$$|\widetilde{M}^{\varepsilon}| = m \cdot |\Omega^{\varepsilon}| + \sum_{k=1}^{m} \sum_{i=1}^{N(\varepsilon)} |G_{kk}^{\varepsilon ii}| + \frac{1}{2} \sum_{k,l=1|k\neq l}^{m} \sum_{i=1}^{N(\varepsilon)} |G_{kl}^{\varepsilon ii}| + \frac{1}{2} \sum_{k,l=1}^{m} \sum_{i,j=1|i\neq j}^{N(\varepsilon)} |G_{kl}^{\varepsilon ij}| \tag{58}$$

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From formulas (13), (25) and (58), it follows that

$$
\begin{aligned}
|\widetilde{M}^{\varepsilon}| \underset{\varepsilon\to 0}{\longrightarrow} m\cdot|\Omega| + 2\pi^2 \Bigg(& \sum_{k=1}^{m} \int_{\Omega} a(b_k^D(x))^3\,\mathrm{d}x + \frac{1}{2}\sum_{k,l=1|k\neq l}^{m} \int_{\Omega} a(b_{kl}^F(x))^3\,\mathrm{d}x \\
& + \sum_{k,l=1}^{m} \frac{1}{2} \int_{\Omega}\int_{\Omega} a(b_{kl}^H(x,y))^3\,\mathrm{d}x\,\mathrm{d}y \Bigg) \\
= \; m\cdot|\Omega| + \sum_{k=1}^{m} \int_{\Omega} \frac{C_k(x)}{D_k(x)}\,\mathrm{d}x & + 2\sum_{k,l=1|k\neq l}^{m} \int_{\Omega} \frac{E_{kl}(x)}{F_{kl}(x)}\,\mathrm{d}x + 2\sum_{k,l=1}^{m} \int_{\Omega}\int_{\Omega} \frac{G_{kl}(x,y)}{H_{kl}(x,y)}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

$$
\int_{\widetilde{M}^{\varepsilon}} f^{\varepsilon}(\tilde{x})\,\mathrm{d}\tilde{x} \underset{\varepsilon\to 0}{\longrightarrow} \sum_{k=1}^{m} \int_{\Omega} f_k(x)\,\mathrm{d}x
$$

i.e.

$$
\lim_{\varepsilon\to 0} L^{\varepsilon} = L \tag{59}
$$

Note that only manifolds of types (14), (16) (i.e. CD, EF and GH-manifolds) make a contribution to L.

Let $\delta>0$ be an arbitrary number and let us fix t. For all $\varepsilon>0$, we have

$$
\|u(\cdot,t)-L\|_0^2 \leqslant 3\cdot\left\{\|Q^{\varepsilon}u^{\varepsilon}(\cdot,t)-u(\cdot,t)\|_0^2 + \|Q^{\varepsilon}u^{\varepsilon}(\cdot,t)-L^{\varepsilon}\|_0^2 + \|L^{\varepsilon}-L\|_0^2\right\} \tag{60}
$$

From Theorem 3 and (59), it follows that there exists $\varepsilon>0$ such that

$$
\|Q^{\varepsilon}u^{\varepsilon}(\cdot,t)-u(\cdot,t)\|_0^2 + \|L^{\varepsilon}-L\|_0^2 + (L^{\varepsilon})^2 \sum_{k=1}^{m}\sum_{i=1}^{N(\varepsilon)} |D_k^{\varepsilon i}| \leqslant \delta
$$

Then

$$
\|u(\cdot,t)-L\|_0^2 \leqslant 3\delta + 3\cdot c_1\cdot\exp\left[\frac{-2t}{c_p}\right]
$$

Passing to the limit as $\delta\to 0$, we have

$$
\|u(\cdot,t)-L\|_0^2 \leqslant 3\cdot c_1\cdot\exp\left[\frac{-2t}{c_p}\right] \quad \forall t>0
$$

Theorem 2 is proved. □

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