NETWORKS AND HETEROGENEOUS MEDIA
©American Institute of Mathematical Sciences
Volume **3**, Number **4**, December **2008**
doi:10.3934/nhm.2008.3.831
pp. **831–862**

THE HOMOGENIZED MODEL OF SMALL OSCILLATIONS OF COMPLEX FLUIDS

M. Berezhnyi, L. Berlyand and E. Khruslov

Institute of Low Temperature Physics and Engineering, Ukrainian Academy of Sciences
Lenin Ave 47, Kharkiv 61164, Ukraine;

Department of Mathematics and Materials Research Institute, Penn State University
University Park, PA-16802-6401, 218 Mc Allister Building, USA;

Institute of Low Temperature Physics and Engineering, Ukrainian Academy of Sciences
Lenin Ave 47, Kharkiv 61164, Ukraine

(Communicated by Antonio DeSimone)

Abstract. We consider the system of equations that describes small non-stationary motions of viscous incompressible fluid with a large number of small rigid interacting particles. This system is a microscopic mathematical model of complex fluids such as colloidal suspensions, polymer solutions etc. We suppose that the system of particles depends on a small parameter ε in such a way that the sizes of particles are of order ε^3, the distances between the nearest particles are of order ε, and the stiffness of the interaction force is of order ε^2.

We study the asymptotic behavior of the microscopic model as $\varepsilon \to 0$ and obtain the homogenized equations that can be considered as a macroscopic model of diluted solutions of interacting colloidal particles.

1. Introduction.

In the last years, in the literature on fluid mechanics we have seen an increasing number of papers devoted to the study of complex fluids. The examples of such fluids (substances) seen in nature are blood, honey, spider's web etc. Nowadays, complex fluids of artificial origin such as magnetic fluids, colloidal suspensions, surfactant fluids, polymer fluids attract the increasing interest of the researchers, because of their potential usefulness in various applications (see [7], [12], [17], [18], [21], [30], [35]).

Generally, these fluids have a specific microstructure. The characteristic feature of this microstructure is the presence of a main fluid and small (of order of angstroms) particles (atoms, molecules) of another substance located in the fluid. These inclusions form flexible elastic chains, clusters, periodic grids and interact between themselves due to the forces of various origins (Van-der-Waals forces, electrostatic forces etc., see [17]). In general, such mixtures show a non-newtonian behavior.

The simplest microscopic model of such substances is a mixture of a newtonian fluid and small solid particles that interact between themselves and with the surrounding fluid. The study of the properties of the fluids in the framework of such

2000 *Mathematics Subject Classification.* Primary: 35B27, 35Q30, 74Q10; Secondary: 76M30, 76M50.

Key words and phrases. Homogenization, non-Newtonian fluids, interaction energy, interacting particles, triangulization condition, Brinkman's law.

a model by using both analytical and numerical methods appears to be an unsurmountable problem because one have to take into consideration the interaction of a great number of the small particles. Therefore it is necessary to develop adequate macroscopic models that can help in studying such fluids. A number of authors, both mathematicians and physicists, have addressed this development ([3],[5], [18], [29], [31]). In particular, oscillations of a viscous incompressible fluid with small solid interacting particles of arbitrary shape are considered in [5] under the assumption that the diameters of particles, the distances between the neighboring particles and the interaction forces are of the same order ε. The asymptotic analysis, as $\varepsilon \to 0$, of this problem is given and the homogenized (macroscopic) model is constructed. It shows that, for ε sufficiently small, the mixture behaves as a one-phase viscoelastic medium.

In [3], a similar problem was studied in the case of very small particles (of order ε^α, $1 < \alpha < 3$). It was shown that the homogenized model is qualitatively the same.

In this paper we study the case, when the diameters of particles are of order ε^3, which is the well-known critical size of inclusions, leading to appearance of an additional potential (named in [10] a "strange term") in the homogenized equation (see also [1], [22], [23]). It turns out that in this critical case the homogenized model has another qualitative form: the model is two-component and the mixture acts as if a main fluid oscillate penetrating through an elastic weightless skeleton. This homogenized model can be considered as a more complex version of Brinkman's law introduced in [9].

A rigorous justification of Brinkman's law requires the study of the homogenization problem for the Stokes equations in perforated domains. This question has been extensively studied in the mathematical literature (see, e.g., [1], [8], [15], [19], [23], [33]). Particularly, the homogenization of the Navier-Stokes equations with the Dirichlet boundary condition in a porous domain modelled as the periodic repetition of an elementary cell of size ε containing a tiny solid obstacle was analyzed in [1]. The convergence of the homogenization process to a Brinkman-type law was proved, with the help of Tartar's energy method, in the case where the solid obstacle is of critical size ε^3 (in the three-dimensional case). If the size of the holes is asymptotically larger than this critical size, then the homogenized problem is governed by Darcy's law; if the size of the holes is asymptotically smaller than the critical size, then the homogenized problem turns out to be the Stokes equations (for more details see [1], [15], [32]).

In [19] and [33], Brinkman's law was also derived by means of a three-scale expansion method. A similar result was obtained in [8] by using the framework of epi-convergence. We also refer to work [23], where it was proved that equations corresponding to Brinkman's law describe the limiting behavior of the Stokes flow in a non-periodically perforated domain for the critical scaling of the holes.

The key feature of our paper is the interaction between particles. Our main purpose is to develop a rigorous mathematical model that shows how the interaction affects the homogenized medium. Namely, for generic non-periodic arrays of particles we describe the homogenized model in terms of the meso-characteristic of a discrete elastic skeleton, which represents local interaction energy on the meso-scale (intermediate scale between the inter-particle distances (microscale) and the domain sizes (macroscale)). We show that in the case of the critical sizes of particles, the homogenized medium exhibits a two-phase behavior.

To this end, we use the Laplace transform to obtain a time-independent version (with spectral parameter λ) of the problem describing small motions of the fluid with solid interacting particles. Then we reduce it to a variational form for $\lambda 0$. In sections 4-7, we study the asymptotic behavior of solution of the variational problem as $\varepsilon \to 0$ by using a method of the computation of a Γ-limit. We find the homogenized variational functional and the system of Euler equations corresponding to this functional. Finally, in section 8 we study the analytic properties of the solutions of these equations with respect to λ, and, applying the inverse Laplace transform, obtain the homogenized non-stationary problem.

2. **The microscopic model.** Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. The domain is occupied by a composite medium, which is a suspension of a large number $N^\varepsilon = O(\frac{1}{\varepsilon^3})$ of small, ball-shaped rigid interacting particles Q^i_ε $(i = \overline{1, N^\varepsilon})$ in a viscous incompressible fluid. We assume that the radii of the particles are of order $O(\varepsilon^3)$, the distances between the neighboring particles are $O(\varepsilon)$, and interaction between the neighboring particles is determined by central forces of order $O(\varepsilon^2)$.

We also assume that the system of particles is in equilibrium when the fluid is at rest. Let $\underline{x}^i_\varepsilon$ be a position of the center of particle Q^i_ε corresponding to its equilibrium state. This equilibrium is determined by the minimum of the interaction potential energy

$$H_\varepsilon(\underline{u}^i_\varepsilon) = H_\varepsilon(0) + \frac{1}{2}\sum_{\substack{i,j\\ j\neq i}} \Big\langle \underline{\underline{C}}^{ij}_\varepsilon[\underline{u}^i - \underline{u}^j], [\underline{u}^i - \underline{u}^j] \Big\rangle + \tag{2.1}$$

$$+ \text{ the terms of higher orders,}$$

where $\underline{u}^i_\varepsilon$ are displacements of the centers of the particles, $\underline{\underline{C}}^{ij}_\varepsilon$ are symmetric positive matrices, the parenthesis $\langle\ ,\ \rangle$ stands for a dot product in $\mathbb{R}^3$.

Since the energy (2.1) is invariant under translation and rotation of the set of particles as a whole, the energy minimization defines many equilibrium states. A unique minimum is specified by the condition that the external boundary interacts with all the particles which are at a distance smaller than $C\varepsilon$ $(C > 0)$ from $\partial\Omega$. Then in addition to the energy (2.1) we need to take into account the energy due to the interaction with the boundary $\partial\Omega$. Formally, the energy can be written in the same form, taking into account that some particles Q^j_ε can lie on the boundary with the correspondent displacements $\underline{u}^j_\varepsilon$ being equal to zero due to the boundary condition on $\partial\Omega$. Thus, in our consideration the system of particles has a unique equilibrium state $\{\underline{x}^i_\varepsilon, i = 1, 2, \ldots, N^\varepsilon\}$.

Introduce the following notations: $\Omega_\varepsilon = \Omega \setminus \bigcup_{i=1}^{N_\varepsilon} Q^i_\varepsilon$ is a domain occupied by the fluid; ρ is the specific density of the fluid; μ is the dynamic viscosity of the fluid; ρ_s is the specific density of particles' substance; $m^i_\varepsilon = \rho_s|Q^i_\varepsilon|$ is the mass of particle Q^i_ε; $\underline{\theta}^i_\varepsilon$ is the rotation vector of Q^i_ε; $I^i_\varepsilon = \frac{2}{5}m^i_\varepsilon(r^i_\varepsilon)^2$ is the inertia moment of Q^i_ε.

Then the linearized system of equations that describe small non-stationary motions of the fluid with solid interacting particles can be written in the form

$$\rho\frac{\partial \underline{v}_\varepsilon}{\partial t} - \mu\Delta\underline{v}_\varepsilon = \nabla p_\varepsilon, \quad \mathrm{div}\underline{v}_\varepsilon = 0, \quad \underline{x} \in \Omega_\varepsilon; \tag{2.2}$$

$$\underline{v}_\varepsilon = \underline{\dot{u}}^i_\varepsilon + \underline{\dot{\theta}}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon) \quad \underline{x} \in \partial Q^i_\varepsilon; \tag{2.3}$$

$$m^i_\varepsilon \underline{\ddot{u}}^i_\varepsilon + \int\limits_{S^i_\varepsilon} \sigma[\underline{v}_\varepsilon]\nu ds = -\nabla_{u^i} H_\varepsilon; \tag{2.4}$$

$$I^i_\varepsilon \underline{\ddot{\theta}}^i_\varepsilon + \int\limits_{S^i_\varepsilon} (\underline{x} - \underline{x}^i_\varepsilon) \times \underline{\sigma}[\underline{v}_\varepsilon]\nu ds = -\nabla_{\theta^i} H_\varepsilon (\equiv 0). \tag{2.5}$$

Here $\underline{v}_\varepsilon = \underline{v}_\varepsilon(\underline{x}, t)$ is the velocity of the fluid, $p_\varepsilon = p_\varepsilon(\underline{x}, t)$ is the pressure, $\underline{u}^i_\varepsilon$ is the displacement of the center of particle Q^i_ε, $\underline{\dot{u}}^i_\varepsilon = \dfrac{d\underline{u}^i_\varepsilon}{dt}$ is the velocity of the center of Q^i_ε, $\underline{\ddot{u}}^i_\varepsilon = \dfrac{d^2\underline{u}^i_\varepsilon}{dt^2}$ is the acceleration of the center of Q^i_ε, and $\underline{\dot{\theta}}^i_\varepsilon$ is the instant angular velocity of Q^i_ε. By $\underline{\nu}$ we denote the unit inner normal to the sphere $S^i_\varepsilon = \partial Q^i_\varepsilon$. The stress tensor in the fluid is denoted by $\underline{\underline{\sigma}}[\underline{v}_\varepsilon]$. The components of the stress tensor are defined as $\underline{\underline{\sigma}}[\underline{v}_\varepsilon]_{ij} = \mu \left[\frac{\partial \underline{v}_{\varepsilon i}}{\partial \underline{x}_j} + \frac{\partial \underline{v}_{\varepsilon j}}{\partial \underline{x}_i} \right] - p_\varepsilon \delta_{ij} \quad (i, j = 1, 2, 3)$. The energy of interaction H_ε is defined by (2.1).

The system (2.2)-(2.5) is supplemented by the initial conditions

$$\underline{v}_\varepsilon(\underline{x}, 0) = \underline{v}_{\varepsilon 0}(\underline{x}), \quad \underline{x} \in \Omega_\varepsilon;$$

$$\underline{u}^i_\varepsilon = 0, \quad \underline{\dot{u}}^i_\varepsilon = \underline{u}^i_{\varepsilon 1}, \quad \underline{\theta}^i_\varepsilon = 0, \quad \underline{\dot{\theta}}^i_\varepsilon = \underline{\theta}^i_{\varepsilon 1} \tag{2.6}$$

and the boundary condition on $\partial\Omega$

$$\underline{v}_\varepsilon(\underline{x}, t) = 0, \quad \underline{x} \in \partial\Omega. \tag{2.7}$$

The homogeneous Dirichlet boundary condition corresponds to the clamped boundary $\partial\Omega$. We choose this condition for the sake of simplicity; other types of boundary condition can be treated in a similar way.

There exists a unique solution $\{\underline{v}_\varepsilon(\underline{x}), \underline{u}^i_\varepsilon, \underline{\theta}^i_\varepsilon, i = 1, 2, \dots N^\varepsilon\}$ of the problem. The main goal of this paper is to study the asymptotic behavior of the solution as $\varepsilon \to 0$. Before formulating the main result we introduce some definitions and assumptions.

3. Local energy characteristics. Formulation of main results.

Let $d^i_\varepsilon = \text{dist}\{Q^i_\varepsilon, \bigcup\limits_{j \neq i} Q^i_\varepsilon \bigcup \partial\Omega\}$ be the distance between Q^i_ε and other particles and boundary $\partial\Omega$, and r^i_ε be the radius of Q^i_ε.

Assume that the following conditions hold.

I. The geometric conditions.

1) There exist positive constants C_1, C_2 and C_3 such that

$$C_1\varepsilon \le d^i_\varepsilon \le C_2\varepsilon; \tag{3.1}$$

$$r^i_\varepsilon \le C_3\varepsilon^3. \tag{3.2}$$

Constants C_1, C_2, C_3 do not depend on ε.

2) The interaction is short-range, so that each point $\underline{x}^i_\varepsilon$ is connected by an edge of some graph Γ to neighbors $\underline{x}^j_\varepsilon$ which are at a distance less than $C\varepsilon$; $C > 0$ is fixed constant which does not depend on ε. Moreover, we consider the graphs which satisfy the so-called *triangulization condition*. Namely, there exists a subgraph $\Gamma' \in \Gamma$ such that Γ' forms a triangularization of $\mathbb{R}^3$, that is it partitions $\mathbb{R}^3$ into simplexes

with angles bounded from below by a constant $C > 0$ which does not depend on ε. Γ' need not contain all the vertices x^i_ε.

Next we restrict all edges Γ on the domain Ω. If an edge of Γ intersects $\partial\Omega$ then we add the intersection point to the graph and denote the obtained graph Γ_Ω.

This triangularization is sufficient for the discrete Korn's inequality (5.3) to hold.

II. The interaction conditions.

1) The particles Q^i_ε and Q^j_ε interact if the distance between them is of order $O(\varepsilon)$. Thus, $\underline{\underline{C}}^{ij}_\varepsilon = 0$, if $\operatorname{dist}(Q^i_\varepsilon, Q^j_\varepsilon) \geq C_1\varepsilon$ $(C_1 0)$. In particular, pairs of particles which are connected by a common edge of a simplex are interacting.

2) The matrix $\underline{\underline{C}}^{ij}_\varepsilon$ of interaction of particles Q^i_ε and Q^j_ε is the matrix of the projection operator on vector $\underline{x}^i_\varepsilon - \underline{x}^j_\varepsilon$ up to a positive scalar multiplier $\dfrac{k^{ij}_\varepsilon}{|\underline{x}^i_\varepsilon - \underline{x}^j_\varepsilon|}$, i.e.

$$\underline{\underline{C}}^{ij}_\varepsilon \underline{u} = k^{ij}_\varepsilon \Big\langle \frac{\underline{u}}{|\underline{x}^i_\varepsilon - \underline{x}^j_\varepsilon|}, \underline{e}_{ij} \Big\rangle \underline{e}_{ij}, \quad \forall \underline{u} \in \mathbb{R}^3. \tag{3.3}$$

Here $|\underline{e}_{ij} = \dfrac{\underline{x}^i_\varepsilon - \underline{x}^j_\varepsilon|}{|\underline{x}^i_\varepsilon - \underline{x}^j_\varepsilon|}$ and k^{ij}_ε is defined as follows

$$k^{ij}_\varepsilon = k^{ij}\varepsilon^2,\ a_1 \leq k^{ij} \leq a_2, \tag{3.4}$$

where the constants $a_2 \geq a_1 > 0$ do not depend on ε.

3.1. **The qualitative characteristic of an elastic skeleton.** We introduce a meso-characteristic which characterizes the local (average on meso-scale) elastic properties of elastic skeleton.

Denote by $K^{\underline{x}}_h = K(\underline{x}, h)$ the cube of side length $h > 0\,(\varepsilon \ll h \ll 1)$ centered at a point $x \in \Omega$. The orientation of the cube is arbitrary but independent of $\underline{x}$ and h. For the sake of definiteness we assume that the edges of this cube are parallel to the coordinate axes.

Let $N(\underline{x}, \varepsilon, h)$ be the set of lattice vector-functions $w_\varepsilon(\underline{x})$. These functions are defined at each point $\underline{x}^i_\varepsilon$, that belongs to the cube $K(\underline{x}, h)$. Their values at the points $\underline{x}^i_\varepsilon$ we denote by $w^i_\varepsilon = w_\varepsilon(\underline{x}^i_\varepsilon)$ $(i = 1, 2, \dots, N^{x,h}_\varepsilon)$.

Consider a minimization problem in the class $N(\underline{x}, \varepsilon, h)$ for the following functional

$$A^\tau_{\varepsilon h}[w_\varepsilon; \underline{x}; T] = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} {}_{K^x_h} \langle \underline{\underline{C}}^{ij}_\varepsilon [w^i_\varepsilon - w^j_\varepsilon], [w^i_\varepsilon - w^j_\varepsilon] \rangle + \tag{3.5}$$

$$+h^{-2-\tau}\varepsilon^3 \sum_i {}_{K^x_h} \left| w^i_\varepsilon - \sum_{p,q=1}^{3} \varphi^{pq}(\underline{x}^i_\varepsilon - \underline{x}) T_{pq} \right|^2 .$$

Here $\varphi_{pq}(\underline{x}) = \frac{1}{2}(x_p \underline{e}^q + x_q \underline{e}^p)$, $T = \{T_{pq}\}$ is a symmetric second rank tensor, the sum $\sum_{K^x_h}$ stands for the summation over all indexes i for which $\underline{x}^i_\varepsilon \in K^x_h$, $0 < \tau < 2$ is a technical parameter.

Lemma 3.1. *There exists a unique lattice function $w = \{w^i\}$ that minimizes the functional (3.5). The minimum of this functional is a quadratic homogeneous function of the tensor $T = \{T_{pq}\}$. Moreover, the following representation holds*

$$\min_{w_\varepsilon \in \mathrm{N}(\underline{x},\varepsilon,h)} A^\tau_{\varepsilon h}[w_\varepsilon; x; T] = \sum_{n,p,q,r=1}^{3} a^\tau_{npqr}(\underline{x},\varepsilon,h) T_{np} T_{qr}, \tag{3.6}$$

where $a^\tau_{npqr}(\underline{x},\varepsilon,h)$ are the components of a fourth rank tensor. These components are defined as follows:

$$a^\tau_{npqr}(\underline{x},\varepsilon,h) = \frac{1}{2}\sum_{i\neq j}{}_{K^x_h}\langle \underline{\underline{C}}^{ij}_\varepsilon [w^{np}(\underline{x}^i_\varepsilon) - w^{np}(\underline{x}^j_\varepsilon)], [w^{qr}(\underline{x}^i_\varepsilon) - w^{qr}(\underline{x}^j_\varepsilon)]\rangle + \tag{3.7}$$

$$+ h^{-2-\tau}\varepsilon^3 \sum_i {}_{K^x_h}\langle w^{np}(\underline{x}^i_\varepsilon) - \varphi^{np}(\underline{x}^i_\varepsilon - x), w^{qr}(\underline{x}^i_\varepsilon) - \varphi^{qr}(\underline{x}^i_\varepsilon - x)\rangle.$$

Hereafter we denote by $w^{np}(\underline{x})$ the lattice vector-function that minimizes functional (3.5) with $T = T^{np} = \frac{1}{2}(\underline{e}^n \otimes \underline{e}^p + \underline{e}^p \otimes \underline{e}^n)$.

The proof of this Lemma is given in [4]. It follows from (3.7) that the tensor $\{a^\tau_{npqr}(\underline{x},\varepsilon,h)\}$ is symmetric

$$a_{npqr} = a_{qrnp} = a_{pnqr} = \dots$$

We call this tensor the local elasticity tensor of the elastic skeleton. This meso-characteristic plays the key role in our considerations since it defines the effective elastic modules in the continuum limit via discrete interactions.

Observe that the first sum in (3.5) represents the interaction energy in K^x_h. The second sum is a penalty term which represents the deviation of the vectors w^i_ε from the linear part $\sum_{p,q=1}^{3} T_{pq}\varphi^{pq}(\underline{x}^i_\varepsilon - \underline{x})$. This linear part can be viewed as a symmetric part of the differential of the second global minimizer $\underline{w}(\underline{x})$ (see Theorem 4.1) when $T_{pq} = e_{pq}[\underline{w}(\underline{x})]$. Therefore we minimize the interaction energy in K^x_h by adding the constraint that the minimizer $\underline{w}^i_\varepsilon$ is "close" to the restriction of the symmetric part of $\nabla \underline{w}(\underline{x})$ on the lattice $\{\underline{x}^i_\varepsilon\}$.

We assume that the following limits exist uniformly for all $x \in \Omega$

$$\lim_{h\to 0}\lim_{\varepsilon\to 0}\frac{1}{h^3}\sum_{i}{}_{K^x_h} r^i_\varepsilon = r(\underline{x}), \tag{3.8}$$

$$\lim_{h\to 0}\varliminf_{\varepsilon\to 0}\frac{a^\tau_{npqr}(\underline{x},\varepsilon,h)}{h^3} = \lim_{h\to 0}\varlimsup_{\varepsilon\to 0}\frac{a^\tau_{npqr}(\underline{x},\varepsilon,h)}{h^3} = a_{npqr}(\underline{x}), \tag{3.9}$$

where $r(\underline{x}) > 0$ is a continuous function and $\{a_{npqr}(\underline{x})\}$ is a continuous, positive defined tensor.

Note, that the existence of limits (3.8)-(3.9) is a general restriction on the spatial distributions of the locations of the particles and their radii. Since we do not require any spatial periodicity, we have to impose some conditions on these distributions. In section 9, we provide an example where limits (3.8)-(3.9) are calculated explicitly.
Remark. If the limits in (3.9) exist for some $\tau 0$, then they exist for any $\tau > 0$ and the limiting tensor $a_{npqr}(\underline{x})$ does not depend on τ (this fact can be proved analogously to [24]). This limiting tensor incorporates the information about the geometric array of the particles and the strength of the interparticle interactions.

Let $\chi_\varepsilon(\underline{x})$ be the characteristic function of the fluid domain Ω_ε, and $\chi^i_\varepsilon(\underline{x})$ be the characteristic function of the particles Q^i_ε.

Using the solution $\{\underline{v}_\varepsilon(\underline{x},t), \underline{u}^i_\varepsilon(\underline{x},t), \underline{\theta}^i_\varepsilon(t), i=1,2,\dots,N_\varepsilon\}$ of the problem (2.2)–(2.7), construct the vector-functions

$$\underline{\tilde{v}}_\varepsilon(\underline{x},t) = \chi_\varepsilon(\underline{x})\underline{v}_\varepsilon(\underline{x},t) + \sum_{i=1}^{N_\varepsilon} \chi^i_\varepsilon(\underline{x})[\underline{\dot{u}}^i_\varepsilon(t) + \underline{\dot{\theta}}^i_\varepsilon(t) \times (\underline{x} - \underline{x}^i_\varepsilon)]. \tag{3.10}$$

The following theorem is the main result of the paper.

Theorem 3.2. *Suppose that conditions I, II, (3.8), (3.9) hold. Suppose that the initial conditions of the problem (2.2)–(2.7) converge as $\varepsilon \to 0$ in the following sense*

$$\underline{\tilde{v}}_\varepsilon(\underline{x},0) \to v_0(\underline{x}) \quad \text{in } L_2(\Omega). \tag{3.11}$$

Then the vector-functions $\underline{v}_\varepsilon(\underline{x},t)$ converge, as $\varepsilon \to 0$, weakly in $L_2(\Omega \times [0,T])$ (for any $T0$) to a vector-function $v(\underline{x},t)$ such that the pair of vector-functions $\{v(\underline{x},t), w(\underline{x},t)\}$ is a solution of the following initial boundary value problem

$$\rho\frac{\partial v}{\partial t} - \mu\Delta v + C(\underline{x})(v-w) = \nabla p, \ \mathrm{div} v = 0, \quad x \in \Omega, \ t > 0. \tag{3.12}$$

$$C(\underline{x})(w-v) - \sum_{n,p,q,r} \frac{\partial}{\partial x_p}\left(a_{npqr}(\underline{x}) e_{qr}\left[\int_0^t w d\tau\right]\right) e^n = 0, \quad x \in \Omega, t > 0, \tag{3.13}$$

$$v(\underline{x},t) = w(\underline{x},t) = 0, \quad x \in \partial\Omega, \tag{3.14}$$

$$v(\underline{x},0) = v_0(\underline{x}), \quad x \in \Omega. \tag{3.15}$$

Here $C(\underline{x}) = 6\pi\mu r(\underline{x})$; the function $r(\underline{x})$ and the tensor $\{a_{npqr}(\underline{x})\}$ are defined by (3.8) and (3.9) respectively.

The proof of theorem 2.1 is given in sections 3-8.

First, in section 3, using Laplace transform, we formulate a stationary version of the problem (2.2)–(2.7) with parameter λ . Then we reduce it to a variational form for $\lambda 0$. In sections 4-7, we study the asymptotic behavior of solution of the variational problem as $\varepsilon \to 0$ by using a method close to the computation of a Γ-limit. We find the homogenized variational functional and the system of Euler equations corresponding to this functional. Finally, in section 8 we study the analytic properties of the solutions of these equations in the parameter λ, and, applying the inverse Laplace transform, obtain the homogenized non-stationary problem (3.12) –(3.15).

In Section 9 we consider the special case of a periodic array of balls Q^i_ε and calculate the tensor $a(\underline{x}) = \{a_{npqr}(\underline{x})\}$ explicitly.

4. **Variational formulation of the problem.** We convert the evolutionary problem (2.2)–(2.7) into the stationary one by using the Laplace transform in time.

For the sake of simplicity, we keep the same notations for the Laplace images as for the corresponding time-dependent quantities: $\underline{v}_\varepsilon(\underline{x},t) \to \underline{v}_\varepsilon(\underline{x},\lambda)$, $p_\varepsilon(\underline{x},t) \to p_\varepsilon(\underline{x},\lambda)$, $\underline{u}^i_\varepsilon(t) \to \underline{u}^i_\varepsilon(\lambda)$, $\underline{\theta}^i_\varepsilon(t) \to \underline{\theta}^i_\varepsilon(\lambda)$ ($\mathrm{Re}\lambda > 0$). Then using (2.1) we obtain the following boundary value problem for Laplace transforms

$$-\mu\Delta\underline{v}_\varepsilon + \lambda\rho\underline{v}_\varepsilon - \nabla p_\varepsilon = \rho\underline{v}_{\varepsilon 0}(\underline{x}), \quad \mathrm{div}\underline{v}_\varepsilon = 0, \quad \underline{x} \in \Omega_\varepsilon, \tag{4.1}$$

$$\underline{v}_\varepsilon(\underline{x}) = \lambda[\underline{u}^i_\varepsilon + \underline{\theta}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon)], \quad \underline{x} \in Q^i_\varepsilon, \tag{4.2}$$

$$\lambda^2 m_\varepsilon^i \underline{u}_\varepsilon^i + \int\limits_{S_\varepsilon^i} \sigma[\underline{v}_\varepsilon]\cdot \underline{\nu} ds = -\sum_{\substack{j\\ j\neq i}} \underline{\underline{C}}_\varepsilon^{ij}[\underline{u}_\varepsilon^i - u_\varepsilon^j] + m_\varepsilon^i u_{\varepsilon 1}^i, \tag{4.3}$$

$$\lambda^2 I_\varepsilon^i \underline{\theta}_\varepsilon^i + \int\limits_{S_\varepsilon^i} (\underline{x} - \underline{x}_\varepsilon^i)\times \sigma[\underline{v}_\varepsilon]\cdot \underline{\nu} ds = I_\varepsilon^i \theta_{\varepsilon 1}^i, \tag{4.4}$$

$$\underline{v}_\varepsilon(\underline{x}) = 0, \quad \underline{x}\in\partial\Omega. \tag{4.5}$$

We extend the velocity function $\underline{v}_\varepsilon(\underline{x},\lambda)$ onto the particles Q_ε^i by equality (4.2) and keep the same notation for the extended function $\underline{v}_\varepsilon = \underline{v}_\varepsilon(\underline{x},\lambda)$. It is easy to see that $\underline{v}_\varepsilon \in \overset{\circ}{H}{}^1(\Omega)$ and that $\operatorname{div}\underline{v}_\varepsilon = 0$ in Ω. We now denote by $\overset{\circ}{J}_\varepsilon(\Omega)$ the class of divergence free vector functions from $\overset{\circ}{H}{}^1(\Omega)$ which are equal to $\underline{a}_\varepsilon^i + \underline{b}_\varepsilon^i \times (\underline{x} - \underline{x}_\varepsilon^i)$ for $\underline{x}\in Q_\varepsilon^i$ with arbitrary constant vectors $\underline{a}_\varepsilon^i$ and $\underline{b}_\varepsilon^i$.

Consider the minimization problem in $\overset{\circ}{J}_\varepsilon(\Omega)$ for the following functional:

$$\Phi_\varepsilon(\underline{v}_\varepsilon) = \int\limits_\Omega \left\{ 2\mu \sum_{k,l=1}^3 e_{kl}^2[\underline{v}_\varepsilon] + \lambda\langle \rho_\varepsilon \underline{v}_\varepsilon, \underline{v}_\varepsilon\rangle - 2\langle \rho_\varepsilon \underline{v}_{\varepsilon 0}, \underline{v}_\varepsilon\rangle \right\} d\underline{x} + \tag{4.6}$$

$$+\frac{1}{\lambda}\sum_{i,j} \langle \underline{\underline{C}}_\varepsilon^{ij}[\underline{v}_\varepsilon(\underline{\underline{x}}_\varepsilon^i) - \underline{v}_\varepsilon(\underline{x}_\varepsilon^j)], [\underline{v}_\varepsilon(\underline{\underline{x}}_\varepsilon^i) - \underline{v}_\varepsilon(\underline{x}_\varepsilon^j)]\rangle,$$

where $\underline{v}_{\varepsilon 0} = \underline{v}_\varepsilon(\underline{x}, 0)$, $\lambda > 0$, $\rho_\varepsilon = \rho\chi_\varepsilon(\underline{x}) + \rho_s \sum_i \chi_\varepsilon^i(\underline{x})$, and $e_{kl}[v] = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right)$ are the components of the strain tensor.

Taking into account the fact that the matrix C_ε^{ij} is non-negative, it is easy to show that there exists a unique minimizer $\underline{v}_\varepsilon = \underline{v}_\varepsilon(\underline{x},\lambda)$ of the functional (4.6) in the class $\overset{\circ}{J}_\varepsilon(\Omega)$. This minimizer is a solution $\{\underline{v}_\varepsilon(x,\lambda)\chi_\varepsilon(x), \underline{u}_\varepsilon^i + \underline{\theta}_\varepsilon^i \times (x - \underline{x}_\varepsilon^i) = \underline{v}_\varepsilon(x,\lambda)\chi_\varepsilon^i(x)\}$ of the boundary value problem (4.1)–(4.5) for $\lambda > 0$.

The main goal is to investigate the asymptotic behavior of the solution $\underline{v}_\varepsilon(x)$ of the minimization problem

$$\Phi_\varepsilon(\underline{v}_\varepsilon) = \min_{\underline{v}'_\varepsilon \in \overset{\circ}{J}_\varepsilon(\Omega)} \Phi_\varepsilon(\underline{v}'_\varepsilon), \tag{4.7}$$

as $\varepsilon \to 0$.

Introduce the functional

$$\Phi_0(\underline{v},\underline{w}) = \int\limits_\Omega \Big\{ 2\mu \sum_{k,l=1}^3 e_{kl}^2[\underline{v}] + \lambda\langle\rho\underline{v},\underline{v}\rangle + \tag{4.8}$$

$$+\langle C(x)[\underline{v}-\underline{w}], [\underline{v}-\underline{w}]\rangle + \frac{1}{\lambda}\sum_{n,p,q,r} a_{npqr}(x) e_{np}[\underline{w}] e_{qr}[\underline{w}] - 2\langle\rho\underline{v}_0,\underline{v}\rangle \Big\} dx,$$

where $\underline{v}_0 = \underline{v}_0(x)$ is the limiting vector function defined by (3.11). We will show that (4.8) is the limiting functional for (4.7) as $\varepsilon \to 0$.

Denote the class of divergence free vector-functions from $\overset{\circ}{H}{}^1(\Omega)$ by $\overset{\circ}{J}(\Omega)$ and introduce the following class of pairs $(\underline{v}', \underline{w}')$: $\overset{\circ}{H}(\Omega) = \overset{\circ}{J}(\Omega) \times \overset{\circ}{H}{}^1(\Omega)$ such that

$\underline{v}' \in \overset{\circ}{J}(\Omega)$ and $\underline{w}' \in \overset{\circ}{H}{}^1(\Omega)$. Consider the minimization problem for the functional (4.8)

$$\Phi_0(\underline{v}, \underline{w}) = \inf_{(\underline{v}', \underline{w}') \in \overset{\circ}{H}(\Omega)} \Phi_0(\underline{v}', \underline{w}'). \tag{4.9}$$

It follows from (4.8) that the solution $(\underline{v}, \underline{w})$ of the problem (4.9) is a generalized solution of the boundary value problem

$$-\mu\Delta\underline{v} + C[\underline{v} - \underline{w}] + \lambda\rho\underline{v} = \rho\underline{v}_0 + \nabla p, \ \mathrm{div}\underline{v} = 0, \quad \underline{x} \in \Omega, \tag{4.10}$$
$$-\frac{1}{\lambda}\sum_{n,p,q,r}\frac{\partial}{\partial x_p}\{a_{npqr}(\underline{x})e_{qr}[\underline{w}]\}\,\underline{e}^n + C[\underline{w} - \underline{v}] = 0, \quad \underline{x} \in \Omega,$$
$$\underline{v} = \underline{w} = 0, \quad \underline{x} \in \partial\Omega. \tag{4.11}$$

(this is valid for $a_{npqr} \in L^\infty(\Omega)$).

For the sake of simplicity, we assume that $a_{npqr} \in C^1(\Omega)$. Then a strong solution of the problem (4.10)–(4.11) exists.

Since $C(\underline{x}) > 0$ and the tensor $\underline{\underline{a}}(\underline{x}) = \{a_{npqr}(\underline{x})\}$ is positive defined, a unique solution of the problem (4.9) exists and it is a solution of the problem (4.10)–(4.11).

Using the solution $\underline{v}_\varepsilon(\underline{x}, \lambda)$ of the minimization problem (4.7), we construct a piecewise linear spline

$$\underline{w}_\varepsilon(\underline{x}, \lambda) = \sum_{i=1}^{N_\varepsilon} \underline{v}_\varepsilon(\underline{x}_\varepsilon^i)\underline{L}_\varepsilon^i(\underline{x}) \equiv \lambda\sum_{i=1}^{N_\varepsilon}\underline{u}_\varepsilon^i\underline{L}_\varepsilon^i, \tag{4.12}$$

where $\underline{u}_\varepsilon^i = \underline{u}_\varepsilon^i(\lambda)$ is defined by equality $\lambda^{-1}\underline{v}_\varepsilon(x, \lambda) = \underline{u}_\varepsilon^i(\lambda) + \underline{\theta}_\varepsilon^i(\lambda) \times (\underline{x} - \underline{x}^i)$ for $\underline{x} \in Q_\varepsilon^i$ and $\underline{L}_\varepsilon^i$ is a finite element of simplexes defined in geometrical condition 2 (Section 2). The function $\underline{L}_\varepsilon^i$ is continuous in $\mathbb{R}^3$, is linear in each simplex of the subgraph Γ', and $\underline{L}_\varepsilon^i(\underline{x}_\varepsilon^j) = \delta_{ij}$.

Theorem 4.1. *Suppose that conditions I, II (Section 2) and conditions (3.8), (3.9) hold. Then the solution of the problem (4.7) converges to the solution $(\underline{v}(\underline{x}, \lambda), \underline{w}(\underline{x}, \lambda))$ of the problem (4.9), as $\varepsilon \to 0$, in the following sense. For any $\lambda, 0$*

$$\underline{v}_\varepsilon(\underline{x}, \lambda) \to \underline{v}(\underline{x}, \lambda) \quad \text{in } L_2(\Omega), \tag{4.13}$$
$$\underline{w}_\varepsilon(\underline{x}, \lambda) \to \underline{w}(\underline{x}, \lambda) \quad \text{in } L_2(\Omega). \tag{4.14}$$

The proof of this theorem is given in §§5-7.

5. **The compactness of solutions of the variational problem (4.7).** Let $\underline{v}_\varepsilon(\underline{x}, \lambda)$, $\varepsilon > 0$, be a solution of the minimization problem (4.7). Since $0 \in \overset{\circ}{J}{}_\varepsilon(\Omega)$, we have $\Phi_\varepsilon(\underline{v}_\varepsilon) \le \Phi_\varepsilon(0) = 0$. Then (4.6) implies

$$\int_\Omega \left\{2\mu\sum_{k.l=1}^3 e_{kl}^2[\underline{v}_\varepsilon]dx + \lambda\langle\rho_\varepsilon\underline{v}_\varepsilon, \underline{v}_\varepsilon\rangle\right\}dx + \frac{1}{\lambda}I_\varepsilon(\underline{v}_\varepsilon, \underline{v}_\varepsilon) \le 2\|\rho_\varepsilon\underline{V}_{\varepsilon 0}\|_{L_2(\Omega)}\|\underline{v}_\varepsilon\|_{L_2(\Omega)}, \tag{5.1}$$

where

$$I_\varepsilon(\underline{v}_\varepsilon, \underline{w}_\varepsilon) = \sum_{j\neq i} \langle \underline{\underline{C}}_\varepsilon^{ij}[\underline{v}_\varepsilon^i - \underline{v}_\varepsilon^j], [\underline{w}_\varepsilon^i - \underline{w}_\varepsilon^j]\rangle, \qquad \underline{v}_\varepsilon^i = \underline{v}_\varepsilon^i(\underline{x}_\varepsilon^i), \quad \underline{w}_\varepsilon^i = \underline{w}_\varepsilon^i(\underline{x}_\varepsilon^i). \tag{5.2}$$

We now apply the second Korn's inequality [28]

$$\|\underline{v}_\varepsilon\|^2_{H^1(\Omega)} \leq C\left(\int_\Omega \sum_{k,l=1}^{3} e^2_{kl}[\underline{v}_\varepsilon]dx + \|\underline{v}_\varepsilon\|^2_{L_2(\Omega)}\right).$$

and the discrete Korn's inequality (see [4])

$$\|\underline{w}_\varepsilon\|^2_{H^1(\Omega)} \leq \frac{4C_2^3}{3a_1C_1}\sum_{i,j}{}' \langle \underline{\underline{C}}_\varepsilon^{ij}[\underline{u}_\varepsilon^i - \underline{u}_\varepsilon^j], [\underline{u}_\varepsilon^i - \underline{u}_\varepsilon^j]\rangle, \tag{5.3}$$

where the constants C_1, C_2 and a_1 are defined in (3.1) and (3.4), respectively, the sum $\sum_{i,j}{}'$ is taken over the pairs (i, j) corresponding to the edges $(\underline{x}_\varepsilon^i, \underline{x}_\varepsilon^j)$ of the simplexes that triangulate Ω. Then, since $\mu > 0$, $\lambda > 0$, $\rho_\varepsilon(x) \geq \min(\rho, \rho_s)0$, (5.1)-(5.3) yield

$$\|\underline{v}_\varepsilon\|^2_{H^1(\Omega)} + \|\underline{v}_\varepsilon\|^2_{H^1(\Omega)} < C,$$

where C does not depend on ε.

Therefore, the sets of vector functions $\{\underline{v}_\varepsilon(\underline{x}, \lambda), \varepsilon 0\}$ and $\{\underline{w}_\varepsilon(\underline{x}, \lambda), \varepsilon > 0\}$ (see (4.12)) are weakly compact in $H^1(\Omega)$. Let us choose subsequences $\underline{v}_{\varepsilon_k}(\underline{x}, \lambda)$ and $\underline{w}_{\varepsilon_k}(\underline{x}, \lambda)$ that converge to some vector functions $\underline{v} \in \overset{\circ}{H}{}^1(\Omega)$ and $\underline{w} \in \overset{\circ}{H}{}^1(\Omega)$, respectively (weakly in $\overset{\circ}{H}{}^1(\Omega)$, as $\varepsilon_k \to 0$). Due to the Embedding Theorem, they converge to $\underline{v}(\underline{x})$ and $\underline{w}(\underline{x})$ strongly in $L_2(\Omega)$. Clearly, $\underline{v}(\underline{x}) \in \overset{\circ}{J}(\Omega)$.

We will show that the pair $\{\underline{v}, \underline{w}\}$ is a minimizer of (4.9). Since this problem has a unique solution and the sets $\{\underline{v}_\varepsilon, \varepsilon 0\}$ and $\{\underline{w}_\varepsilon, \varepsilon > 0\}$ are compact, we conclude that $\underline{v}_\varepsilon \to \underline{v}$ and $\underline{w}_\varepsilon \to \underline{w}$ weakly in $H^1(\Omega)$ and strongly in $L_2(\Omega)$, as $\varepsilon \to 0$. The proof of the fact that the pair $\{\underline{v}, \underline{w}\}$ minimizes functional $\Phi_0[\underline{v}, \underline{w}]$ in $\overset{\circ}{J}(\Omega) \times \overset{\circ}{H}{}^1(\Omega)$ will be given in two steps in Sections 6 and 7. This is done by proving a Γ-convergence result with respect to the convergence (4.13)-(4.14).

The scheme of the proof is the following. Starting from an arbitrary pair $\{\underline{v}', \underline{w}'\} \in \left(\overset{\circ}{J}(\Omega) \times \overset{\circ}{H}{}^1(\Omega)\right) \bigcap C_0^2(\Omega)$ we construct a special test function $\underline{v}_{\varepsilon h}(\underline{x}) \in \overset{\circ}{J}_\varepsilon(\Omega)$, where h is a meso-parameter such that $\varepsilon \ll h \ll 1$. Since $\underline{v}_\varepsilon(\underline{x})$ minimizes (4.7), we have

$$\Phi_\varepsilon(\underline{v}_\varepsilon) \leq \Phi_\varepsilon(\underline{v}_{\varepsilon h}). \tag{5.4}$$

Using the explicit construction of the function $\underline{v}_{\varepsilon h}(\underline{x})$, we show that

$$\overline{\lim_{h\to 0}}\ \overline{\lim_{\varepsilon\to 0}}\ \Phi_\varepsilon(\underline{v}_{\varepsilon h}) \leq \Phi_0(\underline{v}', \underline{w}'). \tag{5.5}$$

It follows from (5.4) – (5.5) that

$$\overline{\lim_{\varepsilon\to 0}}\ \Phi_\varepsilon(\underline{v}_\varepsilon) \leq \Phi_0[\underline{v}', \underline{w}'], \quad \forall \underline{v}' \in \overset{\circ}{J}(\Omega),\ \ \underline{w}' \in \overset{\circ}{H}{}^1(\Omega). \tag{5.6}$$

Next, in Section 7 we show that if the functions $\underline{v} \in \overset{\circ}{J}(\Omega)$ and $\underline{w} \in \overset{\circ}{H}{}^1(\Omega)$ are the weak limits of $\underline{v}_\varepsilon(\underline{x})$ and $\underline{w}_\varepsilon(\underline{x})$, respectively (as $\varepsilon = \varepsilon_k \to 0$), then the inverse inequality holds:

$$\varliminf_{\varepsilon=\varepsilon_k\to 0} \Phi_\varepsilon(\underline{v}_\varepsilon) \geq \Phi_0(\underline{v},\underline{w}). \tag{5.7}$$

It follows from (5.6) and (5.7) that for any pair $(\underline{v}',\underline{w}') \in \overset{\circ}{H}(\Omega) = \overset{\circ}{J}(\Omega) \times \overset{\circ}{H}{}^1(\Omega)$,

$$\Phi_0(\underline{v},\underline{w}) \leq \Phi_0(\underline{v}',\underline{w}').$$

Thus the pair $(\underline{v},\underline{w})$ is a solution of the minimization problem (4.9).

6. **The proof of upper bound (5.5).** Cover the domain Ω by cubes $K_\alpha = K(\underline{x}^\alpha, h)$ with the edges of length h, which are parallel to coordinate axes. The centers $\underline{x}^\alpha$ of the cubes form a cubic lattice of the $h-\delta$ $(0<\varepsilon\ll\delta\ll h)$, so that the cubes overlap. Due to the overlap of the cubes, we can further select smaller cubes $K'_\alpha = K(\underline{x}^\alpha, h)$ with the same center $\underline{x}_\alpha$ and edges of length $h' = h - 2\delta$. Clearly, $K'_\alpha = K_\alpha \setminus \bigcup_{\beta\neq\alpha} K_\beta$.

Let $\{\varphi_h^\alpha(\underline{x}),\quad \alpha = 1,2,\ldots\}$ be the partition of unity which corresponds to the covering $\bigcup_\alpha K_\alpha$: 1) $0 \leq \varphi_h^\alpha(\underline{x}) \leq 1$, $\quad \varphi_h^\alpha(\underline{x}) \in C^2(\mathbb{R}^3)$; 2) $\varphi_h^\alpha(\underline{x}) = 1$ in K'_α and $\varphi_h^\alpha(\underline{x}) = 0$ in $\mathbb{R}^3 \setminus K_\alpha$; 3) $\sum_\alpha \varphi_h^\alpha(\underline{x}) = 1$ in $\mathbb{R}^3$; 4) $|\nabla\varphi_h^\alpha(\underline{x})| < \dfrac{C}{\delta}$, where C does not depend on h and δ.

Let $\underline{w}(\underline{x})$ be an arbitrary vector-function in $C_0^2(\Omega)$. Introduce the discrete vector function $\underline{w}^{\varepsilon h}(\underline{x})$ defined at each point $\underline{x}_\varepsilon^i \in \Omega$, $i = 1,\ldots,N_\varepsilon$,

$$\underline{w}^{\varepsilon h}(\underline{x}_\varepsilon^i) = \sum_\alpha \Bigg\{ \underline{w}(\underline{x}^\alpha) + \sum_{n,p=1}^3 e_{np}[\underline{w}(\underline{x}^\alpha)]\underline{w}^{\alpha np}(\underline{x}_\varepsilon^i) + \tag{6.1}$$
$$+ \sum_{n,p=1}^3 \omega_{np}[\underline{w}(\underline{x}^\alpha)]\underline{\psi}^{np}(\underline{x}_\varepsilon^i - \underline{x}^\alpha)\Bigg\}\varphi_h^\alpha(\underline{x}_\varepsilon^i).$$

Here $\underline{\psi}^{np}(\underline{x}) = \frac{1}{2}(\underline{e}^n x_p - \underline{e}^p x_n)$ and $\{e_{np}[\underline{w}(\underline{x}^\alpha)]\}$, $\{w_{np}[\underline{w}(\underline{x}^\alpha)]\}$ are, respectively, the symmetric and skew-symmetric parts of the tensor $\nabla\underline{w}(\underline{x})$ respectively at the point $\underline{x}^\alpha$: $e_{np}[\underline{w}(\underline{x}^\alpha)] = \frac{1}{2}\left(\frac{\partial w_n}{\partial x_p}(\underline{x}^\alpha) + \frac{\partial w_p}{\partial x_n}(\underline{x}^\alpha)\right)$; $\omega_{np}[\underline{w}(\underline{x}^\alpha)] = \frac{1}{2}\left(\frac{\partial w_n}{\partial x_p}(\underline{x}^\alpha) - \frac{\partial w_p}{\partial x_n}(\underline{x}^\alpha)\right)$, $\underline{\psi}^{np}(\underline{x}) = \frac{1}{2}(\underline{e}^n x_p - \underline{e}^p x_n)$. The vector function $\underline{w}^{\alpha np}(\underline{x})$ is a discrete vector function that minimizes the functional $A_{\varepsilon h}^\tau(\underline{w};\underline{x}^\alpha;T)$ (see (3.5)) in the cube $K(\underline{x}^\alpha,h)$ with $T = T^{np} = \dfrac{1}{2}(\underline{e}^n \otimes \underline{e}^p + \underline{e}^p \otimes \underline{e}^n)$; therefore it is close to φ^{np}.

Remark. We construct this function in order to satisfy the following conditions:

1) for sufficiently small h and $\varepsilon < \hat{\varepsilon}(h)$, the function $\underline{w}^{\varepsilon h}(\underline{x})$ is close (in some sense) to the restriction of $\underline{w}(\underline{x})$ on the lattice $\{\underline{x}_\varepsilon^i\}$;

2) for $h \ll 1$ and $\varepsilon \ll \hat{\varepsilon}(h)$, the function $\underline{w}^{\varepsilon h}(\underline{x})$ is an "almost" minimizer of the functional (3.5) in every cube $K(\underline{x}^\alpha,h)$ when $T_{np} = e_{np}[\underline{w}(\underline{x}^\alpha)]$.

Note that any function $\underline{w}(\underline{x}) \in C^2\big(K(\underline{x}^\alpha,h)\big)$ can be represented in the form

$$\underline{w}(\underline{x}) = \underline{w}(\underline{x}^\alpha) + \sum_{n,p=1}^{3} \Big(e_{np}[\underline{w}(\underline{x}^\alpha)]\varphi^{np}(\underline{x} - \underline{x}^\alpha) + \tag{6.2}$$
$$+\omega_{np}[\underline{w}(\underline{x}^\alpha)]\underline{\psi}^{np}(\underline{x} - \underline{x}^\alpha)\Big) + \underline{O}(h^2),$$

and that the term

$$\omega_{np}[\underline{w}(\underline{x}^\alpha)]\underline{\psi}^{np}(\underline{x} - \underline{x}^\alpha) = \underline{b}^{\alpha np} \times (\underline{x} - \underline{x}^\alpha) \tag{6.3}$$

($\underline{b}^{\alpha np}$ is a constant vector) does not contribute to the elastic energy. Thus 1)–2) in Remark hold. Hereafter $\underline{O}(h^k)$ stands for a vector function with the norm of order $O(h^k)$.

To go further, we need the following theorem on the representation of divergence free functions [13].

Theorem 6.1. *Let G be a domain in $\mathbb{R}^3$ that is a homeomorphic image of a ball. Let $J(G)$ be a subspace of $L_2(G)$ that is the closure of the set of smooth divergence free vector functions. Then for any $\underline{u}(\underline{x}) \in J(G)$, the following representation holds:*

$$\underline{u}(\underline{x}) = \operatorname{curl} \underline{\tilde{u}}(\underline{x})$$

where the vector function $\underline{\tilde{u}}(\underline{x})$ ia such that $\underline{\tilde{u}}(\underline{x}) \in H^1(G)$, $\operatorname{div} \underline{\tilde{u}} = 0$ and $\langle \underline{\tilde{u}}, \underline{n} \rangle = 0$ on ∂G. The function $\underline{\tilde{u}}(\underline{x})$ is defined uniquely by these three conditions.

If $\underline{u}(\underline{x}) \in J(G) \bigcap H^m(G)$, then $\underline{\tilde{u}}(\underline{x}) \in H^{m+1}(G)$, and the following inequality holds: $\|\underline{\tilde{u}}\|_{H^{m+1}(G)} \le C\|\underline{u}\|_{H^m(G)}$. Here the constant C depends only on the domain G.

Let $G = B_a$ be a ball of radius a. Then Theorem 6.1 and the scaling considerations give the following estimates

$$\|\underline{\tilde{u}}\|_{L_2(B_a)} \le Ca\|\underline{u}\|_{L_2(B_a)}, \tag{6.4}$$

$$\|D^\alpha \underline{\tilde{u}}\|_{L_2(B_a)} \le C \left(a^{1-|\alpha|}\|\underline{u}\|_{L_2(B_a)} + \sum_{|\beta|=|\alpha|-1} \|D^\beta \underline{u}\|_{L_2(B_a)} \right), \tag{6.5}$$

where C does not depend on a and $\underline{u} \in J(B_a)$, $|\alpha| = 1, 2$.

We denote by $B^i_{1\varepsilon}$ and $B^i_{2\varepsilon}$ the balls of radii $r^i_{1\varepsilon}$ and $r^i_{2\varepsilon}$ with a common center at the point $\underline{x}^i_\varepsilon$. Choose $r^i_{1\varepsilon}$ equal to the doubled radius of the ball Q^i_ε (see Sec.2) and $r^i_{2\varepsilon} = \dfrac{d^i_\varepsilon}{2}$, where d^i_ε is the distance from Q^i_ε to $\bigcup_{j\ne i} Q^j_\varepsilon \bigcup \partial\Omega$.

Consider the following boundary value problem in $\mathbb{R}^3 \setminus Q^i_\varepsilon$:

$$\mu\Delta v(\underline{x}) = \nabla p(\underline{x}), \quad \operatorname{div} v(\underline{x}) = 0, \quad x \in \mathbb{R}^3 \setminus Q^i_\varepsilon; \tag{6.6}$$
$$v(\underline{x}) = e^k + q^{ik} \times (\underline{x} - \underline{x}^i_\varepsilon), \quad x \in \partial Q^i_\varepsilon = S^i_\varepsilon; \tag{6.7}$$
$$\int_{\partial Q^i_\varepsilon} (\underline{x} - \underline{x}^i_\varepsilon) \times \underline{\sigma}[v]\nu ds = 0; \tag{6.8}$$
$$v(\underline{x}) \in H^1(\mathbb{R}^3 \setminus Q^i_\varepsilon). \tag{6.9}$$

Here e^k is a unit vector of axis x_k, q^{ik} is a constant vector, $\sigma[v]$ is the stress tensor with components $\sigma_{kl}[v] = \mu\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right) - p\delta_{kl}$.

There exists a unique solution of this problem. We will denote it by $(v^{ik}(\underline{x}), q^{ik})$. We will continue the velocity vector $v^{ik}(\underline{x})$ on Q^i_ε by equality (6.7).

Since the particles Q^i_ε are balls (and, hence, the above problem is symmetric), the vector $\underline{q}^{ik}$ in the boundary condition (6.7) equals zero and the boundary condition (6.8) is satisfied automatically and thus can be dropped. Therefore, problem (6.6) – (6.9) is equivalent to the Stokes problem (the problem of viscous incompressible flow around the ball).

The solution of this problem is well-known (see [20]). Using the exact form of this solution, it is easy to show that

$$\int\limits_{\mathbb{R}^3} 2\mu \sum_{p,q=1}^{3} e_{pq}[v^{ik}]e_{pq}[v^{il}]dx = 6\pi\mu r^i_\varepsilon \delta_{kl}, \tag{6.10}$$

$$|D^\alpha v^{ik}_l(\underline{x})| \leq \frac{A_1 r^i_\varepsilon}{|\underline{x}-\underline{x}^i_\varepsilon|^{1+\alpha}}, \quad |p^{i,k}(\underline{x})| \leq \frac{A_2 r^i_\varepsilon}{|\underline{x}-\underline{x}^i_\varepsilon|^2}, \tag{6.11}$$

and

$$\int\limits_{B^i_{2\varepsilon}} |D^\alpha v^{ik}_l(\underline{x})|^2 dx \leq A_3 \left[(r^i_\varepsilon)^{3-2|\alpha|} + (r^i_\varepsilon)^2 (d^i_\varepsilon)^{1-2|\alpha|}\right], \tag{6.12}$$

where r^i_ε is the radius of the ball Q^i_ε, δ_{kl} is a Kronecker delta, $v^{ik}_l(\underline{x})$ are the components of the solution $\underline{v}^{ik}(\underline{x})$ of problem (6.6) – (6.9), $|\alpha| = 0, 1, 2$, and $\underline{x}$ in (6.11) is outside the ball $B^i_{1\varepsilon}$.

Let $\underline{v}(\underline{x})$ be a divergence free vector function of class $C^2_0(\Omega)$ and $\underline{v}^{i,k}(\underline{x})$ be the solution of problem (6.6)–(6.9) extended onto Q^i_ε by equality (6.7) such that $\text{div}\underline{v}^{i,k} = 0$ in $\mathbb{R}^3$.

In accordance with Theorem 6.1, we introduce the vector-functions $\underline{\tilde{v}}^i(\underline{x})$ and $\underline{\tilde{v}}^{ik}(\underline{x})$ by equalities

$$\text{curl}\underline{\tilde{v}}^i(\underline{x}) = \underline{v}(\underline{x}^i_\varepsilon) - \underline{v}(\underline{x}), \quad x \in B^i_{1\varepsilon}; \quad \text{curl}\underline{\tilde{v}}^{ik}(\underline{x}) = \underline{v}^{ik}(\underline{x}), \quad x \in B^i_{2\varepsilon}. \tag{6.13}$$

We now construct a test vector function $\underline{v}^{\varepsilon h}(x) \in \overset{\circ}{J}_\varepsilon(\Omega)$ using given $\underline{v}(\underline{x}) \in J^\circ(\Omega) \bigcap C^2_0(\Omega)$ and $\underline{w}(\underline{x}) \in C^2_0(\Omega)$:

$$\underline{v}_{\varepsilon h}(\underline{x}) = \underline{v}(\underline{x}) + \sum_{i=1}^{N_\varepsilon} \text{curl}[\underline{\tilde{v}}^i(\underline{x})\varphi^i_{1\varepsilon}(\underline{x})] + \tag{6.14}$$

$$+ \sum_{i=1}^{N_\varepsilon}\sum_{k=1}^{3} [w^{\varepsilon h}_k(\underline{x}^i_\varepsilon) - v_k(\underline{x}^i_\varepsilon)]\text{curl}[\underline{\tilde{v}}^{ik}(\underline{x})\varphi^i_{2\varepsilon}(\underline{x})].$$

Here $\varphi^i_{1\varepsilon}(\underline{x}) = \varphi\left(\frac{|\underline{x}-\underline{x}^i_\varepsilon|}{r^i_{1\varepsilon}}\right)$, $\quad \varphi^i_{2\varepsilon}(\underline{x}) = \varphi\left(\frac{|\underline{x}-\underline{x}^i_\varepsilon|}{r^i_{2\varepsilon}}\right)$.

The function $\varphi(r) \in C^2_0(0, \infty)$ is equal to 1 as $r < 1/2$ and 0 as $r > 1$. The functions $v_k(\underline{x})$ and $w^{\varepsilon h}_k(\underline{x})$ are, respectively, the components of the given vector function $\underline{v}(\underline{x})$ and of the lattice vector function $\underline{w}^{\varepsilon h}(\underline{x})$ constructed according to (6.1).

It follows from (6.7), (6.13)-(6.14) and the properties of the functions $\varphi^i_{1\varepsilon}(\underline{x})$, $\varphi^i_{2\varepsilon}(\underline{x})$ that $\underline{v}^{\varepsilon h}(\underline{x}) \in \overset{\circ}{H}{}^1(\Omega)$, $\quad \text{div}\underline{v}^{\varepsilon h}(\underline{x}) = 0$, $\quad x \in \Omega$, and $\underline{v}^{\varepsilon h}(\underline{x}) = \underline{w}^{\varepsilon h}(\underline{x}^i_\varepsilon)$, $\quad x \in Q^i_\varepsilon$. Thus $\underline{v}^{\varepsilon h}(\underline{x}) \in \overset{\circ}{J}_\varepsilon(\Omega)$.

Remark. Vector-function (6.14) possesses the following properties:

1) for sufficiently small h and $\varepsilon < \hat{\varepsilon}(h)$, $\underline{v}^{\varepsilon h}(\underline{x})$ is close (in L_2-norm) to the vector function $\underline{v}(\underline{x})$;

2) $\underline{v}^{\varepsilon h}(\underline{x})$ coincides with the first given vector-function $\underline{v}(\underline{x})$ far from the particles while its restriction on the lattice $\{\underline{x}^i_\varepsilon\}$ coincides with the discrete vector-function $\underline{w}^{\varepsilon h}(\underline{x})$ that approximates (in some sense) the second given vector-function $\underline{w}(\underline{x})$ and "almost" minimizes functional (3.5) on every cube $K(\underline{x}^\alpha, h)$ when $T_{np} = e_{np}[\underline{w}(\underline{x}^\alpha)]$.

The latter property allows us to pass to the limit as $\varepsilon \to 0$ and to compute the limiting functional via meso-characteristics. In fact, this procedure corresponds to what is usually referred to as the Γ-upper limit (for more details see, for example, [2], [11], [14], [16], [34] and references therein).

To calculate $\Phi_\varepsilon(\underline{v}^{\varepsilon h})$ (see (4.6)) for small h, δ, and ε ($\varepsilon \ll \delta = h^{1+\tau/2} \ll h \ll 1$), we need estimates on the solution $\underline{w}^{np}_{\varepsilon h}(\underline{x})$ of the minimization problem for the functional (3.5).

Lemma 6.2. *For sufficiently small h, $\delta = h^{1+\tau/2}$ and $\varepsilon < \hat{\varepsilon}(h, \delta)$, the lattice functions $\underline{w}^{\alpha np}_{\varepsilon h}(\underline{x})$ that minimize the functional (3.5) for $T = T^{np}$ satisfy the following estimates:*

$$\sum_{i,j} {}_{K_\alpha} \langle \underline{\underline{C}}^{ij}_\varepsilon [\underline{w}^{\alpha np}_{\varepsilon h}(\underline{x}^i_\varepsilon) - \underline{w}^{\alpha np}_{\varepsilon h}(\underline{x}^j_\varepsilon)], [\underline{w}^{\alpha np}_{\varepsilon h}(\underline{x}^i_\varepsilon) - \underline{w}^{\alpha np}_{\varepsilon h}(\underline{x}^j_\varepsilon)]\rangle = O(h^3); \tag{6.15}$$

$$\sum_{i} {}_{K_\alpha} \left|\underline{w}^{\alpha np}_{\varepsilon h}(\underline{x}^i_\varepsilon) - \varphi^{np}(\underline{x}^i_\varepsilon - \underline{x}^\alpha)\right|^2 \varepsilon^3 = O(h^{5+\tau}); \tag{6.16}$$

$$\sum_{i,j} {}_{K_\alpha \setminus K'_\alpha} \langle \underline{\underline{C}}^{ij}_\varepsilon [\underline{w}^{\alpha np}_{\varepsilon h}(\underline{x}^i_\varepsilon) - \underline{w}^{\alpha np}_{\varepsilon h}(\underline{x}^j_\varepsilon)], [\underline{w}^{\alpha np}_{\varepsilon h}(\underline{x}^i_\varepsilon) - \underline{w}^{\alpha np}_{\varepsilon h}(\underline{x}^j_\varepsilon)]\rangle = o(h^3); \tag{6.17}$$

$$\sum_{i,j} {}_{K_\alpha \setminus K'_\alpha} \left|\underline{w}^{\alpha np}_{\varepsilon h}(\underline{x}^i_\varepsilon) - \varphi^{np}(\underline{x}^i_\varepsilon - \underline{x}^\alpha)\right|^2 \varepsilon^3 = o(h^{5+\tau}). \tag{6.18}$$

Here $K'_\alpha = K(x^\alpha, h)$, $h' = h - 2\delta$, and the sum $\sum_G$ is taken over $\underline{x}^i_\varepsilon \in G$, $\forall G \subset \Omega$.

The proof of this lemma is given in [4].

Next, we rearrange (6.14) to distinguish the terms that give non-vanishing contribution to the integral term in $\Phi_\varepsilon(\underline{v}^{\varepsilon h})$, as $\varepsilon \to 0$ and $h \to 0$:

$$\underline{v}_{\varepsilon h}(\underline{x}) = \underline{v}(\underline{x}) + \sum_{i=1}^{N_\varepsilon}\sum_{k=1}^{3}[w_k(\underline{x}^i_\varepsilon) - v_k(\underline{x}^i_\varepsilon)]\underline{v}^{ik}(\underline{x})\varphi^i_{2\varepsilon}(\underline{x}) + \sum_{s=1}^{5}\underline{\delta}^s_{\varepsilon h}(\underline{x}). \tag{6.19}$$

Here we use the equality $\mathrm{curl}(\varphi\underline{v}) = \varphi\mathrm{curl}\underline{v} + \underline{v} \times \nabla\varphi$ to obtain

$$\underline{\delta}^1_{\varepsilon h}(\underline{x}) = \sum_{i=1}^{N_\varepsilon}\sum_{k=1}^{3}[w^{\varepsilon h}_k(\underline{x}^i_\varepsilon) - w_k(\underline{x}^i_\varepsilon)]\underline{v}^{ik}(\underline{x})\varphi^i_{2\varepsilon}(\underline{x}),$$

$$\underline{\delta}^2_{\varepsilon h}(\underline{x}) = \sum_{i=1}^{N_\varepsilon}\sum_{k=1}^{3}[w_k(\underline{x}^i_\varepsilon) - v_k(\underline{x}^i_\varepsilon)]\underline{\tilde{v}}^{ik}(\underline{x}) \times \nabla\varphi^i_{2\varepsilon}(\underline{x}),$$

$$\underline{\delta}^3_{\varepsilon h}(\underline{x}) = \sum_{i=1}^{N_\varepsilon}\sum_{k=1}^{3}[w^{\varepsilon h}_k(\underline{x}^i_\varepsilon) - w_k(\underline{x}^i_\varepsilon)]\underline{\tilde{v}}^{ik}(\underline{x}) \times \nabla\varphi^i_{2\varepsilon}(\underline{x}), \tag{6.20}$$

$$\underline{\delta}^4_{\varepsilon h}(\underline{x}) = \sum_{i=1}^{N_\varepsilon}[\underline{v}(\underline{x}^i_\varepsilon) - \underline{v}(\underline{x})]\varphi^i_{1\varepsilon}(\underline{x}), \quad \underline{\delta}^5_{\varepsilon h}(\underline{x}) = \sum_{i=1}^{N_\varepsilon}\underline{\tilde{v}}^i(\underline{x}) \times \nabla\varphi^i_{1\varepsilon}(\underline{x}).$$

Since the supports of the functions $\varphi^i_{2\varepsilon}(\underline{x})$ do not intersect, we have

$$\int\limits_\Omega 2\mu \sum_{k.l=1}^{3} e^2_{kl}[\underline{v}_{\varepsilon h}]dx = \int\limits_\Omega 2\mu \sum_{k.l=1}^{3} e^2_{kl}[\underline{v}]dx + \tag{6.21}$$

$$+\sum_{i=1}^{N_\varepsilon}\sum_{p,q}\underline{\underline{C}}^{i\varepsilon}_{pq}(w_p(\underline{x}^i_\varepsilon) - v_p\underline{x}^i_\varepsilon))(w_q(\underline{x}^i_\varepsilon) - v_q\underline{x}^i_\varepsilon)) + \Delta(\varepsilon, h),$$

where

$$\underline{\underline{C}}^{i\varepsilon}_{pq} = \int\limits_\Omega 2\mu \sum_{k,l=1}^{3} e_{kl}[\underline{v}^{ip}\varphi^i_{2\varepsilon}]e_{kl}[\underline{v}^{iq}\varphi^i_{2\varepsilon}]dx;$$

$$\Delta(\varepsilon, h) = \sum_{r,s=1}^{5}\int\limits_\Omega 2\mu \sum_{k,l=1}^{3} e_{kl}[\underline{\delta}^r_{\varepsilon h}]e_{kl}[\underline{\delta}^s_{\varepsilon h}]dx + \tag{6.22}$$

$$\sum_{s=1}^{5}\int\limits_\Omega 4\mu \sum_{k,l=1}^{3} e_{kl}[\underline{v} + \underline{w}_\varepsilon]e_{kl}[\underline{\delta}^s_{\varepsilon h}]dx + \int\limits_\Omega 4\mu \sum_{k,l=1}^{3} e_{kl}[\underline{v}]e_{kl}[\underline{w}_\varepsilon]dx.$$

Here we have used the notation $\underline{w}_\varepsilon = \sum_{s=1}^5 \underline{\delta}^s_{\varepsilon h}(x)$.

Using lemma 6.2 and inequalities (6.11)-(6.12), we can show that

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\,\Delta(\varepsilon, h) = 0. \tag{6.23}$$

We demonstrate the proof of (6.23) only for two terms in the first sum in (6.22) which involve $\underline{\delta}^1_{\varepsilon h}$ and $\underline{\delta}^2_{\varepsilon h}$. Since the supports $B^i_{2\varepsilon}$ of $\varphi^i_{2\varepsilon}(\underline{x})$ do not intersect, from (6.20) it follows that

$$\int\limits_\Omega \sum_{k,l=1}^{3} e^2_{kl}(\underline{\delta}^1_{\varepsilon h})dx \leq 2\sum_{i=1}^{N_\varepsilon}\sum_{p=1}^{3}|\underline{w}^{\varepsilon h}_p(\underline{x}^i_\varepsilon) - \underline{w}_p(\underline{x}^i_\varepsilon)|^2 \int\limits_{B^i_{2\varepsilon}} \sum_{k,l=1}^{3} e^2_{kl}[\underline{v}^{ip}\varphi^i_{2\varepsilon}] \leq$$

$$\leq 2\sum_{i=1}^{N_\varepsilon}\sum_{p=1}^{3}|\underline{w}^{\varepsilon h}_p(\underline{x}^i_\varepsilon) - \underline{w}_p(\underline{x}^i_\varepsilon)|^2 \int\limits_{B^i_{2\varepsilon}} \left\{|\nabla\underline{v}^{ip}|^2\varphi^i_{2\varepsilon} + |\underline{v}^{ip}|^2|\nabla\varphi^i_{2\varepsilon}|^2\right\} dx.$$

Using (6.12), conditions (3.1), (3.2) and properties of $\varphi^i_{2\varepsilon}(\underline{x})$ ($\varphi^i_{2\varepsilon} \leq 1$ and $|\nabla\varphi^i_{2\varepsilon}| < C(d^i_\varepsilon)^{-1}$), we get

$$\int\limits_\Omega \sum_{k,l} e^2_{kl}(\underline{\delta}^1_{kl}) \leq C_1\sum_{i=1}^{N_\varepsilon}|\underline{w}^{\varepsilon h}(\underline{x}^i_\varepsilon) - \underline{w}_p(\underline{x}^i_\varepsilon)|^2\varepsilon^3. \tag{6.24}$$

From (6.1) and (6.2), it follows that for $x^i_\varepsilon \in K(\underline{x}^\alpha, h)$ we have

$$|\underline{w}_{\varepsilon h}(\underline{x}^i_\varepsilon) - \underline{w}(\underline{x}^i_\varepsilon)|^2 \leq C_2|\underline{w}^{\alpha np}(\underline{x}^i_\varepsilon) - \underline{\varphi}^{np}(\underline{x}^i_\varepsilon - \underline{x}^\alpha)|^2 + O(h^4).$$

Substituting this inequality into (6.24), we obtain

$$\int\limits_{\Omega} \sum_{k,l=1}^{3} e_{kl}(\underline{\delta}^1_{\varepsilon h}) \leq \sum_{\alpha} \sum_{i=1}^{N_\varepsilon} {}_{K_h^\alpha} |\underline{w}^{\alpha n p}(\underline{x}^i_\varepsilon) - \underline{\varphi}^{np}(\underline{x}^i_\varepsilon - \underline{x}^\alpha)|^2 \varepsilon^3 + O(h^4).$$

Now, using lemma 6.2 (estimate (6.16)), we have that for $h \ll 1$ and $\varepsilon \leq \hat{\varepsilon}(h)$,

$$\int\limits_{\Omega} \sum_{k,l=1}^{3} e^2_{kl}(\underline{\delta}^1_{\varepsilon h}) = O(h^{2+\tau}) \qquad \text{and} \qquad \lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} \int\limits_{\Omega} \sum_{k,l=1}^{3} e^2_{kl}(\underline{\delta}^1_{kl}) = 0.$$

To estimate the contribution of $\underline{\delta}^2_{\varepsilon h}(\underline{x})$, we use (6.4)-(6.5) and (6.11)- (6.12). Taking into account the properties of $\varphi^i_{2\varepsilon}(\underline{x})$ and using (6.4)–(6.5), (6.12), and conditions (3.1)- (3.2), we get

$$\int\limits_{\Omega} \sum_{k,l=1}^{3} e^2_{kl}[\underline{\delta}^2_{\varepsilon h}]dx \leq C_1 \sum_{k=1}^{3} \sum_{i=1}^{N_\varepsilon} \left(\|\nabla \underline{\tilde{v}}^{ik}\|^2_{L_2(B^i_{2\varepsilon})} (d^\varepsilon_i)^{-2} + \|\underline{\tilde{v}}^{ik}\|^2_{L_2(B^i_{2\varepsilon})} (d^i_\varepsilon)^{-4} \right) \leq$$

$$\leq C_1 \sum_{k=1}^{3} \sum_{i=1}^{N_\varepsilon} \left(\|\underline{v}^{ik}\|^2_{L_2(B^i_{2\varepsilon})} (d^\varepsilon_i)^{-2} \right) = \sum_{i=1}^{N_\varepsilon} O(\varepsilon^5).$$

Since $N_\varepsilon = O(\varepsilon^{-3})$, we deduce that $\lim\limits_{\varepsilon\to 0} \int\limits_{\Omega} \sum\limits_{k,l=1}^{3} e^2_{kl}(\underline{\delta}^2_{\varepsilon h}) = 0$.

The remaining terms in $\Delta(\varepsilon, h)$ can be estimated in the same way. Analogously, using (6.1), (6.19), (6.20), inequalities (6.4), (6.11)-(6.12) and lemma 6.2, we obtain

$$\lim_{\varepsilon\to 0} \|\underline{v}_{\varepsilon h} - \underline{v}\|_{L_2(\Omega)} = 0 \tag{6.25}$$

which is used for the estimation of the second and third terms in the integral in (4.6).

Finally, using (6.11)-(6.12) and the properties of $\varphi^i_{2\varepsilon}(\underline{x})$, we derive the following asymptotic relations for the numbers $C^{i\varepsilon}_{pq}$ (see (6.22))

$$C^{i\varepsilon}_{pq} = 6\pi\mu r^i_\varepsilon \delta_{kl} + O(\varepsilon^4) \quad (\varepsilon \to 0). \tag{6.26}$$

Combining (6.21), (6.23), (6.25), (6.26) and taking into account conditions (3.1), (3.2), (3.8) and (3.9), we get

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} \int\limits_{\Omega} \left\{ 2\mu \sum_{k,l=1}^{3} e^2_{kl}[\underline{v}_{\varepsilon h}] + \lambda \langle \rho_\varepsilon \underline{v}_{\varepsilon h}, \underline{v}_{\varepsilon h} \rangle - 2\langle \rho_\varepsilon \underline{V}_{\varepsilon 0}, \underline{v}_{\varepsilon h} \rangle \right\} dx =$$

$$= \int\limits_{\Omega} \left\{ 2\mu \sum_{k,l=1}^{3} e^2_{kl}[\underline{v}] + \lambda \langle \rho \underline{v}, \underline{v} \rangle + \langle C(\underline{x})[\underline{v} - \underline{w}], [\underline{v} - \underline{w}] \rangle - 2\langle \rho \underline{v}_0, \underline{v} \rangle \right\} dx. \tag{6.27}$$

Here $C(\underline{x}) = 6\pi\mu r(\underline{x})$, $r(\underline{x})$ is defined in (3.8), and $\underline{v}_0(\underline{x})$ is defined in (3.11).

Now we consider the sum $I_\varepsilon[\underline{v}_{\varepsilon h}] = I_\varepsilon(\underline{v}_{\varepsilon h}, \underline{v}_{\varepsilon h})$ (see (5.2)) in the functional $\Phi_\varepsilon(\underline{v}_{\varepsilon h})$ defined in (4.6). According to (6.14), $\underline{v}_{\varepsilon h}(\underline{x}^i_\varepsilon) = \underline{w}^{\varepsilon h}(\underline{x}^i_\varepsilon)$. Thus,

$$I_\varepsilon[\underline{v}_{\varepsilon h}] = \sum_{i,j} \langle \underline{\underline{C}}^{ij}_\varepsilon [\underline{w}^{\varepsilon h}(\underline{x}^i_\varepsilon) - \underline{w}^{\varepsilon h}(\underline{x}^j_\varepsilon)], [\underline{w}^{\varepsilon h}(\underline{x}^i_\varepsilon) - \underline{w}^{\varepsilon h}(\underline{x}^j_\varepsilon)] \rangle. \tag{6.28}$$

Denote by $\underline{\hat{w}}^\alpha(x^i_\varepsilon)$, $x^i_\varepsilon \in K^\alpha_h$, the lattice function in the braces in (6.1). Then we can write

$$\underline{w}^{\varepsilon h}(\underline{x}^i_\varepsilon) - \underline{w}^{\varepsilon h}(\underline{x}^j_\varepsilon) = \sum_\alpha [\underline{\hat{w}}^\alpha(\underline{x}^i_\varepsilon) - \underline{\hat{w}}^\alpha(\underline{x}^j_\varepsilon)]\varphi^\alpha_h(\underline{x}^i_\varepsilon) + \sum_\alpha \underline{\hat{w}}^\alpha(\underline{x}^j)[\varphi^\alpha_h(\underline{x}^i_\varepsilon) - \varphi^\alpha_h(\underline{x}^j_\varepsilon)].$$

Taking into account that $\sum_\alpha \varphi^\alpha_h(\underline{x}) \equiv 1$ for $\underline{x} \in \Omega$, we get

$$\underline{w}^{\varepsilon h}(\underline{x}^i_\varepsilon) - \underline{w}^{\varepsilon h}(\underline{x}^j_\varepsilon) = \sum_\alpha [\underline{\hat{w}}^\alpha(\underline{x}^i_\varepsilon) - \underline{\hat{w}}^\alpha(\underline{x}^j_\varepsilon)]\varphi^\alpha_h(\underline{x}^i_\varepsilon) + \tag{6.29}$$
$$+ \sum_\alpha [\underline{\hat{w}}^\alpha(\underline{x}^j_\varepsilon) - \underline{w}(\underline{x}^j_\varepsilon)][\varphi^\alpha_h(\underline{x}^i_\varepsilon) - \varphi^\alpha_h(\underline{x}^j_\varepsilon)],$$

where $\underline{w}(\underline{x}) \in C^2_0(\Omega)$ defines the lattice vector function $\underline{w}^{\varepsilon h}(\underline{x}^i_\varepsilon)$, $i = 1, \ldots, N_\varepsilon$, according to (6.1).

It follows from (6.28) and (6.29) that

$$I_\varepsilon[\underline{v}_{\varepsilon h}] = \sum_{\alpha,\beta}\sum_{i,j} \langle \underline{\underline{C}}^{ij}_{1\varepsilon}[\underline{\hat{w}}^\alpha(\underline{x}^i_\varepsilon) - \underline{\hat{w}}^\alpha(\underline{x}^j_\varepsilon)], [\underline{\hat{w}}^\beta(\underline{x}^i_\varepsilon) - \underline{\hat{w}}^\beta(\underline{x}^j_\varepsilon)]\rangle \varphi^\alpha_h(\underline{x}^i_\varepsilon)\varphi^\beta_h(\underline{x}^j_\varepsilon) +$$
$$+ \sum_{\alpha,\beta}\sum_{i,j} \langle \underline{\underline{C}}^{ij}_\varepsilon \underline{\Delta}^\alpha(\underline{x}^j_\varepsilon), \underline{\Delta}^\beta(\underline{x}^j_\varepsilon)\rangle [\varphi^\alpha_h(\underline{x}^i_\varepsilon) - \varphi^\alpha_h(\underline{x}^j_\varepsilon)][\varphi^\beta_h(\underline{x}^i_\varepsilon) - \varphi^\beta_h(\underline{x}^j_\varepsilon)] + \tag{6.30}$$
$$+2\sum_{\alpha,\beta}\sum_{i,j} \langle \underline{\underline{C}}^{ij}_\varepsilon [\underline{\hat{w}}^\alpha(\underline{x}^i_\varepsilon) - \underline{\hat{w}}^\alpha(\underline{x}^j_\varepsilon)], \underline{\Delta}^\beta(\underline{x}^j_\varepsilon)\rangle [\varphi^\beta_h(\underline{x}^i_\varepsilon) - \varphi^\beta_h(\underline{x}^i_\varepsilon)]\varphi^\alpha_h(\underline{x}^i_\varepsilon) \equiv$$
$$\equiv \sum\nolimits_1^{\varepsilon h} + \sum\nolimits_2^{\varepsilon h} + \sum\nolimits_3^{\varepsilon h},$$

where $\underline{\Delta}^\alpha(\underline{x}^j_\varepsilon) = \underline{\hat{w}}^\alpha(\underline{x}^j_\varepsilon) - \underline{w}(\underline{x}^j_\varepsilon)$.

Now we want to show that

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}} \sum\nolimits_1^{\varepsilon h} \le \int_\Omega \sum_{n,p,q,r} a_{npqr}(\underline{x}) e_{np}[\underline{w}] e_{qr}[\underline{w}] dx,$$
$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}} \sum\nolimits_2^{\varepsilon h} = 0, \quad \lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}} \sum\nolimits_3^{\varepsilon h} = 0. \tag{6.31}$$

Notice that the third equality follows from the previous two ones, due to the inequality

$$\left|\sum\nolimits_3^{\varepsilon h}\right| \le \left(\sum\nolimits_1^{\varepsilon h}\right)^{1/2}\left(\sum\nolimits_2^{\varepsilon h}\right)^{1/2}.$$

Taking into account the properties of the partition of unity $\{\varphi^\alpha_h(\underline{x})\}$and (3.3), we have

$$\sum\nolimits_1^{\varepsilon h} = \sum_\alpha \sum_{i,j} {}_{K_\alpha}\langle \underline{\underline{C}}^{ij}_\varepsilon [\underline{\hat{w}}^\alpha(\underline{x}^i_\varepsilon) - \underline{\hat{w}}^\alpha(\underline{x}^j_\varepsilon)], [\underline{\hat{w}}^\alpha(\underline{x}^i_\varepsilon) - \underline{w}^\alpha(\underline{x}^j_\varepsilon)]\rangle \varphi^\alpha_h(\underline{x}^i_\varepsilon)\varphi^\alpha_h(\underline{x}^j_\varepsilon) +$$
$$+ \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \sum_{i,j} {}_{K_\alpha \cap K_\beta} \left|\langle \underline{\underline{C}}^{ij}_\varepsilon [\underline{\hat{w}}^\alpha(\underline{x}^i_\varepsilon) - \underline{\hat{w}}^\alpha(\underline{x}^j_\varepsilon)], [\underline{\hat{w}}^\beta(\underline{x}^i_\varepsilon) - \underline{w}^\beta(\underline{x}^j_\varepsilon)]\rangle\right| \varphi^\alpha_h(\underline{x}^i_\varepsilon)\varphi^\beta_h(\underline{x}^j_\varepsilon) \le$$
$$\le \sum_\alpha \sum_{i,j} {}_{K_\alpha}\langle \underline{\underline{C}}^{ij}_\varepsilon [\underline{\hat{w}}^\alpha(\underline{x}^i_\varepsilon) - \underline{\hat{w}}^\alpha(\underline{x}^j_\varepsilon)], [\underline{\hat{w}}^\alpha(\underline{x}^i_\varepsilon) - \underline{\hat{w}}^\alpha(\underline{x}^j_\varepsilon)]\rangle + \tag{6.32}$$
$$+ \sum_\alpha \sum_{i,j} {}_{K_\alpha \setminus K'_\alpha}\langle \underline{\underline{C}}^{ij}_\varepsilon [\underline{\hat{w}}^\alpha(\underline{x}^i_\varepsilon) - \underline{\hat{w}}^\alpha(\underline{x}^j_\varepsilon)], [\underline{\hat{w}}^\alpha(\underline{x}^i_\varepsilon) - \underline{w}^\alpha(\underline{x}^j_\varepsilon)]\rangle = \sum\nolimits_{11}^{\varepsilon h} + \sum\nolimits_{12}^{\varepsilon h}.$$

Recalling the definition of $\underline{\hat{w}}^\alpha(\underline{x})$ (see the braces in (6.1)) and taking into account (6.3), we write

$$\underline{\hat{w}}^\alpha(\underline{x}_\varepsilon^i)-\underline{\hat{w}}^\alpha(\underline{x}_\varepsilon^j)=\sum_{n,p=1}^{3} e_{np}[\underline{w}^\alpha(\underline{x}^\alpha)][\underline{w}^{\alpha np}(\underline{x}_\varepsilon^i)-\underline{w}^{\alpha np}(\underline{x}_\varepsilon^j)]+\underline{b}^\alpha\times(\underline{x}_\varepsilon^i-\underline{x}_\varepsilon^j). \quad (6.33)$$

Then (6.33) and (3.3)-(3.4) imply

$$\underline{\underline{C}}_\varepsilon^{ij}[\underline{\hat{w}}^\alpha(\underline{x}_\varepsilon^i)-\underline{\hat{w}}^\alpha(\underline{x}_\varepsilon^j)]=\sum_{n,p=1}^{3} e_{np}[\underline{w}(\underline{x}^\alpha)\underline{\underline{C}}_\varepsilon^{ij}[\underline{w}^{\alpha np}(\underline{x}_\varepsilon^i)-\underline{w}^{\alpha np}(\underline{x}_\varepsilon^j)]+\underline{O}(\varepsilon^4). \quad (6.34)$$

Now substitute (6.33) and (6.34) into (6.32). Using the estimate (6.17) and conditions (3.2) and II_1, we conclude that

$$\overline{\lim_{\varepsilon\to0}}\sum\nolimits_{12}^{\varepsilon h}=o(1)\quad(h\to0),. \quad (6.35)$$

It follows from (3.7) that

$$\sum\nolimits_{11}^{\varepsilon h}=\sum_\alpha\sum_{n,p,q,r} e_{np}[\underline{w}(\underline{x}^\alpha)]e_{qr}[\underline{w}(\underline{x}^\alpha)]\sum_{i,j}{}_{K_\alpha}\langle\underline{\underline{C}}_\varepsilon^{ij}[\underline{w}^{\alpha np}(\underline{x}_\varepsilon^i)-\underline{w}^{\alpha np}(\underline{x}_\varepsilon^i)], \quad (6.36)$$

$$[\underline{w}^{\alpha qr}(\underline{x}_\varepsilon^i)-\underline{w}^{\alpha qr}(\underline{x}_\varepsilon^i)]\rangle+O(\varepsilon)\le\sum_\alpha\sum_{n,p,q,r} e_{np}[\underline{w}(\underline{x}^\alpha)]e_{qr}[\underline{w}(\underline{x}^\alpha)]a_{npqr}^\tau(\underline{x},\varepsilon,h)+O(\varepsilon).$$

The regularity of $\underline{w}(x)$, condition (3.9) and (6.36) imply that

$$\lim_{h\to0}\overline{\lim_{\varepsilon\to0}}\sum\nolimits_{11}^{\varepsilon h}\le\int a_{npqr}(\underline{x})e_{np}[\underline{w}(\underline{x})e_{qr}[\underline{w}(\underline{x})dx. \quad (6.37)$$

Taking into account (6.32), (6.35) and (6.37), the first inequality in (6.31) follows.

Now we estimate $\sum_2^{\varepsilon h}$. Using the definition of $\underline{\hat{w}}^\alpha(\underline{x}^j)$ and equality (6.2) we have

$$\underline{\Delta}^\alpha(\underline{x}_\varepsilon^j)=\underline{\hat{w}}^\alpha(\underline{x}^i)-\underline{w}(\underline{x}^j)=\sum_{n,p=1}^{3} e_{np}[\underline{w}(\underline{x}^\alpha)][\underline{w}^{\alpha np}(\underline{x}_\varepsilon^j)-\varphi^{np}(\underline{x}_\varepsilon^j-\underline{x}^\alpha)]+\underline{O}(h^2).$$

Using this equality and the definition of $\sum_2^{\varepsilon h}$ (see (6.30)), (3.3)-(3.4) and the properties of the partition of unity (the intersection multiplicity of $K_\alpha\bigcap K_\beta$ is not more than 8), we get

$$\sum\nolimits_{2}^{\varepsilon h}\le C_1\varepsilon\sum_\alpha\sum_{\substack{i,j\\ i\ne j}}{}'\sum_{n,p}|\underline{w}^{\alpha np}(\underline{x}_\varepsilon^j)-\underline{\varphi}^{np}(\underline{x}_\varepsilon^j-\underline{x}^\alpha)|^2|\varphi_h^\alpha(\underline{x}_\varepsilon^i)-\varphi_h^\alpha(\underline{x}_\varepsilon^j)|^2+ \quad (6.38)$$

$$+C_2\varepsilon h^4\sum_\alpha\sum_{i,j}{}'|\varphi_h^\alpha(\underline{x}_\varepsilon^i)-\varphi_h^\alpha(\underline{x}_\varepsilon^j)|^2$$

Since $\underline{\underline{C}}_\varepsilon^{ij}=0$, if $\mathrm{dist}(Q_\varepsilon^i,Q_\varepsilon^j)\ge C\varepsilon$, the sums $\sum\limits_{i,j}{}'$ are taken over the pairs (i,j) for which $|\underline{x}_\varepsilon^i-\underline{x}_\varepsilon^j|<C\varepsilon$.

It follows from the properties of $\varphi_h^\alpha(\underline{x})$ that $\varphi^\alpha(\underline{x}_\varepsilon^i)-\varphi^\alpha(\underline{x}_\varepsilon^j)=0$ for $\underline{x}_\varepsilon^i,\underline{x}_\varepsilon^j\in K'_\alpha$ and $|\varphi^\alpha(\underline{x}_\varepsilon^i)-\varphi^\alpha(\underline{x}_\varepsilon^j)|\le C\frac{\varepsilon}{\delta}$ if $\underline{x}_\varepsilon^i\in K_\alpha\setminus K'_\alpha$ or $\underline{x}_\varepsilon^j\in K_\alpha\setminus K'_\alpha$.

Then (6.38) implies

$$\sum\nolimits_2^{\varepsilon h} \leq C\delta^{-2}\sum_{\alpha}\sum_{i,j}{}_{K_\alpha\setminus K'_\alpha}|\underline{w}^{\alpha np}(\underline{x}^i_\varepsilon)-\underline{\varphi}^{np}(\underline{x}^j_\varepsilon-\underline{x}^\alpha)|^2\varepsilon^3+ \tag{6.39}$$
$$+C\varepsilon h^4\sum_{\alpha}\sum_{i,j}{}_{K_\alpha\setminus K'_\alpha}|\varphi^\alpha_h(\underline{x}^i_\varepsilon)-\varphi^\alpha_h(\underline{x}^j_\varepsilon)|^2,$$

where the constants C do not depend on ε, δ and h.

It is easy to see that the second sum in (6.39) is of order $O(h^3/\delta)$ and the first sum is of order $o(h^{2+\tau}/\delta^2)$ due to the estimate (6.18).

Choosing $\delta = h^{1+\frac{\tau}{2}}$ with $0<\tau<2$ gives the second equality in (6.31). As we noted above, the third inequality in (6.31) follows from the first and the second ones. Using (6.30) and (6.31), we obtain

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\, I_\varepsilon[v_{\varepsilon h}] \leq \int_\Omega \sum_{n,p,q,r} a_{npqr}(\underline{x})e_{np}[\underline{w}]e_{qr}[\underline{w}]dx. \tag{6.40}$$

Finally, (6.27) and (6.40) yield the required inequality (5.5).

7. **The proof of inequality (5.7).** Assume that $\underline{v}(\underline{x}) \in \overset{\circ}{J}(\Omega)$ and $\underline{w}(\underline{x}) \in \overset{\circ}{H}{}^1(\Omega)$ are, respectively, the weak limits in $H^1(\Omega)$, as $\varepsilon=\varepsilon_k\to 0$, of the solution $\underline{v}_\varepsilon(\underline{x})$ of the problem (4.7) and the vector function $\underline{w}_\varepsilon(\underline{x})$ defined by (4.12). For the sake of simplicity we first assume that $\underline{v}(\underline{x}) \in \overset{\circ}{J}(\Omega)\bigcap C^2_0(\Omega)$ and $\underline{w}(\underline{x})\in C^2_0(\Omega)$. We partition the domain Ω by non-intersecting cubes $K_\alpha = K(\underline{x}^\alpha, h)$ whose centers $\underline{x}^\alpha$ form a cubic lattice of period h and edges are parallel to the coordinate axes. In each cube introduce the following discrete vector function

$$\underline{w}^\alpha_\varepsilon(\underline{x}^i_\varepsilon) = \underline{w}_\varepsilon(\underline{x}^i_\varepsilon) - (\underline{w}(\underline{x}^\alpha)+\underline{\theta}^\alpha\times(\underline{x}^i_\varepsilon-\underline{x}^\alpha)), \tag{7.1}$$

where $\underline{x}^i_\varepsilon\in K_\alpha$, $\underline{\theta}^\alpha$ is a constant vector defined by $\underline{\theta}^\alpha = \sum_{n,p}\underline{b}^{\alpha np}$, and vectors $\underline{b}^{\alpha np}$ are defined by (6.3).

We now explain the main idea of the construction of $\underline{w}^\alpha_\varepsilon(\underline{x}^i_\varepsilon)$. First, the RHS of (7.1) is close to the discretization of the symmetric part of $\nabla\underline{w}(\underline{x})$ in K_α. This is sufficient to make the penalty term in (3.5) small. Second, due to (7.1) the interaction term $I_\varepsilon[\underline{w}^\alpha_\varepsilon]$ in (3.5) is such that $I_\varepsilon[\underline{w}^\alpha_\varepsilon] = I_\varepsilon[\underline{w}_\varepsilon]$ in K_α.

In accordance with (3.6)-(3.7) we have

$$A^\tau[\underline{w}^\alpha_\varepsilon;\underline{x};\underline{\underline{T}}] \geq \sum_{n,p,q,r} a^\tau_{npqr}(\underline{x}^\alpha,\varepsilon,h)T_{np}T_{qr}$$

for any tensor $\underline{\underline{T}} = \{T_{np}\}^3_{n,p=1}$. Choose $T_{np} = e_{np}[\underline{w}(\underline{x}^\alpha)]$, then substitute (7.1) into (3.8) to obtain

$$\frac{1}{2}\sum_{\substack{i,j\\ j\neq i}}{}_{K_\alpha}\langle \underline{\underline{C}}^{ij}_\varepsilon[\underline{w}_\varepsilon(\underline{x}^i_\varepsilon)-\underline{w}_\varepsilon(\underline{x}^j_\varepsilon)],[\underline{w}_\varepsilon(\underline{x}^i_\varepsilon)-\underline{w}_\varepsilon(\underline{x}^j_\varepsilon)]\rangle+ \tag{7.2}$$
$$+h^{-2-\tau}\varepsilon^3\sum_i{}_{K_\alpha}\left|\underline{w}^\alpha_\varepsilon(\underline{x}^i_\varepsilon)-\sum_{n,p=1}^3 e_{np}[\underline{w}(\underline{x}^\alpha)]\underline{\varphi}^{np}(\underline{x}^i_\varepsilon-\underline{x}^\alpha)\right|^2\geq$$
$$\geq\sum_{n,p,q,r}a^\tau_{npqr}(\underline{x}^\alpha,\varepsilon,h)e_{np}[\underline{w}(\underline{x}^\alpha)]e_{qr}[\underline{w}(\underline{x}^\alpha)].$$

Consider the second term in the LHS of (7.2). Taking into account (7.1) and (6.2), we have

$$h^{-2-\tau}\varepsilon^3\sum_i K_\alpha\left|\underline{w}^\alpha_\varepsilon(\underline{x}^i_\varepsilon)-\sum_{n,p=1}^3 e_{np}[\underline{w}(\underline{x}^\alpha)]\underline{\varphi}^{np}(\underline{x}^i_\varepsilon-\underline{x}^\alpha)\right|^2\le \tag{7.3}$$

$$\le 2h^{-2-\tau}\sum_i K_\alpha|\underline{w}_\varepsilon(\underline{x}^i_\varepsilon)-\underline{w}(\underline{x}^i_\varepsilon)|^2\varepsilon^3+O(h^{5-\tau}).$$

Since $\underline{w}(\underline{x})\in C^2_0$ and $\underline{w}_\varepsilon(\underline{x})\in H^1(\Omega)\bigcap C$ is bounded uniformly on ε in $H^1(\Omega)$, we have

$$\sum_i K_\alpha|\underline{w}_\varepsilon(\underline{x}^i_\varepsilon)-\underline{w}(\underline{x}^i_\varepsilon)|^2\varepsilon^3<C\int_{K_\alpha}|\underline{w}_\varepsilon(\underline{x})-\underline{w}(\underline{x})|^2dx,$$

where C does not depend on ε.

Since the Embedding Theorem implies $w_\varepsilon\to w$ in $L_2(\Omega)$ as $\varepsilon=\varepsilon_k\to 0$, we obtain

$$\lim_{\varepsilon=\varepsilon_k\to 0}h^{-2-\tau}\sum_i K_\alpha|\underline{w}_\varepsilon(\underline{x}^i_\varepsilon)-\underline{w}(\underline{x}^i_\varepsilon)|^2\varepsilon^3=0. \tag{7.4}$$

Thus, (7.2), (7.3) and (7.4) imply that

$$\lim_{h\to 0}\varliminf_{\varepsilon=\varepsilon_k\to 0}\sum_{i,j}\langle\underline{\underline{C}}^{ij}_\varepsilon[\underline{w}_\varepsilon(\underline{x}^i_\varepsilon)-\underline{w}_\varepsilon(\underline{x}^j_\varepsilon)],[\underline{w}_\varepsilon(\underline{x}^i_\varepsilon)-\underline{w}_\varepsilon(\underline{x}^j_\varepsilon)]\rangle\ge$$

$$\ge\lim_{h\to 0}\varliminf_{\varepsilon=\varepsilon_k\to 0}\sum_\alpha\sum_{n,p,q,r}a^\tau_{npqr}(\underline{x}^\alpha,\varepsilon,h)e_{np}[\underline{w}(\underline{x}^\alpha]e_{qr}[\underline{w}(\underline{x}^\alpha].$$

Next we use condition (3.9) and take into account that $\underline{w}(\underline{x})\in C^2_0(\Omega)$ and $\underline{w}_\varepsilon(\underline{x}^i_\varepsilon)=\underline{v}_\varepsilon(\underline{x}^i_\varepsilon)$ to obtain the lower bound for the sum in the functional (4.6):

$$\lim_{h\to 0}\varliminf_{\varepsilon=\varepsilon_k\to 0}\sum_{i,j}\langle\underline{\underline{C}}^{ij}_\varepsilon[\underline{v}_\varepsilon(\underline{x}^i_\varepsilon)-\underline{v}_\varepsilon(\underline{x}^j_\varepsilon)],[\underline{v}_\varepsilon(\underline{x}^i_\varepsilon)-\underline{v}_\varepsilon(\underline{x}^j_\varepsilon)]\rangle\ge \tag{7.5}$$

$$\ge\int_\Omega\sum_{n,p,q,r}a_{npqr}(\underline{x})e_{np}[\underline{w}(\underline{x})]e_{qr}[\underline{w}(\underline{x})]dx.$$

It remains to estimate the integral in (4.6). Let $\underline{v}_\varepsilon(\underline{x})$ be a minimizer of (4.6) in the class $\overset{\circ}{J}_\varepsilon(\Omega)$. We denote by $\overset{\circ}{J}_{1\varepsilon}(\Omega)$ the class of divergence free vector functions from $\overset{\circ}{H}{}^1(\Omega)$ that are equal to $\underline{v}_\varepsilon(\underline{x}^i_\varepsilon)+\underline{b}^i_\varepsilon\times(\underline{x}-\underline{x}^i_\varepsilon)$ on Q^i_ε. Here the vectors $\underline{v}_\varepsilon(\underline{x}^i_\varepsilon)$ are given and the vectors $\underline{b}^i_\varepsilon$ are arbitrary. Let $\underline{v}^1_\varepsilon(\underline{x})$ be a minimizer of the problem

$$\Phi_{1\varepsilon}[\underline{v}^1_\varepsilon]=\min_{\underline{v}'_\varepsilon\in\overset{\circ}{J}_{1\varepsilon}(\Omega)}\Phi_{1\varepsilon}[\underline{v}'_\varepsilon], \tag{7.6}$$

where

$$\Phi_{1\varepsilon}[\underline{v}'_\varepsilon]=\int_\Omega\left\{2\mu\sum_{k,l}e^2_{kl}[\underline{v}'_\varepsilon]+\lambda\langle\rho_\varepsilon\underline{v}'_\varepsilon,\underline{v}'_\varepsilon\rangle-2\langle\rho_\varepsilon\underline{v}_{\varepsilon 0},\underline{v}'_\varepsilon\rangle\right\}dx. \tag{7.7}$$

From (4.6), (7.6) and the definition of $\overset{\circ}{J}_{1\varepsilon}(\Omega)$ we conclude that $\underline{v}^1_\varepsilon(\underline{x})\equiv\underline{v}_\varepsilon(\underline{x})$. We seek the minimizer $\underline{v}^1_\varepsilon(\underline{x})$ in the form

$$\underline{v}^1_\varepsilon(\underline{x}) = \underline{v}_{1\varepsilon}(\underline{x}) + \underline{\zeta}_\varepsilon(\underline{x}) \ (\equiv \underline{v}_\varepsilon), \tag{7.8}$$

where the function $\underline{v}_{1\varepsilon}(\underline{x})$ is defined by equality (6.14), where vectors $\underline{w}^{\varepsilon h}(\underline{x}^i_\varepsilon)$ are replaced by vectors $\underline{v}_\varepsilon(\underline{x}^i_\varepsilon)$.

Using equalities (6.13) and (6.7) we verify that $\underline{v}_{1\varepsilon}(\underline{x})$ equals $\underline{v}_\varepsilon(\underline{x}^i_\varepsilon)+\underline{b}^i_\varepsilon\times(\underline{x}-\underline{x}^i_\varepsilon)$ on the particles Q^i_ε, i.e. $\underline{v}_{1\varepsilon}(\underline{x}) \in \overset{\circ}{J}_{1\varepsilon}(\Omega)$. Combining this fact with (7.6), (7.7) and (7.8) we conclude that $\underline{\zeta}_\varepsilon(\underline{x}) \in \overset{\circ}{J}_{0\varepsilon}(\Omega)$, where the class $\overset{\circ}{J}_{0\varepsilon}(\Omega)$ consists of divergence free vector functions from $H^1(\Omega)$ equal to $\underline{b}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon)$ on the particles, with arbitrary vectors $\underline{b}^i_\varepsilon$. Moreover, the function $\underline{\zeta}_\varepsilon(\underline{x})$ minimizes the following functional in $\overset{\circ}{J}_{0\varepsilon}(\Omega)$

$$\Phi_{0\varepsilon}[\underline{\zeta}_\varepsilon] = \int_\Omega \Big\{ 2\mu \sum_{k,l} e^2_{kl}[\underline{\zeta}_\varepsilon] + 4\mu \sum_{k,l} e^2_{kl}[\underline{v}_{1\varepsilon}]e_{kl}[\underline{\zeta}_\varepsilon] + \lambda\langle\rho_\varepsilon\underline{\zeta}_\varepsilon, \underline{\zeta}_\varepsilon\rangle \tag{7.9}$$
$$+2\lambda\langle\underline{v}_{1\varepsilon}, \underline{\zeta}_\varepsilon\rangle - 2\langle\rho_\varepsilon\underline{v}_{\varepsilon 0}, \underline{\zeta}_\varepsilon\rangle \Big\} dx.$$

Let us show that

$$\lim_{\varepsilon=\varepsilon_k\to 0} \Phi_{0\varepsilon}[\underline{\zeta}_\varepsilon] = 0. \tag{7.10}$$

First, we reduce $\underline{v}_{1\varepsilon}(\underline{x})$ to the form (6.19), (6.20), where $\underline{w}^{\varepsilon h}_k(\underline{x}^i_\varepsilon)$ are replaced by $\underline{v}_{\varepsilon k}(\underline{x}^i_\varepsilon)$. Taking into account this representation, inequalities (6.11), (6.12) and the weak convergence in $H^1(\Omega)$ of $\underline{w}_\varepsilon(\underline{x})$ (see (4.12)) to $\underline{w}(\underline{x})$ as $\varepsilon = \varepsilon_k \to 0$, we get

$$\lim_{\varepsilon=\varepsilon_k\to 0} \|\underline{v}_{1\varepsilon} - \underline{v}\|_{L_2(\Omega)} = 0. \tag{7.11}$$

Since $\underline{v}_\varepsilon$ converges to $\underline{v}(\underline{x})$ weakly in $H^1(\Omega)$ as $\varepsilon = \varepsilon_k \to 0$ (therefore, $\underline{v}^1_\varepsilon(x) \to \underline{v}(x)$ strongly in $L_2(\Omega)$), (7.8) implies that

$$\lim_{\varepsilon=\varepsilon_k\to 0} \|\underline{\zeta}_\varepsilon\|_{L_2(\Omega)} = 0. \tag{7.12}$$

Taking into account (3.10) and (3.11), we conclude that the integrals of the third, forth and fifth terms in the integrand in (7.9) vanish as $\varepsilon = \varepsilon_k \to 0$.

It remains to estimate the integrals of the first two terms. Since $\Phi_{0\varepsilon}[\underline{\zeta}_\varepsilon] \le \Phi_{0\varepsilon}[0] = 0$, taking into account (7.9) we obtain

$$\int_\Omega 2\mu \sum_{k,l=1}^3 e^2_{kl}[\underline{\zeta}_\varepsilon]dx \le \left| \int_\Omega 4\mu \sum_{k,l=1}^3 e_{kl}[\underline{v}_{1\varepsilon}]e_{kl}[\underline{\zeta}_\varepsilon]dx \right| + o(1), \tag{7.13}$$

as $\varepsilon = \varepsilon_k \to 0$.

Integrate the integral in the right-hand side of this inequality by parts. Since $\underline{v}_{1\varepsilon} \in \overset{\circ}{J}_\varepsilon(\Omega) \bigcap C^2(\Omega \setminus \bigcup_i Q^i_\varepsilon)$ and the integral in (7.13) is taken over the domain $\Omega \setminus \bigcup_i Q^i_\varepsilon$, we have

$$\int_\Omega 4\mu \sum_{k,l=1}^3 e_{kl}[\underline{v}_{1\varepsilon}]e_{kl}[\underline{\zeta}_\varepsilon]dx = -2 \int_{\Omega\setminus\bigcup_i Q^i_\varepsilon} \mu\langle\Delta\underline{v}_{1\varepsilon}, \underline{\zeta}_\varepsilon\rangle dx + 2\sum_{i=1}^{N_\varepsilon} \int_{S^i_\varepsilon} \langle\underline{\underline{\sigma}}^0[\underline{v}_{1\varepsilon}]\nu, \underline{\zeta}_\varepsilon\rangle ds. \tag{7.14}$$

Here the components of the tensor $\underline{\sigma}^0[\underline{v}]$ are defined by $\sigma^0_{kl}[\underline{v}] = 2\mu e_{kl}[\underline{v}]$. We now use explicit form of $\underline{v}_{1\varepsilon}(\underline{x})$ (given by (6.14) and an equivalent formula (6.19), where the functions $\underline{w}^{\varepsilon h}(\underline{x}^i_\varepsilon)$ are replaced by $\underline{v}_\varepsilon(\underline{x}^i_\varepsilon)$). Thus, (6.19) and (6.20) imply that

$$\int\limits_{\Omega\setminus\bigcup_i Q^i_\varepsilon} \langle \Delta \underline{v}_{1\varepsilon}, \underline{\zeta}_\varepsilon\rangle dx = \sum_{l=1}^{6} I^\varepsilon_l, \tag{7.15}$$

where

$$I^\varepsilon_1 = \int\limits_{\Omega\setminus\bigcup_i Q^i_\varepsilon} \sum_i \sum_k \underline{u}_{\varepsilon k}(\underline{x}^i_\varepsilon)\langle \Delta \underline{v}^{i,k}, \underline{\zeta}_\varepsilon\rangle \varphi^i_{2\varepsilon} dx, \quad I^\varepsilon_2 = \int\limits_{\Omega\setminus\bigcup_i Q^i_\varepsilon} \langle \Delta \underline{v}(\underline{x}), \underline{\zeta}_\varepsilon\rangle dx,$$

$$I^\varepsilon_3 = \int\limits_{\Omega\setminus\bigcup_i Q^i_\varepsilon} \sum_i \sum_k \underline{u}_{\varepsilon k} \left\langle \sum_{\substack{|\alpha+\beta|=2\\ |\alpha|<2}} C_{\alpha\beta} D^\alpha \underline{v}^{i,k} D^\beta \varphi^i_{2\varepsilon}, \underline{\zeta}_\varepsilon \right\rangle dx,$$

$$I^\varepsilon_4 = \int\limits_{\Omega\setminus\bigcup_i Q^i_\varepsilon} \sum_i \sum_k \underline{u}_{\varepsilon k} \left\langle \sum_{|\alpha+\beta|=2} C_{\alpha\beta} D^\alpha \underline{\tilde{v}}^{i,k} \times \nabla D^\beta \varphi^i_{2\varepsilon}, \underline{\zeta}_\varepsilon \right\rangle dx, \tag{7.16}$$

$$I^\varepsilon_5 = \int\limits_{\Omega\setminus\bigcup_i Q^i_\varepsilon} \sum_i \left\langle \sum_{|\alpha+\beta|=2} C_{\alpha\beta} D^\alpha [\underline{v}(\underline{x}^i_\varepsilon) - \underline{v}(\underline{x})] \nabla D^\beta \varphi^i_{1\varepsilon}, \underline{\zeta}_\varepsilon \right\rangle dx,$$

$$I^\varepsilon_6 = \int\limits_{\Omega\setminus\bigcup_i Q^i_\varepsilon} \sum_i \left\langle \sum_{|\alpha+\beta|=2} C_{\alpha\beta} D^\alpha \underline{\tilde{v}}^i \times \nabla D^\beta \varphi^i_{1\varepsilon}, \underline{\zeta}_\varepsilon \right\rangle dx.$$

Here $\underline{u}_{\varepsilon k}(\underline{x}^i_\varepsilon) = \underline{v}_{\varepsilon k}(\underline{x}^i_\varepsilon) - \underline{v}_k(\underline{x}^i_\varepsilon)$ and $C_{\alpha\beta}$ are nonnegative integer-valued coefficients. Taking into account (7.12), the boundedness of the vector functions (4.12) in $H^1(\Omega)$, and inequalities (6.11), (6.12), we can get

$$\lim_{\varepsilon=\varepsilon_k\to 0} I^\varepsilon_l = 0 \quad (l = 2, \ldots, 6). \tag{7.17}$$

This happens to be the case for I^ε_2, since $\underline{v}(\underline{x}) \in C^2(\Omega)$. Demonstrate this fact for I^ε_3. Since $|\alpha| < 2$ (and, thus, $|\beta| > 1$), then $|D^\alpha \underline{v}^{ik}| = O(\varepsilon^{2-\alpha})$ on the support of the functions $D^\beta \varphi^i_{2\varepsilon}(\underline{x})$ due to inequality (6.11) and condition (3.2). Due to the properties of the functions $\varphi^i_{2\varepsilon}(\underline{x})$ we have $D^\beta \varphi^i_{2\varepsilon}(\underline{x}) = O(\varepsilon^{-\beta})$. Thus, we arrive at the estimate

$$|I^\varepsilon_3| \le C \sum_i |\underline{v}_\varepsilon(\underline{x}^i_\varepsilon) - \underline{v}(\underline{x}^i_\varepsilon)| \int\limits_{B^i_{2\varepsilon}} |\underline{\zeta}_\varepsilon|\, dx.$$

Further, applying the Schwartz inequality, we get

$$|I^\varepsilon_3| \le C \sum_i |\underline{v}_\varepsilon(\underline{x}^i_\varepsilon) - \underline{v}(\underline{x}^i_\varepsilon)||B^i_{2\varepsilon}|^{1/2} \left(\int\limits_{B^i_{2\varepsilon}} |\underline{\zeta}_\varepsilon|^2 dx \right)^{1/2} \le \tag{7.18}$$

$$\le C \left(\sum_i |\underline{v}_\varepsilon(\underline{x}^i_\varepsilon) - \underline{v}(\underline{x}^i_\varepsilon)|^2 |B^i_{2\varepsilon}| \right)^{1/2} \left(\int\limits_{\Omega} |\underline{\zeta}_\varepsilon|^2\, dx \right)^{1/2}.$$

Since the functions (4.12) are bounded, uniformly in ε, in $H^1(\Omega)$ and they converge to $\underline{w}(\underline{x}) \in C^2(\Omega)$ strongly in $L_2(\Omega)$ as $\varepsilon = \varepsilon_k \to 0$, taking into account condition I_1 ($|B^i_{2\varepsilon}| < C\varepsilon^3$), we obtain

$$\sum_i |\underline{v}_\varepsilon(\underline{x}^i_\varepsilon) - \underline{v}(\underline{x}^i_\varepsilon)|^2 |B^i_{2\varepsilon}| < C. \tag{7.19}$$

where C does not depend on ε.

The required equality (7.17) for I^ε_3 follows from (7.18), (7.19) and (7.12). Analogously, using (6.13), equality (7.17) is obtained for other I^ε_l ($l = 4, 5, 6$).

The integral I^ε_1 is not small, but the sum of I^ε_1 and the surface integrals in the RHS of (7.14) vanishes as $\varepsilon = \varepsilon_k \to 0$.

Indeed, recall that the vector function $\underline{v}^{i,k}(\underline{x})$ is a solution of problem (6.6) – (6.9). Therefore, we have

$$I^\varepsilon_1 = \frac{1}{\mu} \int\limits_{\Omega\setminus Q^i_\varepsilon} \sum_i \sum_k u_{\varepsilon k}(\underline{x}^i_\varepsilon) \langle \nabla p^{i,k}_\varepsilon, \underline{\zeta}_\varepsilon \rangle \varphi^i_{2\varepsilon} dx.$$

Integrating by parts and using the properties of the functions $\varphi^i_{2\varepsilon}(x)$ and the divergence free constraint for the function $\underline{\zeta}_\varepsilon(x)$ lead to

$$I^\varepsilon_1 = \frac{1}{\mu} \sum_i \sum_k u_{\varepsilon k}(\underline{x}^i_\varepsilon) \int\limits_{S^i_\varepsilon} \langle p^{ik}_\varepsilon \underline{\nu}, \underline{\zeta}_\varepsilon \rangle ds - \tag{7.20}$$

$$-\frac{1}{\mu} \sum_i \sum_k \underline{u}_{\varepsilon k}(\underline{x}^i_\varepsilon) \int\limits_{B^i_{2\varepsilon}\setminus B'^i_{2\varepsilon}} p^{ik}_\varepsilon \langle \nabla \varphi^i_{2\varepsilon}, \underline{\zeta}_\varepsilon \rangle dx,$$

where ν is the inner normal on S^i_ε and $B'^i_{2\varepsilon}$ is the ball of radius $r'^i_\varepsilon = \dfrac{r^i_\varepsilon}{2}$ concentric to $B^i_{2\varepsilon}$ (thus, in accordance with condition (3.1), $r'^i_\varepsilon > C\varepsilon$).

Now we estimate the second sum in the right-hand side of (7.20). Since $\underline{u}^i_{\varepsilon k}(\underline{x}^i_\varepsilon) = \underline{v}_{\varepsilon k}(\underline{x}) - \underline{v}_k(\underline{x}^i_\varepsilon)$, taking into account the properties of the functions $\varphi^i_{2\varepsilon}(\underline{x})$ and (6.11), to get

$$\left| \sum_i \sum_k \underline{u}^i_{\varepsilon k}(\underline{x}^i_\varepsilon) \int\limits_{B^i_{2\varepsilon}\setminus B'^i_{2\varepsilon}} p^{i,k}_\varepsilon \langle \nabla \varphi^i_{2\varepsilon}, \underline{\zeta}_\varepsilon \rangle \, dx \right| \leq$$

$$\leq C \sum_i |\underline{v}_\varepsilon(\underline{x}^i_\varepsilon) - \underline{v}(\underline{x}^i_\varepsilon)| |B^i_{2\varepsilon} \setminus B'^i_{2\varepsilon}|^{1/2} \left\{ \int\limits_{B^i_{2\varepsilon}\setminus B'^i_{2\varepsilon}} |\underline{\zeta}_\varepsilon|^2 dx \right\}^{1/2} \leq$$

$$\leq C \left\{ \sum_i |\underline{v}_\varepsilon(\underline{x}^i_\varepsilon) - \underline{v}(\underline{x}^i_\varepsilon)|^2 |B^i_{2\varepsilon}| \right\}^{1/2} \left\{ \int\limits_\Omega |\underline{\zeta}_\varepsilon|^2 dx \right\}^{1/2}.$$

Then (7.12) implies

$$\lim_{\varepsilon=\varepsilon_k\to 0} \sum_i \sum_k \underline{u}_{\varepsilon k}(\underline{x}^i_\varepsilon) \int\limits_{B^i_{2\varepsilon}\setminus B'^i_{2\varepsilon}} p^{i,k}_\varepsilon \langle \nabla \varphi^i_{2\varepsilon}, \underline{\zeta}_\varepsilon \rangle dx = 0. \tag{7.21}$$

Recall that $\underline{\zeta}_{\varepsilon}(\underline{x}) = \underline{b}^i_{\varepsilon} \times (\underline{x} - \underline{x}^i_{\varepsilon})$ on Q^i_{ε}. According to (6.19), $\underline{v}_{1\varepsilon}(\underline{x}) = \underline{v}(\underline{x}^i_{\varepsilon}) + \sum_i (\underline{v}_{\varepsilon k}(\underline{x}^i_{\varepsilon}) - \underline{v}_k(\underline{x}^i_{\varepsilon}))\underline{v}^{ik}(\underline{x}) \equiv \underline{v}(\underline{x}^i_{\varepsilon}) + \sum_k \underline{u}_{\varepsilon k}(\underline{x}^i_{\varepsilon})\underline{v}^{ik}(\underline{x})$ in a neighborhood of Q^i_{ε} and thus $\underline{\underline{\sigma}}^0[\underline{v}_{1\varepsilon}] = \sum_k \underline{u}_{\varepsilon k}(\underline{x}^i_{\varepsilon})\underline{\underline{\sigma}}^0[v^{ik}]$ on S^i_{ε}.

Therefore, combining (7.14) – (7.17), (7.20) and (7.21), we get

$$\int_{\Omega} 4\mu \sum_{k,l=1}^{3} e_{kl}[\underline{v}_{1\varepsilon}]e_{kl}[\underline{\zeta}_{\varepsilon}]dx =$$

$$= -2\sum_{i=1}^{N_{\varepsilon}} \sum_k \underline{u}_{\varepsilon k}(\underline{x}^i_{\varepsilon}) \int_{S^i_{\varepsilon}} \langle (\underline{x} - \underline{x}^i_{\varepsilon}) \times \underline{\underline{\sigma}}[\underline{v}^{ik}]\nu, \underline{b}^i_{\varepsilon}\rangle ds + o(1), \quad \text{as } \varepsilon = \varepsilon_k \to 0.$$

Due to (6.8), the first term in the RHS of this equality vanishes and therefore $\lim\limits_{\varepsilon=\varepsilon_k\to 0} \int_{\Omega} 4\mu \sum e_{kl}[\underline{v}_{1\varepsilon}]e_{kl}[\underline{\zeta}_{\varepsilon}]dx = 0$.

Taking into account (7.13), we have $\lim\limits_{\varepsilon=\varepsilon_k\to 0} \int_{\Omega} 2\mu \sum_{k,l=1}^{3} e^2_{kl}[\underline{\zeta}_{\varepsilon}]dx = 0$. Thus, the required equality (7.10) is proved.

Due to (7.8) the following equality holds

$$\Phi_{1\varepsilon}[\underline{v}_{\varepsilon}] = \Phi_{1\varepsilon}[\underline{v}_{1\varepsilon}] + \Phi_{0\varepsilon}[\underline{\zeta}_{\varepsilon}], \tag{7.22}$$

where the functionals $\Phi_{1\varepsilon}[\underline{v}_{\varepsilon}]$ and $\Phi_{0\varepsilon}[\underline{\zeta}_{\varepsilon}]$ are defined in (7.7) and (7.9). Taking into account the form of the vector function $\underline{v}_{1\varepsilon}(\underline{x})$ (see (6.19) and (6.20), where $w_k(\underline{x}^i_{\varepsilon})$ are replaced by $v_{\varepsilon k}(\underline{x}^i_{\varepsilon})$) and (7.11), we obtain

$$\Phi_{1\varepsilon}[\underline{v}_{1\varepsilon}] = \int_{\Omega} \left\{ 2\mu \sum_{k,l=1}^{3} e^2_{kl}[\underline{v}] + \lambda\langle \rho\underline{v}, \underline{v}\rangle - 2\langle \rho\underline{v}_{\varepsilon 0}, \underline{v}\rangle \right\} dx +$$

$$+ \sum_i \sum_{p,q} 6\pi\mu r^i_{\varepsilon}\delta_{pq}[\underline{v}_q(\underline{x}^i_{\varepsilon}) - \underline{w}_q(\underline{x}^i_{\varepsilon})][\underline{v}_p(\underline{x}^i_{\varepsilon}) - \underline{w}_p(\underline{x}^i_{\varepsilon})] + o(1),$$

as $\varepsilon = \varepsilon_k \to 0$.

Therefore, using the conditions (3.8), (3.11), and the regularity of the vector functions $\underline{v}(\underline{x})$ and $\underline{w}(\underline{x})$, we obtain

$$\lim_{\varepsilon=\varepsilon_k\to 0} \Phi_{1\varepsilon}[\underline{v}_{1\varepsilon}] = \int_{\Omega} \Big\{ 2\mu \sum_{k,l=1}^{3} e_{kl}[\underline{v}] + \lambda\langle \rho\underline{v}, \underline{v}\rangle + \tag{7.23}$$

$$+ \langle C(\underline{x})[\underline{v} - \underline{w}], [\underline{v} - \underline{w}]\rangle - 2\langle \rho\underline{v}_0, \underline{v}\rangle \Big\} dx.$$

Finally, combining (7.22), (7.23), (7.10), (7.5), (4.6) and (4.8) leads to inequality (5.7).

We have proved this inequality for smooth limiting vector functions $\underline{v}(\underline{x})$, $\underline{w}(\underline{x}) \in C^2(\Omega)$. Since the regularity for these vector functions is not known in advance, the complete proof is more technical, but its scheme is the same: introduce smooth approximations $\underline{v}_\sigma(x)$ and $\underline{w}_\sigma(x)$ of $\underline{v}(\underline{x})$ and $\underline{w}(\underline{x})$, obtain the corresponding inequalities for the vector functions $\underline{v}_\sigma(\underline{x})$ and $\underline{w}_\sigma(\underline{x})$, and pass to the limit as $\sigma \to 0$. This can be done similarly to [4].

Theorem 4.1 is proved.

8. **Analytic properties of solutions of problems (4.1) – (4.5) and (4.10) – (4.11). Completion of the proof of main theorem. 1.** Let us represent problem (4.1) – (4.5) in an operator form. For this purpose we introduce a space $\overset{\circ}{J}_\varepsilon(\Omega)$. Let $\overset{\circ}{\tilde{J}}_\varepsilon(\Omega)$ be the set of vector functions $\underline{u}_\varepsilon(\underline{x})$ such that the following conditions hold: $\underline{u}_\varepsilon(\underline{x}) \in \overset{\circ}{H}{}^1(\Omega) \bigcap H^2(\Omega)$, $\text{div}\underline{u}_\varepsilon(\underline{x}) = 0$, $\underline{u}_\varepsilon(\underline{x}) = \underline{a}^i_\varepsilon \times \underline{b}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon)$ as $\underline{x} \in Q^i_\varepsilon$ $(i = 1, 2, \dots, N_\varepsilon)$, where $\underline{a}^i_\varepsilon$ and $\underline{b}^i_\varepsilon$ are arbitrary vectors. Denote by $\overset{\circ}{J}_\varepsilon(\Omega)$ the closure in $L_2(\Omega)$ of the set $\overset{\circ}{\tilde{J}}_\varepsilon(\Omega)$ and by $G_\varepsilon(\Omega)$ the set of vector functions from $L_2(\Omega)$ such that they can be represented in the form $\underline{g}_\varepsilon(\underline{x}) = \text{grad}\varphi(\underline{x}) + \sum_{i=1}^{N_\varepsilon} \underline{\psi}^i_\varepsilon(\underline{x})\chi_\varepsilon(\underline{x})$. Here $\varphi(\underline{x})$ are the functions from $H^1(\Omega)$ and $\chi^i_\varepsilon(\underline{x})$ are the characteristic functions of the sets Q^i_ε $(i = 1, 2, \dots, N_\varepsilon)$. The vector functions $\underline{\psi}^i_\varepsilon(\underline{x}) \in L_2(Q^i_\varepsilon)$ are orthogonal in $L_2(Q^i_\varepsilon)$ to all vector functions of the form $\underline{a}^i_\varepsilon + \underline{b}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon)$.

Lemma 8.1. *The following decomposition of the space $L_2(\Omega)$ into an orthogonal sum holds*

$$L_2(\Omega) = \overset{\circ}{J}_\varepsilon(\Omega) \oplus G_\varepsilon(\Omega).$$

Introduce the operator $\hat{A}_\varepsilon$ that maps $\overset{\circ}{\tilde{J}}_\varepsilon(\Omega)$ into $L_2(\Omega)$ according to the following rule

$$\hat{A}_\varepsilon \underline{u}_\varepsilon = \begin{cases} -\mu\Delta\underline{u}_\varepsilon, & x \in \Omega_\varepsilon, \\ \underline{A}^i_0[\underline{u}_\varepsilon] + \underline{B}^i_0[\underline{u}_\varepsilon] \times (\underline{x} - \underline{x}^i_\varepsilon), & x \in Q^i_\varepsilon \ (i = 1, 2, \dots, N_\varepsilon), \end{cases}$$

where the constant vectors $\underline{A}^i_0[\underline{u}_\varepsilon]$ and $\underline{B}^i_0[\underline{u}_\varepsilon]$ are defined by $\underline{A}^i_0[\underline{u}_\varepsilon] = \frac{\rho_s}{m^i_\varepsilon} \int\limits_{S^i_\varepsilon} \underline{\underline{\sigma}}^0[\underline{u}_\varepsilon] \cdot \underline{\nu} ds$, $\underline{B}^i_0[\underline{u}_\varepsilon] = \rho_s (I^i_\varepsilon)^{-1} \int\limits_{S^i_\varepsilon} (\underline{x} - \underline{x}^i_\varepsilon) \times (\underline{\underline{\sigma}}^0[\underline{u}_\varepsilon] \cdot \underline{\nu})\, ds$. Let $\tilde{A}_\varepsilon = P_\varepsilon \hat{A}_\varepsilon$, where P_ε is the orthogonal projection in $L_2(\Omega)$ on $\overset{\circ}{J}_\varepsilon(\Omega)$.

The following lemma can be proved (see [5]).

Lemma 8.2. *The operator $\tilde{A}_\varepsilon$ is symmetric and positive definite in the space $\overset{\circ}{J}_\varepsilon(\Omega)$.*

Denote by A_ε the Friedreich's extension of $\tilde{A}_\varepsilon$ ([27]).

It easy to show that A_ε is self-adjoint and invertible in $\overset{\circ}{J}_\varepsilon(\Omega)$ and that A_ε^{-1} is compact. The operator A_ε describes the fluid; now we construct operators that describe the elastic skeleton.

Introduce in $\overset{\circ}{J}_\varepsilon(\Omega)$ the bounded operators $B_{k\varepsilon} = P_\varepsilon \hat{B}_{k\varepsilon}$ $(k = 1, 2)$, where $\hat{B}_{1\varepsilon}$, $\hat{B}_{2\varepsilon}$ map $\overset{\circ}{J}_\varepsilon(\Omega)$ into $L_2(\Omega)$ according to the formulas $\hat{B}_{1\varepsilon}\underline{u}_\varepsilon(\underline{x}) = \rho_\varepsilon(\underline{x})\underline{u}_\varepsilon(\underline{x}) =$

$\rho\chi_\varepsilon(\underline{x})\underline{u}_\varepsilon(\underline{x}) + \rho_s \sum_{i=1}^{N_\varepsilon} \chi_\varepsilon^i(\underline{x})\underline{u}_\varepsilon(\underline{x})$ and $\hat{B}_{2\varepsilon}\underline{u}_\varepsilon(\underline{x}) = \sum_{i,j} \chi_\varepsilon^i(\underline{x})\rho_s(m_\varepsilon^i)^{-1}\underline{\underline{C}}_\varepsilon^{ij}[\underline{u}_\varepsilon^i - \underline{u}_\varepsilon^j]$ respectively, where $\chi_\varepsilon^i(\underline{x})$ are the characteristic functions of Q_ε^i, $\underline{u}_\varepsilon^i = \underline{u}_\varepsilon(\underline{x}_\varepsilon^i)$.

The following lemma can be easily proved.

Lemma 8.3. *The operators $B_{k\varepsilon}$ $(k = 1, 2)$ are self-adjoint in $\overset{\circ}{J}_\varepsilon(\Omega)$. $B_{1\varepsilon}$ is positive definite and $B_{2\varepsilon}$ is non-negative.*

Introduce the function $\varphi_\varepsilon(x) = \rho\chi_\varepsilon(\underline{x})\underline{v}_{0\varepsilon}(\underline{x}) + \sum \rho_s\chi_s^i(x)[u_{\varepsilon i}^i + \theta_{\varepsilon i}^i \times (\underline{x} - \underline{x}_\varepsilon^i) \in L_2(\Omega)$, which corresponds to the RHS in equations (4.1), (4.2). Let $g_\varepsilon(\underline{x})$ be the projection of $\varphi_\varepsilon(x)$ on $\overset{\circ}{J}_\varepsilon(\Omega)$. Clearly, the function $g_\varepsilon(\underline{x})$ is bounded in the norm of $L_2(\Omega)$ uniformly in ε.

It follows from the definition of the operators A_ε, $B_{1\varepsilon}$, $B_{2\varepsilon}$ and $\underline{g}_\varepsilon(\underline{x})$ that problem (4.1) – (4.5) can be represented in an operator form in the space $\overset{\circ}{J}_\varepsilon(\Omega)$, as follows:

$$A_\varepsilon\underline{v}_\varepsilon + \lambda B_{1\varepsilon}\underline{v}_\varepsilon + \frac{1}{\lambda}B_{2\varepsilon}\underline{v}_\varepsilon = \underline{g}_\varepsilon. \tag{8.1}$$

Due to the properties of the operator A_ε equation (8.1) can be written in the form $(I + L_\varepsilon(\lambda))\underline{v}_\varepsilon = A_\varepsilon^{-1}\underline{g}_\varepsilon$, where $L_\varepsilon(\lambda) = \lambda A_\varepsilon^{-1}B_{1\varepsilon} + \frac{1}{\lambda}A_\varepsilon^{-1}B_{2\varepsilon}$ is a compact operator for any $\lambda \neq 0$.

The operator-function $C_\varepsilon(\lambda) = I + L_\varepsilon(\lambda)$ is analytic in $C \setminus \{0\}$ and, due to the properties of operators A_ε, $B_{1\varepsilon}$ and $B_{2\varepsilon}$, the operator $C_\varepsilon(\lambda)$ has a bounded inverse for any $\lambda > 0$. According to [26], $C_\varepsilon(\lambda)$ is a regular pencil in $C \setminus \{0\}$ and, therefore, its resolvent $C_\varepsilon^{-1}(\lambda)$ is meromorphic in $C \setminus 0$. Moreover, using the properties of the operators A_ε, $B_{1\varepsilon}$ and $B_{2\varepsilon}$, we can show that $C_\varepsilon^{-1}(\lambda)$ is analytic in the domain $\{\text{Re}\lambda > 0\}$.

Let us show that $\underline{v}_\varepsilon(\lambda)$ satisfies the following estimate (as $|\lambda| \to \infty$ and $|\arg\lambda - \pi| > \delta$)

$$\|\underline{v}_\varepsilon(\lambda)\|_{L_2(\Omega)} \leq \frac{C_\varepsilon}{|\lambda|}, \tag{8.2}$$

where $C_\varepsilon < \infty$, if $|\lambda|\hat{\lambda}(\varepsilon, \delta)$.

Take the scalar product in $L_2(\Omega)$ of equation (8.1) with $\underline{v}_\varepsilon(\lambda)$, and separate the real and imaginary parts:

$$(A_\varepsilon\underline{v}_\varepsilon, \underline{v}_\varepsilon) + \text{Re}\lambda(B_{1\varepsilon}\underline{v}_\varepsilon, \underline{v}_\varepsilon) + \frac{\text{Re}\lambda}{|\lambda|^2}(B_{2\varepsilon}\underline{v}_\varepsilon, \underline{v}_\varepsilon) = \text{Re}(\underline{g}_\varepsilon, \underline{v}_\varepsilon),$$

$$\text{Im}\lambda(B_{1\varepsilon}\underline{v}_\varepsilon, \underline{v}_\varepsilon) - \frac{\text{Im}\lambda}{|\lambda|^2}(B_{2\varepsilon}\underline{v}_\varepsilon, \underline{v}_\varepsilon) = \text{Im}(\underline{g}_\varepsilon, \underline{v}_\varepsilon). \tag{8.3}$$

Taking into account the non-negativity of the operators A_ε and $B_{2\varepsilon}$, the fact that the operator $B_{1\varepsilon}$ is positive definite uniformly in ε (i.e. $(B_{1\varepsilon}\underline{u}, \underline{u}) \geq \beta\|\underline{u}\|^2$, $\beta > 0$), from the first equality in (8.3) we obtain

$$\|\underline{v}_\varepsilon\|_{L_2(\Omega)} \leq \frac{1}{\beta\text{Re}\lambda}\|\underline{g}_\varepsilon\|_{L_2(\Omega)} \tag{8.4}$$

for $\text{Re}\lambda > 0$.

From the second equation in (8.3) we get (taking into account that $B_{2\varepsilon}$ is a bounded operator)

$$\|\underline{v}_\varepsilon\|_{L_2(\Omega)} \le \frac{1}{|\mathrm{Im}\lambda|}\|\underline{g}_\varepsilon\|_{L_2(\Omega)}\left(\beta - \frac{\|B_{2\varepsilon}\|}{|\lambda|^2}\right)^{-1}, \tag{8.5}$$

for $|\arg\lambda - \pi| > \delta > 0$. Since g_ε is bounded uniformly in ε and $\|B_{2\varepsilon}\| \le C_{1\varepsilon} < \infty$, (8.4), (8.5) imply that inequality (8.2) holds as $|\lambda| \to \infty$ and $|\arg\lambda - \pi| > \delta$. It means that the solution $\underline{v}_\varepsilon(\underline{x}, \lambda)$ of problem (4.1) – (4.5) is analytic in the domain $\Lambda_{\varepsilon\delta} = \{\lambda \in \mathbb{C} : |\arg\lambda - \pi| > \delta,\ |\lambda|\hat{\lambda}(\varepsilon,\delta)\} \bigcup \{\mathrm{Re}\lambda > 0\}$.

2. Consider now the homogenized problem (4.10)–(4.11) for $\lambda \in \mathbb{C}$. To this end we write it in an operator form in a Hilbert space defined as follows. Denote by $\overset{\circ}{\tilde{H}}(\Omega)$ the set of the pairs $(\underline{v}, \underline{w})$, where $\underline{v} \in \overset{\circ}{J}(\Omega)\bigcap H^2(\Omega)$ and $\underline{w} \in \overset{\circ}{H}{}^1(\Omega)\bigcap H^2(\Omega)$. Thus $\overset{\circ}{\tilde{H}}(\Omega) = \left(\overset{\circ}{J}(\Omega)\bigcap H^2(\Omega)\right) \times \left(\overset{\circ}{H}{}^1(\Omega)\bigcap H^2(\Omega)\right)$. Denote by $\overset{\circ}{H}(\Omega)$ the closure of this set in $(L_2(\Omega))^2$. This is a Hilbert space with the scalar product defined as in $(L_2(\Omega))^2$. The following orthogonal decomposition (analogous to the Weil decomposition) holds

$$(L_2(\Omega))^2 = \overset{\circ}{H}(\Omega) \oplus G(\Omega), \tag{8.6}$$

where we denote by $G(\Omega)$ a subspace in $(L_2(\Omega))^2$, which consists of the pairs of the vector functions of the form $(\mathrm{grad}\varphi, 0)$. Here $\varphi(\underline{x})$ is an arbitrary function from $H^1(\Omega)$.

Introduce the linear operators $\hat{A}$, $\hat{B}_1$ and $\hat{B}_2$ in $\overset{\circ}{H}(\Omega)$ that map $\overset{\circ}{\tilde{H}}(\Omega)$ into $(L_2(\Omega))^2$:

$$\hat{A}\begin{bmatrix}\underline{v}\\ \underline{w}\end{bmatrix} = \begin{bmatrix}-\mu\Delta\underline{v} + C(x)\underline{v} - C(x)\underline{w}\\ -C(x)\underline{v} + C(x)\underline{w}\end{bmatrix}, \tag{8.7}$$

$$\hat{B}_1\begin{bmatrix}\underline{v}\\ \underline{w}\end{bmatrix} = \begin{bmatrix}\rho\underline{v}\\ 0\end{bmatrix}, \quad \hat{B}_2\begin{bmatrix}\underline{v}\\ \underline{w}\end{bmatrix} = \begin{bmatrix}0\\ -\sum\limits_{n,p,q,r}\dfrac{\partial}{\partial x_p}\left\{a_{npqr}(\underline{x})e_{qr}[\underline{w}]\right\}\underline{e}^n\end{bmatrix}. \tag{8.8}$$

Here we use the fact that $a_{npqr}(\underline{x}) \in C^1_\varepsilon(\Omega)$.

Let P be the projection operator in $L_2(\Omega)$ onto the space $\overset{\circ}{H}(\Omega)$. Introduce the operators $A = P\hat{A}$, $B_1 = P\hat{B}_1$, $B_2 = P\hat{B}_2$ that map $\overset{\circ}{\tilde{H}}(\Omega)$ into $\overset{\circ}{H}(\Omega)$. According to (8.6) $B_1 = \hat{B}_1$ and $B_2 = \hat{B}_2$.

We keep the same notation for the Friedreich's extension of operators A, B_1 and B_2. In the standard way, one can show that the operators B_1 and B_2 are self-adjoint and non-negative. The operator A is positive definite, invertible, and the operator A^{-1} is compact.

It follows from (8.7) – (8.8) that problem (4.10)–(4.11) can be written in the following operator form (in $\overset{\circ}{H}(\Omega)$):

$$A\underline{u} + \lambda B_1\underline{u} + \frac{1}{\lambda}B_2\underline{u} = \underline{g}, \tag{8.9}$$

where $\underline{g} = \begin{bmatrix}\rho\underline{v}_0\\ 0\end{bmatrix} \in H^\circ(\Omega)$. Due to the properties of the operator A, this equation can be rewritten in the form $(I + L(\lambda))\underline{u} = A^{-1}\underline{g}$, where $L(\lambda) = \lambda A^{-1}B_1 +$

$\dfrac{1}{\lambda}A^{-1}B_2$.Taking into account the fact that the operator A is positive definite and B_1 and B_2 are non-negative, one can show that equation (8.9) has a unique solution for Re$\lambda \geq 0$. Moreover, this solution is analytic in the right half-plane.

To obtain the estimates in some angle analogous to (8.2) for the solution of equation (8.9), we use directly equalities (4.10)-(4.11).

Let λ be from the right half-plane (Re$\lambda \geq 0$). Then the problem (4.10)-(4.11) has a unique solution $(\underline{v}, \underline{w})$. Multiply the first equation (4.10) by $\bar{\underline{v}}(\underline{x}) \in \overset{\circ}{J}(\Omega)$ and the second equation (4.10) by $\bar{\underline{w}}(\underline{x}) \in \overset{\circ}{H}{}^1(\Omega)$, where the bar stands for complex conjugation. Next, integrate them over Ω and sum up the obtained equalities. Then separate the real and imaginary parts to get (after integrating by parts and taking into account that $\bar{\underline{v}}(\underline{x}) \in \overset{\circ}{J}(\Omega)$, $\bar{\underline{w}}(\underline{x}) \in \overset{\circ}{H}{}^1(\Omega), C(\underline{x}) > 0$)

$$\int\limits_\Omega \Big\{ 2\mu \sum_{k,l=1}^{3} e_{kl}[\underline{v}] e_{kl}[\bar{\underline{v}}] + \langle C(x)[\underline{v} - \underline{w}], \bar{\underline{v}} - \bar{\underline{w}} \rangle + \mathrm{Re}\lambda \rho \langle \underline{v}, \bar{\underline{v}} \rangle + \\ + \frac{\mathrm{Re}\lambda}{|\lambda|^2} \sum a_{npqr}(\underline{x}) e_{np}[\underline{w}] e_{qr}[\bar{\underline{w}}] \Big\} dx = \mathrm{Re} \int\limits_\Omega \rho \langle \underline{v}_0, \bar{\underline{v}} \rangle dx \tag{8.10}$$

and

$$\mathrm{Im}\lambda \int\limits_\Omega \rho \langle \underline{v}, \bar{\underline{v}} \rangle dx - \frac{\mathrm{Im}\lambda}{|\lambda|^2} \int\limits_\Omega \sum a_{npqr}(\underline{x}) e_{np}[\underline{w}] e_{qr}[\bar{\underline{w}}] dx = \mathrm{Im} \int\limits_\Omega \rho \langle \underline{v}_0, \bar{\underline{v}} \rangle dx. \tag{8.11}$$

It follows from (8.10) that for Re$\lambda > 0$

$$\|\underline{v}\|_{L_2(\Omega)} \leq \frac{1}{\mathrm{Re}\lambda} \|\underline{v}_0\|_{L_2(\Omega)}, \tag{8.12}$$

Multiply equation (4.10) by $\bar{\underline{w}}(x)$ and integrate over Ω. Then separate the real and imaginary parts to get

$$\frac{\mathrm{Re}\lambda}{|\lambda|^2} \int\limits_\Omega \sum_{n,p,q,r=1}^{3} a_{npqr} e_{np}[\underline{w}] e_{qr}[\bar{\underline{w}}] dx + \int\limits_\Omega \langle C\underline{w}, \bar{\underline{w}} \rangle dx = \mathrm{Re} \int\limits_\Omega \langle C\underline{v}, \bar{\underline{w}} \rangle dx \tag{8.13}$$

and

$$-\frac{\mathrm{Im}\lambda}{|\lambda|^2} \int\limits_\Omega \sum a_{npqr} e_{np}[\underline{w}] e_{qr}[\bar{\underline{w}}] dx = \mathrm{Im} \int\limits_\Omega \langle C\underline{v}, \bar{\underline{w}} \rangle dx. \tag{8.14}$$

Since $C(x) > 0$, from (8.13)-(8.14) it follows that for Im$\lambda \neq 0$,

$$\|\underline{w}\|_{L_2(\Omega)} \leq C \left(1 + \frac{|\mathrm{Re}\lambda|}{|\mathrm{Im}\lambda|} \right) \|\underline{v}\|_{L_2(\Omega)}. \tag{8.15}$$

Taking into account the positiveness of ρ, from (8.11) and (8.14) we get

$$\|\underline{v}\|_{L_2(\Omega)} \leq \frac{C}{|\mathrm{Im}\lambda|} \left(\|\underline{v}_0\|_{L_2(\Omega)} + \|\underline{w}\|_{L_2(\Omega)} \right).$$

Combining this inequality with (8.15) we get that for $|\mathrm{Re}\lambda| \leq |\mathrm{Im}\lambda|$,

$$\|\underline{v}\|_{L_2(\Omega)} \leq \frac{C_1}{|\mathrm{Im}\lambda|}, \tag{8.16}$$

where C_1 does not depend on λ.

Thus, taking into account (8.12), (8.15) and (8.16), we get that in the domain $\text{Re}\lambda - |\text{Im}\lambda|$, the following inequalities hold:

$$\|\underline{v}\|_{L_2(\Omega)} \leq \frac{C}{|\lambda|}, \quad \|\underline{w}\|_{L_2(\Omega)} \leq \frac{C}{|\lambda|}. \tag{8.17}$$

From (8.17) it follows that the solution $(\underline{v}(\lambda), \underline{w}(\lambda))$ of problem (4.10)–(4.11) is analytic in the infinite domain Λ that contains the half plane $\text{Re}\lambda > 0$ and such that its boundary asymptotically tends to the rays $\text{Re}\lambda = -|\text{Im}\lambda|$.

Let L_δ and $L_{\varepsilon\delta}$ be the contours that intersect the real axis at some point $\sigma > 0$, lie in the domains Λ and $\Lambda_{\varepsilon\delta}$, respectively, and tend asymptotically to the lines $\arg\lambda = \pm\left(\frac{\pi}{2} + \delta\right)$ as $|\lambda| \to \infty$.

For $t > 0$ consider the integrals

$$\underline{v}_\varepsilon(\underline{x}, t) = \frac{1}{2\pi i} \int\limits_{L_{\varepsilon\delta}} \underline{v}_\varepsilon(\underline{x}, \lambda) e^{\lambda t} d\lambda, \tag{8.18}$$

$$\underline{v}(\underline{x}, t) = \frac{1}{2\pi i} \int\limits_{L_\delta} \underline{v}(\underline{x}, \lambda) e^{\lambda t} d\lambda, \quad \underline{w}(\underline{x}, t) = \frac{1}{2\pi i} \int\limits_{L_\delta} \underline{w}(\underline{x}, \lambda) e^{\lambda t} d\lambda. \tag{8.19}$$

Taking into account (8.2) and (8.17), it is easy to show that this improper integrals are convergent (in the L_2 norm with respect to $\underline{x}$). From this, recalling (4.1) – (4.5) and (4.10)–(4.11), we conclude that the functions $\underline{v}_\varepsilon(\underline{x}, t)$ and $\big(\underline{v}(\underline{x}, t), \underline{w}(\underline{x}, t)\big)$ are the solutions of problems (2.2) – (2.5) and (3.12) – (3.15), respectively.

Note that the analyticity of $\underline{v}_\varepsilon(\underline{x}, \lambda)$ and $\underline{v}(\underline{x}, \lambda)$ allows the deformation of contours in the integrals (8.18)–(8.19), so that they can be written as

$$\underline{v}_\varepsilon(\underline{x}, t) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \underline{v}_\varepsilon(\underline{x}, \lambda) e^{\lambda t} d\lambda, \tag{8.20}$$

$$\underline{v}(\underline{x}, t) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \underline{v}(\underline{x}, \lambda) e^{\lambda t} d\lambda. \tag{8.21}$$

It was proved in Section 3 and Section 6 that for any $\lambda > 0$ the solution $\underline{v}_\varepsilon$ of problem (4.1) – (4.5) converges to the solution $\underline{v}$ of problem (4.10)–(4.11) in $L_2(\Omega)$. Thus, taking into account the analyticity of $\underline{v}_\varepsilon(x, \lambda)$ in $\text{Re}\lambda 0$ and the uniform bound (8.4), with the help of the Vitaly theorem we conclude that $\underline{v}_\varepsilon(x, \lambda)$ converges to $\underline{v}(\underline{x}, \lambda)$ in $L_2(\Omega)$, uniformly with respect to λ in any compact subset of the domain $\text{Re}\lambda \geq \sigma > 0$. This, together with (8.4), (8.17), (8.20), (8.21), gives the statement of Theorem 3.2.

9. **A periodic structure.** We now present a case when conditions I, II, III are satisfied and the tensors $\{\underline{C}_{kl}(\underline{x})\}$ and $\{a_{npqr}(\underline{x})\}$ can be calculated explicitly.

Consider a periodic array formed by the balls Q_ε^i of the same radius $r_\varepsilon^i = r\varepsilon^3$ ($r > 0$), whose centers $\underline{x}_\varepsilon^i$ form a cubic lattice. For the sake of simplicity we assume that each vertex of this lattice is connected by a spring to its nearest neighbors NN (the edges of the unit periodicity cube), to its next nearest neighbors NNN (the diagonals of the faces of the cube) and to next-to-next neighbors $NNNN$ (the diagonals of the cube). So, each vertex is connected to $3^3 - 1 = 26$ vertices in the lattice. The elastic constants k^{ij} (see (3.4)) of these springs are k_1, k_2, k_3 respectively, see Figure 1).

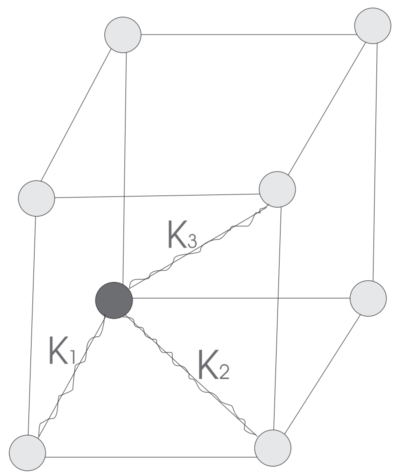

FIGURE 1. The basic periodic cell

In this figure, a fixed ball Q^i_ε with the center at the point $\underline{x}^i_\varepsilon$ is shown as a dark ball and all its neighbors Q^j_ε are shown as lighter balls.

Theorem 9.1. *For the cubic lattice described above (see also Fig. 1) the elastic modules $a_{npqr}(\underline{x})$ in (3.7) are constants and are given by the following formulas:*

$$a_{nnnn} = k_1 + \sqrt{2}k_2 + \frac{4}{9}\sqrt{3}k_3, \quad a_{nnpp} = a_{npnp} = \frac{1}{2}\sqrt{2}k_2 + \frac{4}{9}\sqrt{3}k_3, \; n, p = \overline{1,3},$$

$a_{npqr} = 0$ *in all other cases.*

This theorem can be proved in much the same way as Theorem 4 in [4].

It is clear that in this case, $C(\underline{x}) = 6\pi\mu r$. If we set $k_1 = \frac{k_2}{\sqrt{2}} + \frac{8k_3}{3\sqrt{3}}$, then we obtain the isotropic model of an elastic skeleton (i.e., the equality $a_{nnnn} = 2a_{npnp} + a_{nnpp}$ holds). In this case, equations (3.12) – (3.13) can be written as follows:

$$\rho\frac{\partial v}{\partial t} - \mu\Delta v + 6\pi\mu r(v - \dot{u}) = \nabla p; \quad \mathrm{div} v = 0 \quad x \in \Omega, t > 0,$$

$$6\pi\mu r(\dot{u} - v) - a\Delta u(\underline{x}, t) - (a + b)\nabla\, \mathrm{div} u(\underline{x}, t) = 0, \quad x \in \Omega, t > 0.$$

Here $a = a_{npnp}$, $b = a_{nnpp}$, and $\underline{u}(\underline{x}, t) = \int\limits_0^t \underline{w}(\underline{x}, \tau)\, d\tau$ is the displacement of the elastic skeleton.

Acknowledgements. The authors are grateful to both referees for useful comments and suggestions which led to an improvement of the manuscript. We also thank D. Shepelsky for very helpful discussions and criticism of the manuscript. Part of this work was done when E.Khruslov was visiting Penn State University. His visit was supported by NSF grant DMS-9971999. The work of L.Berlyand was supported in part by NSF grants DMS-9971999, DMS-0204637 and DMS-0708324. The work of M. Berezhnyi is partially supported by the research grant of The President of Ukraine for young scientists and by grant $N20-2007$ of the National Academy of Sciences of Ukraine for young scientists.

REFERENCES

[1] Grégory Allaire, *Homogenization of the Navier-Stokes Equations in Open Sets Perforated with Tiny Holes. I. Abstract Framework, a Volume Distribution of Holes. II. Noncritical sizes of the holes for a volume distribution and surface distribution of holes.*, Arch. Rational Mech. Anal., **113** (1990), 209–259; **113** (1990), 261–298 (1079190).

[2] B. Amaziane , M. Goncharenko and L. Pankratov , Γ_D*-convergence for a class of quasilinear elliptic equations in thin structures*, Math. Meth. Appl. Sci., **28** (2005), 1847–1865.

[3] M. A. Berezhnyi, *The asymptotic behaviour of viscous incompressible fluid small oscillations with solid interacting particles*, Journal of Mathematical Physics, Analysis, Geometry, **3** (2007), 135–156.

[4] M. Berezhnyy and L.Berlyand, *Continuum limit for three-dimensional mass-spring networks and discrete Korn's inequality*, Journal of the Mechanics and Physics of Solids, **54** (2006), 635–669.

[5] L. V. Berlyand and E. Ya. Khruslov, *Homogenized non-Newtonian viscoelastic rheology of a suspension of interacting particles in a viscous Newtonian fluid*, SIAM, Journal of Applied Mathematics, **64** (2004), 1002–1034.

[6] L. Berlyand and E. Ya Khruslov, *The asymptotic behavior of viscous incompressible fluid small oscillations with solid interacting particles*, "The Pennsylvania State University," USA, N 2003-16, 2003.

[7] R. Byron Bird and C. F. Curtiss, *Surprising Polymeric Liquids*, Physics Today, **37** (1984), 36–55.

[8] A. Brillard, *Asymptotic analysis of incompressible and viscous fluid flow through porous media. Brinkman's law via epi-convergence methods*, Ann. Fac. Sci. Toulouse, **8** (1986/87), 225–252.

[9] H. C. Brinkman, *A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles*, Appl. Sci. Res., **A1** (1947), 27–34.

[10] D. Cioranescu and F. Murat, *Un terme etrange venu d'ailleurs*, in "Nonlinear Partial Differential Equations and their Applications, College de France Seminar," **II** (Paris, 1979/1980), 98–138, 389–390, Res. Notes in Math., 60, Pitman, Boston, Mass.-London, 1982.

[11] G. Dal Maso, "An Introduction to Γ-convergence," Birkhäuser, Boston, 1993.

[12] D. J. Evans, H. J. M. Hanley and S. Hess, *Non-Newtonian phenomena in simple fluids*, Physics Today, **37** (1984), 26–35.

[13] R. P. Gilbert and A. Mikelic, *Homogenizing the acoustic properties of the seabed*, I. Lakshmikantham's legacy: a tribute on his 75th birthday, Nonlinear Analysis, Ser. A: Theory Methods, **40** (2000), 185–212.

[14] E. De Giorgi and S. Spagnolo, *Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine*, (Italian), Boll. Un. Mat. Ital, **8** (1973), 391–411.

[15] U. Hornung, "Homogenization and Porous Media," Interdisciplinary Applied Mathematics, 6, Springer-Verlag, New York, 1997.

[16] A. A. Kovalevskij, *Conditions of the* Γ*-convergence and homogenization of integral functionals with different domains of the definition*, Doklady Akademii Nauk Ukrainskoj SSR, **4** (1991), 5–8 (in Russian).

[17] Ronald G. Larson, "The Structure and the Rheology of Complex Fluids," New York-Oxford, Oxford University Press, 1999.

[18] A. I. Leonov, *On rheology of filled polymers*, J. Rheol, **34** (1990), 1039–1068.

[19] T. Levy, *Fluid flow through an array of fixed particles*, Int. J. Engin. Sci., **21** (1983), 11–23.

[20] L. G. Loitsyanskiy, "Mechanics of Liquids and Gases," Begell House, New York, 1996.

[21] T. C. Lubensky and P. A. Pinkus, *Superpolymers, ultraweak solid bodies and aggregates*, Physics Today, **37** (1984), 44–59.

[22] V. A. Marchenko and E. Ya. Khruslov, *Boundary value problems with fine-grained boundary*, Mat. Sb. (N.S.), **65** (1964), 458–472 (in Russian).

[23] V. A. Marchenko and E. Ya. Khruslov, "Boundary Problems in Domains with Finely Granulated Boundaries" (in Russian), Naukova Dumka, Kiev, 1974.

[24] V. Marchenko and E. Khruslov, "Homogenization of Partial Differential Equations," Translated from the 2005 Russian original by M. Goncharenko and D. Shepelsky, Progress in Mathematical Physics, 46, Birkhäuser, Boston, Inc., Boston, MA, 2006.

[25] A. I. Marcushevich, "Theory of Analytic Functions," Nauka, Moscow, 1968. (in Russian)

[26] A. S. Markus, "Introduction to the Spectral Theory of Polynomial Operator Pencils," Translated from the Russian by H. H. McFaden. Translation edited by Ben Silver, With an appendix by M. V. Keldysh, Translations of Mathematical Monographs, 71, American Mathematical Society, Providence, RI, USA, 1988.

[27] S. G. Mikhlin, "The Problem of the Minimum of a Quadratic Functional," Translated by A. Feinstein, Holden-Day Series in Mathematical Physics Holden-Day, Inc., San Francisco, Calif.-London-Amsterdam 1965.

[28] O. A. Oleinic, A. S. Shamaev and G. A. Iosif'yan, "Mathematical Problems in Elasticity and Homogenization," in "Studies in Mathematics and its Applications", vol. **26**, North-Holland Publishing Co., Amsterdam (1982).

[29] J. N. Pernin and E. Jacquet, *Elasticity and viscoelasticity in highly heterogeneous composite medium: threshold phenomenon and homogenization*, Intern. J. of Eng. Sci., **39** (2001), 1655–1689.

[30] W. B. Russel, D. A. Saville and W. R. Schowalter, "Colloidal Dispersions," Cambridge Iniversity Press, Cambridge, 1989.

[31] J. Sanchez-Hubert, *Asymptotic study of the macroscopic behaviour of solid-liquid mixture*, Math. Methods Appl. Sci., **2** (1980), 1–11.

[32] E. Sanchez-Palencia, "Non-homogeneous Media and Vibration Theory," Lecture Notes in Physics, 127, Springer-Verlag, Berlin-New York, 1980.

[33] E. Sanchez-Palencia, *On the asymptotic of the fluid flow past an array of fixed obstacles*, Int. J. Engin. Sci., **20** (1982), 1291–1301.

[34] S. Spagnolo, *Sulla convergenza di soluzioni di equazioni paraboliche ed ellitiche*, (Italian), Ann. Sc. Norm. Sup. Pisa. Cl. Sci, **22** (1968), 571–597.

[35] M. J. Wang, *Effect of polymer-filler and filler-filler interactions on dynamic properties of filled vulcanizates*, Rubb. Chem. Technol., **71** (1998), 520–589.

Received June 2007; revised February 2008.

E-mail address: `berezhny@ilt.kharkov.ua`
E-mail address: `berlyand@math.psu.edu`
E-mail address: `khruslov@ilt.kharkov.ua`