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Continuum limit for three-dimensional mass-spring networks and discrete Korn's inequality

M. Berezhnyy[a], L. Berlyand[b,*]

[a]*Institute of Low Temperature and Engineering, Ukrainian Academy of Science, Lenin Ave 47, Kharkiv 61164, Ukraine*
[b]*Department of Mathematics and Materials Research Institute, Penn State University, University Park, PA-16802-6401, 218 Mc Allister Building, USA*

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Abstract

In a bounded domain $\Omega \subset \mathbb{R}^3$ we consider a discrete network of a large number of concentrated masses (particles) connected by elastic springs. We provide sufficient conditions on the geometry of the array of particles, under which the network admits a rigorous continuum limit. Our proof is based on the discrete Korn's inequality. Proof of this inequality is the key point of our consideration. In particular, we derive an explicit upper bound on the Korn's constant. For generic non-periodic arrays of particles we describe the continuum limit in terms of the local energy characteristic on the mesoscale (intermediate scale between the interparticle distances (small scale) and the domain sizes (large scale)), which represents local energy in the neighborhood of a point. For a periodic array of particles we compute coefficients in the limiting continuum problems in terms of the elastic constants of the springs.

Keywords: Homogenization; Elasticity; Mesoscale; Triangulized network; Discrete Korn's inequality

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1. Formulation of the problem and main result

Consider a system of N point particles (N is a large number) located in a smooth bounded domain $\Omega \in \mathbb{R}^3$. Introduce a small parameter $\varepsilon = \frac{1}{\sqrt[3]{N}} \to 0$ and denote by $\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}$ the locations of

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[*] Corresponding author. Tel.: +1 814 863 9683; fax: +1 814 865 3735.
E-mail addresses: berezhny@ilt.kharkov.ua (M. Berezhnyy), berlyand@math.psu.edu (L. Berlyand).

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particles (we will omit the superscript ε where it will not cause any confusion). We say that the particles $\mathbf{x}_\mathbf{i}^{(\varepsilon)}$ and $\mathbf{x}_\mathbf{j}^{(\varepsilon)}$ are neighboring particles if the distance between them is of order ε. More precisely, there exist constants $c_1 \geqslant c_2 > 0$ which do not depend on ε such that

$$c_2\varepsilon \leqslant |\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\mathbf{j}^{(\varepsilon)}| \leqslant c_1\varepsilon. \tag{1.1}$$

We now introduce the mass-spring network. Assume that each particle has a finite mass $m_i^{(\varepsilon)} = M_i \cdot \varepsilon^3$ $(0 < m_1 \leqslant M_i \leqslant m_2 < \infty)$, where m_1 and m_2 do not depend on ε. Particles are assumed to be of infinitely small volume; that is, we consider a system of point masses. Some neighboring particles (distances between neighbors are of order ε) are connected by elastic springs, so that the system of particles and springs form a three-dimensional graph (network) whose edges correspond to the springs and the vertices correspond to particles. The only condition on these edges is that the triangulization condition in the sense of Definition 1 (see below) is satisfied. The properties of this graph which are essential in our consideration will be specified later (e.g., the degree of a vertex, which is the number of edges adjacent to a given vertex). Throughout this paper we assume that each particle has a finite number of neighbors which does not depend on ε.

We define the interparticle interaction as follows: denote the displacement of the ith particle by $\mathbf{u}_\mathbf{i}^{(\varepsilon)}(t), i \in \overline{1, N}$. The displacements of the particles which are connected by elastic springs are small in the following sense:

$$|\mathbf{u}_\mathbf{i}^{(\varepsilon)} - \mathbf{u}_\mathbf{j}^{(\varepsilon)}| \leqslant c\varepsilon. \tag{1.2}$$

This assumption is crucial for our consideration, and the applicability of the resulting homogenized formulas depends on whether this assumption is actually satisfied. Violation of (1.2) may lead to instabilities (Friesecke and Theil, 2002) where a lattice model with quadratic springs contains a geometric non-linearity.

Assume that the springs provide a linear elastic interaction. Then the elastic energy of two neighboring particles is given by

$$\langle C_\varepsilon^{ij}(\mathbf{u}_\mathbf{i}^{(\varepsilon)} - \mathbf{u}_\mathbf{j}^{(\varepsilon)}), \mathbf{u}_\mathbf{i}^{(\varepsilon)} - \mathbf{u}_\mathbf{j}^{(\varepsilon)}\rangle,$$

where $\langle\cdot,\cdot\rangle$ stands for a dot product in R_3 and the matrix of elastic constants C_ε^{ij} for ith and jth particles ($C_\varepsilon^{ij} \equiv 0$ if these particles do not interact) is determined by a scalar spring constant k_ε^{ij}:

$$C_\varepsilon^{ij}(\mathbf{u}_\mathbf{i}^{(\varepsilon)} - \mathbf{u}_\mathbf{j}^{(\varepsilon)}) = k_\varepsilon^{ij}\left\langle \frac{\mathbf{u}_\mathbf{i}^{(\varepsilon)} - \mathbf{u}_\mathbf{j}^{(\varepsilon)}}{|\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\mathbf{j}^{(\varepsilon)}|}, \mathbf{e}_{ij}^{(\varepsilon)}\right\rangle \mathbf{e}_{ij}^{(\varepsilon)}, \quad \mathbf{e}_{ij}^{(\varepsilon)} = \frac{\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\mathbf{j}^{(\varepsilon)}}{|\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\mathbf{j}^{(\varepsilon)}|}, \tag{1.3}$$

which means that we consider the central interaction between neighboring particles. Then the potential energy of the network is given by

$$H(\mathbf{u}_\mathbf{1}^{(\varepsilon)}, \ldots, \mathbf{u}_\mathbf{N}^{(\varepsilon)}) = H_0 + \frac{1}{2}\sum_{i,j}\langle C_\varepsilon^{ij}(\mathbf{u}_\mathbf{i}^{(\varepsilon)} - \mathbf{u}_\mathbf{j}^{(\varepsilon)}), \mathbf{u}_\mathbf{i}^{(\varepsilon)} - \mathbf{u}_\mathbf{j}^{(\varepsilon)}\rangle, \tag{1.4}$$

where the summation is taken over all pairs of interacting particles, and H_0 is an arbitrary constant. We consider the elastic constants of the form

$$k_\varepsilon^{ij} = k^{ij}\varepsilon^2, \quad k_1 \leqslant k^{ij} \leqslant k_2, \tag{1.5}$$

where the constants $k_2 \geqslant k_1 > 0$ do not depend on ε. Since the displacements $\mathbf{u}_\mathbf{i}^\varepsilon$ are of order ε and the number of terms in the sum (1.4) is of order ε^{-3}, the scaling factor ε^2 in (1.5) implies that $H(\mathbf{u}_\mathbf{1}^{(\varepsilon)},\ldots,\mathbf{u}_\mathbf{N}^{(\varepsilon)})$ in (1.4) is finite. Note that the condition $k_1>0$ is crucial in our consideration. If $k_1=0$ or $k_1\to 0$ as $\varepsilon\to 0$, then an instability (due to the blow up of the Korn's constant in (3.5)) may appear; for an example, see (Friesecke and Theil, 2002).

Since the energy (1.4) is invariant under translations of the set of the particles as a whole, the energy minimization defines many equilibrium states. The unique minimum is defined by the condition that all particles that are at a distance smaller than $c_1\varepsilon$ (see (1.1)) from the clamped boundary $\partial\Omega$ (the corresponding displacements $\mathbf{u}_\mathbf{i}^{(\varepsilon)}$ are equal to zero). Thus, in our consideration the system of particles has the unique equilibrium state $\{\mathbf{x}_\mathbf{i}^{(\varepsilon)}\}_{i=1}^N$. Denote by M the number of particles which are located in the boundary layer $U(\partial\Omega, c_1\varepsilon)$, $M=O(\varepsilon^{-2})\ll N=\varepsilon^{-3}$. In what follows we will call such particles *boundary particles.*

Then the motion of the network is described by a system of $3N$ ODEs:

$$m_i^{(\varepsilon)}\ddot{\mathbf{u}}_i^{(\varepsilon)} = -\nabla_{\mathbf{u}_\mathbf{i}^{(\varepsilon)}} H(\mathbf{u}_\mathbf{1}^{(\varepsilon)},\ldots,\mathbf{u}_\mathbf{N}^{(\varepsilon)}), \quad i\in\overline{1,N}, \tag{1.6}$$

subject to the following initial and boundary conditions, respectively:

$$\mathbf{u}_\mathbf{i}^{(\varepsilon)}(0)=\mathbf{0}, \quad \dot{\mathbf{u}}_i^{(\varepsilon)}(0)=\mathbf{a}_\mathbf{i}^{(\varepsilon)}, \quad i\in\overline{1,N}, \tag{1.7}$$

$$\mathbf{u}_\mathbf{i}^{(\varepsilon)}(t)\equiv 0, \quad t>0 \quad (i\in\overline{1,M}), \tag{1.8}$$

where $\mathbf{a}_\mathbf{i}^{(\varepsilon)}$ $(i\in\overline{1,N})$ is a set of given constants such that the kinetic energy

$$\sum_{i=1}^N m_i^{(\varepsilon)}|\mathbf{a}_\mathbf{i}^{(\varepsilon)}|^2\leqslant C \tag{1.9}$$

is bounded uniformly in ε.

Condition (1.7) means that the initial displacements of particles are zero and the system is driven by the initial velocities. Condition (1.8) means that the boundary particles are clamped. Here homogeneous Dirichlet boundary conditions are assumed for technical simplicity only. Generalization for non-homogeneous boundary conditions, as well as for other type of boundary conditions (Neumann, Robin), is straightforward as is usually the case in linear homogenization problems when the homogenization limit (the PDE or the constitutive equation) does not depend on the type of external boundary conditions.

The goal of our study is to derive the continuum limit for the problem (1.6)–(1.8) as $\varepsilon\to 0$. The understanding of the relationship between the atomistic (discrete) and the continuum moduli of solids dates back to the work of Cauchy–Born (Born and Huang, 1954), where linear elasticity equations were derived based on physical considerations for periodic mass-spring lattices. However, the compactness of the family of solutions of discrete problems was not established there. Such compactness is necessary for a rigorous justification of the continuum limit, which, in particular, provides the limits of validity for the formulas for macroscopic elastic moduli obtained previously by the Cauchy–Born rule.

More recently, the applicability of the Cauchy–Born rule under various assumptions was investigated by a number of authors. In the work of Blanc et al. (2002) the continuum limit was derived based on the hypothesis that the microscopic displacements are equal to the macroscopic ones (see also the work of E. and Ming, 2004). The validity and failure of the Cauchy–Born rule for a two-dimensional mass-spring lattice was studied by Friesecke and Theil (2002). The examples of failure of this rule underline the necessity of a mathematical justification of its limits of validity.

Several one-dimensional non-linear problems were studied by Braides et al. (1999), as well as Truskinovsky (1996). Friesecke and James (2000) studied the passage from atomistic to continuum for thin films.

Our work addresses the following novel aspects. First, we rigorously establish the compactness of the solutions of the discrete problems and therefore justify the continuum limit for a *non-periodic* three-dimensional spacial array of point masses. Second, our proof is based on the discrete Korn's inequality (3.5), where $\mathbf{u}_\varepsilon(x)$ is a vector-function defined in the domain Ω as a linear interpolation of the Laplace transform in t of the discrete vector-function $\mathbf{u}_\mathbf{i}^\varepsilon(t), t>0$ which solves (1.6)–(1.8). We prove this inequality for generic arrays of particles (satisfying the so-called non-periodic triangulization condition introduced in Section 2) and obtain an upper bound on the Korn's constant in terms of the geometric characteristics of the network c_1, c_2 and the physical parameter k_1 which are defined in (1.1)–(1.2) and (1.5) respectively. In particular, this bound allows us to see when this constant goes to infinity, which may lead to instabilities analogous to those discussed in the work of Friesecke and Theil (2002).

We derive and justify the following continuum limit:

$$\begin{cases} \rho(x)\dfrac{\partial^2\mathbf{u}(x,t)}{\partial t^2} - \displaystyle\sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_q} a_{npqr}(x)\varepsilon_{np}[\mathbf{u}(x,t)]\mathbf{e}_\mathbf{r} = \mathbf{0}, \quad x\in\Omega, \quad t>0, \\ \mathbf{u}(x,t) = \mathbf{0}, \quad x\in\partial\Omega, \quad t\geqslant 0, \\ \mathbf{u}(x,0) = \mathbf{0}, \dfrac{\partial\mathbf{u}(x,t)}{\partial t}\Big|_{t=0} = \mathbf{a}(x), \quad x\in\Omega, \end{cases}$$

where $\mathbf{e}_r(r=\overline{1,3})$ form an orthonormal basis in $\mathbb{R}^3$ and $\varepsilon_{np}[\mathbf{u}] = 1/2(\partial u_n/\partial x_p + \partial u_p/\partial x_n)$ is the strain tensor. The functions $a_{npqr}(x), \rho(x)$ and the vector-function $\mathbf{a}(x)$ are determined in Sections 2 and 4 (formulas ((2.7), (4.1) and (4.2)).

For generic non-periodic arrays of particles the elastic coefficients $a_{npqr}(x)$ are determined via mesocharacteristics introduced in Section 2. For periodic lattices, coefficients $a_{npqr}(x)$ are constants and are calculated explicitly in terms of the spring constant for a cubic lattice (Section 6).

Finally, we mention here the work of Vogelius (1991), where the continuum limit was rigorously derived for planar electrical non-periodic networks. The compactness of solutions in this problem does not required Korn's inequality since this problem is scalar. These results were subsequently extended by Krasniansky (1997) for more general scalar problems.

2. Mesocharacteristic

We now introduce a mesocharacteristic which describes the elastic interaction on a mesoscale h, where $\varepsilon \ll h \ll diam\,\Omega$. Here we assume h does not depend on ε, and the consecutive limit is zero:

$$\lim_{h\to 0}\lim_{\varepsilon\to 0}\frac{\varepsilon}{h} = 0.$$

These assumptions reflect the physics of the problem. Indeed, while the mesoscale (the size of a representative piece of an inhomogeneous medium) does not depend on the microscale (size of an inhomogeneity, in our case the length of the springs), the microscale is an order of magnitude smaller than the mesoscale.

To this end, we assume that the array of particles is such that the points $\mathbf{x}_\mathbf{i}^{(\varepsilon)}$ partition the domain Ω into simplices (pyramids) $P_{\alpha\varepsilon}$ whose edges are the springs between interacting particles. The angles of the simplices are uniformly bounded from below by a strictly positive constant which does not depend on ε. The edges of these simplices are not necessarily equal and the simplices are not necessarily identical. The only condition on $\mathbf{x}_\mathbf{i}^{(\varepsilon)}$ is (1.1).

Definition 1. The network which satisfies this condition is called a *triangulized network*.

We will show that this partition's condition (non-periodic triangulization) is sufficient for the discrete Korn's inequality (3.5) to hold.

We introduce the following notations:

$$\varphi^{\mathbf{np}}(\mathbf{x}) = \tfrac{1}{2}(\mathbf{e}^n x_p + \mathbf{e}^p x_n), \quad \psi^{\mathbf{np}}(\mathbf{x}) = \tfrac{1}{2}(\mathbf{e}^n x_p - \mathbf{e}^p x_n). \tag{2.1}$$

Consider a point $\mathbf{y} \in \Omega$ and introduce a cube $K_h^\mathbf{y}$ of side length h centered at $\mathbf{y}$, where h is the mesoscale defined above. Introduce the following functional (called the *mesocharacteristic*):

$$E_{K_h^\mathbf{y}}^\varepsilon(\mathbf{v}) = \frac{1}{2}\sum_{i,j\,K_h^\mathbf{y}} \langle C_\varepsilon^{ij}(\mathbf{v_i} - \mathbf{v_j}), \mathbf{v_i} - \mathbf{v_j}\rangle + h^{-2-\gamma}\varepsilon^3 \sum_{i\,K_h^\mathbf{y}} \left| \mathbf{v_i} - \sum_{j,k=1}^{3} T_{jk}\varphi^{jk}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{y}) \right|^2, \tag{2.2}$$

where $\sum_{i,jK_h^\mathbf{y}}$ means the summation over all pairs of interacting particles which are located inside the cube $K_h^\mathbf{y}$ (the total number of particles in $K_h^\mathbf{y}$ is denoted by p). Denote by $\mathbf{v}$ the collection of displacement vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ so that $\mathbf{v} = (\mathbf{v_1}, \ldots, \mathbf{v_p})$ and let $\{T_{jk}\}_{j,k=1}^3$ be an arbitrary second rank tensor with constant components such that $T_{jk} = T_{kj}$. The first sum in (2.2) represents the elastic energy in $K_h^\mathbf{y}$. The second sum is a penalty term which represents the deviation of the vectors $\mathbf{v_i}$ from the linear part $\sum_{j,k=1}^3 T_{jk}\varphi^{jk}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{y})$. The third sum in (2.2) can be viewed as a linear part (differential) of a homogenized vector-function $\mathbf{u}(x)$ (see (3.13)), and $0<\gamma<2$ is a technical parameter. We seek the minimum of this functional among all vectors $\mathbf{v_i}$ which correspond to particles $\mathbf{x}_\mathbf{i}^{(\varepsilon)} \in K_h^\mathbf{y}$, $i = 1, \ldots, p$. The minimizing displacement vectors are denoted by $\{\mathbf{w_i}\}_{i=1}^p$; they exist since C_ε^{ij} is a positive definite operator due to (1.3):

$$\min_\mathbf{v} E_{K_h^\mathbf{y}}^\varepsilon(\mathbf{v}) = E_{K_h^\mathbf{y}}^\varepsilon(\mathbf{w}), \quad \mathbf{w} = (\mathbf{w_1}, \ldots, \mathbf{w_p}).$$

Next, we consider a specific set of tensors defined via the Kronecker delta notation:

$$\underline{\underline{T^{(np)}}} = \tfrac{1}{2}(\mathbf{e}^n \otimes \mathbf{e}^p + \mathbf{e}^p \otimes \mathbf{e}^n) = \{T_{jk}^{(np)} = \tfrac{1}{2}(\delta_{jn}\delta_{kp} + \delta_{jp}\delta_{kn})\}_{j,k\in 1}^3, \quad n,p = \overline{1,3}.$$

If the vectors $\{\mathbf{w}_\mathbf{i}^\mathbf{np}\}_{i=1}^p$ minimize the functional $E_{K_h^\mathbf{y}}^{\varepsilon(np)}(\mathbf{v})$ in which $T_{jk} = T_{jk}^{(np)}$, then the sets of vectors $\{\mathbf{w}_\mathbf{i}^\mathbf{np}\}$ and $\{\mathbf{w_i}\}$ satisfy the following algebraic systems, respectively (the Euler–Lagrange equations):

$$\sum_{j\,K_h^\mathbf{y}}^{i} C_\varepsilon^{ij}(\mathbf{w}_\mathbf{i}^\mathbf{np} - \mathbf{w}_\mathbf{j}^\mathbf{np}) + h^{-2-\gamma}\varepsilon^3\mathbf{w}_\mathbf{i}^\mathbf{np} = h^{-2-\gamma}\varepsilon^3\varphi^\mathbf{np}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{y}), \tag{2.3}$$

$$\sum\nolimits_{j\ K_h^{\mathbf{y}}}^{i} C_\varepsilon^{ij}(\mathbf{w_i}-\mathbf{w_j}) + h^{-2-\gamma}\varepsilon^3\mathbf{w_i} = h^{-2-\gamma}\varepsilon^3 \sum_{j,k=1}^{3} T_{jk}\varphi^{\mathbf{jk}}(\mathbf{x_i}^{(\varepsilon)}-\mathbf{y}) \tag{2.4}$$

for all particles $\mathbf{x_i}^{(\varepsilon)} \in K_h^{\mathbf{y}}$, where $\sum\nolimits_{j K_h^{\mathbf{y}}}^{i}$ stands for the summation over all particles which are located in $K_h^{\mathbf{y}}$ and interact (that is, are connected by the springs) with a given particle $\mathbf{x_i}^{(\varepsilon)}$.

It follows from (2.3)–(2.4) that $\mathbf{w_i} = \sum_{n,p=1}^{3}\mathbf{w_i^{np}}T_{np}$. Substitute this formula into (2.2). Then,

$$\min_{\mathbf{v}} E^{\varepsilon}_{K_h^{y}}(\mathbf{v}) = E^{\varepsilon}_{K_h^{y}}(\mathbf{w}) = \sum_{n,p,q,r=1}^{3} a_{npqr}(\mathbf{y},\varepsilon,h,\gamma)T_{np}T_{qr}, \tag{2.5}$$

where

$$\begin{aligned} a_{npqr}(\mathbf{y},\varepsilon,h,\gamma) = {} & \frac{1}{2}\sum_{i,j\ K_h^{\mathbf{y}}} \langle C_\varepsilon^{ij}(\mathbf{w_i^{np}}-\mathbf{w_j^{np}}), \mathbf{w_i^{qr}}-\mathbf{w_j^{qr}}\rangle \\ & + h^{-2-\gamma}\varepsilon^3 \sum_{i\ K_h^{\mathbf{y}}} \langle \mathbf{w_i^{np}} - \varphi^{\mathbf{np}}(\mathbf{x_i}^{(\varepsilon)}-\mathbf{y}), \mathbf{w_i^{qr}} - \varphi^{\mathbf{qr}}(\mathbf{x_i}^{(\varepsilon)}-\mathbf{y})\rangle. \end{aligned} \tag{2.6}$$

These coefficients play an important role in our consideration since they define the effective elastic moduli in the continuum limit. Namely, we define a fourth rank tensor $a_{npqr}(\mathbf{y}), \mathbf{y} \in \Omega$, as follows:

$$a_{npqr}(\mathbf{y}) = \lim_{h\to 0}\ \overline{\lim_{\varepsilon\to 0}} \frac{a_{npqr}(\mathbf{y},\varepsilon,h,\gamma)}{h^3} = \lim_{h\to 0}\ \underline{\lim}_{\varepsilon\to 0} \frac{a_{npqr}(\mathbf{y},\varepsilon,h,\gamma)}{h^3}. \tag{2.7}$$

We consider spatial arrays of particles such that the limits (2.7) exist for any $\gamma \in (0,2)$. In Section 6, we provide an example where the limits (2.7) are calculated explicitly.

3. Compactness of the family of solutions and Korn's inequality

We begin by converting the evolutionary problem (1.6)–(1.8) into a stationary one by using the Laplace transform in t. We denote the Laplace transform of $\mathbf{u_i}^{(\varepsilon)}(t)$ by $\mathbf{u_i}^{(\varepsilon)}(\lambda)$, and by a slight abuse of notation simply write $\mathbf{u_i}^{(\varepsilon)}$ instead of $\mathbf{u_i}^{(\varepsilon)}(\lambda)$:

$$\mathbf{u_i}^{(\varepsilon)}(\lambda) = (L\mathbf{u_i}^{(\varepsilon)})(\lambda) = \int_0^\infty \mathbf{u_i}^{(\varepsilon)}(t)e^{-\lambda t}\,dt.$$

Applying the Laplace transform to (1.6)–(1.8) we obtain

$$m_i^{(\varepsilon)}\lambda^2\mathbf{u_i}^{(\varepsilon)} + \nabla_{\mathbf{u_i}^{(\varepsilon)}} H(\mathbf{u_1}^{(\varepsilon)},\ldots,\mathbf{u_N}^{(\varepsilon)}) - m_i^{(\varepsilon)}\mathbf{a_i}^{(\varepsilon)} = 0, \quad i \in \overline{1,N}, \tag{3.1}$$

$$\mathbf{u_i}^{(\varepsilon)}(\lambda) \equiv 0, \quad i \in \overline{1,M}.$$

Fix $\lambda>0$. Taking into account (1.4), we observe that the solutions of (3.1) minimize the following quadratic functional:

$$\Phi_\varepsilon(\mathbf{u}^\varepsilon)=\frac{1}{2}\sum_{i,j}\langle C_\varepsilon^{ij}(\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}),\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}\rangle+\lambda^2\sum_{i=1}^N m_i^{(\varepsilon)}|\mathbf{u}_\mathbf{i}^{(\varepsilon)}|^2-2\sum_{i=1}^N m_i^{(\varepsilon)}\langle\mathbf{a}_\mathbf{i}^{(\varepsilon)},\mathbf{u}_\mathbf{i}^{(\varepsilon)}\rangle, \tag{3.2}$$

where $\mathbf{u}^\varepsilon=(\mathbf{u}_\mathbf{1}^{(\varepsilon)},\dots,\mathbf{u}_\mathbf{N}^{(\varepsilon)})$. In order to establish the compactness of $\mathbf{u}_\varepsilon(x)$, we need a uniform in ε bound on the interaction energy:

$$\sum_{i,j}\langle C_\varepsilon^{ij}(\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}),\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}\rangle\leqslant C,$$

where C does not depend on ε. It follows from (3.2) that $\Phi_\varepsilon(\mathbf{u}^\varepsilon)\leqslant\Phi_\varepsilon(\mathbf{0})=0$. Next, using the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\sum_{i,j}\langle C_\varepsilon^{ij}(\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}),\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}\rangle+\lambda^2\sum_{i=1}^N m_i^{(\varepsilon)}|\mathbf{u}_\mathbf{i}^{(\varepsilon)}|^2\leqslant 2\sum_{i=1}^N m_i^{(\varepsilon)}|\langle\mathbf{a}_\mathbf{i}^{(\varepsilon)},\mathbf{u}_\mathbf{i}^{(\varepsilon)}\rangle|\\
&\leqslant 2\left(\sum_{i=1}^N m_i^{(\varepsilon)}|\mathbf{a}_\mathbf{i}^{(\varepsilon)}|^2\right)^{1/2}\left(\sum_{i=1}^N m_i^{(\varepsilon)}|\mathbf{u}_\mathbf{i}^{(\varepsilon)}|^2\right)^{1/2}\leqslant 2\left(\sum_{i=1}^N m_i^{(\varepsilon)}|\mathbf{a}_\mathbf{i}^{(\varepsilon)}|^2\right)^{1/2}\left(\sum_{i=1}^N m_i^{(\varepsilon)}|\mathbf{u}_\mathbf{i}^{(\varepsilon)}|^2\right.\\
&\left.+\frac{1}{2\lambda^2}\sum_{i,j}\langle C_\varepsilon^{ij}(\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}),\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}\rangle\right)^{1/2}=\frac{2}{\lambda}\left(\sum_{i=1}^N m_i^{(\varepsilon)}|\mathbf{a}_\mathbf{i}^{(\varepsilon)}|^2\right)^{1/2}\\
&\cdot\left(\frac{1}{2}\sum_{i,j}\langle C_\varepsilon^{ij}(\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}),\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}\rangle+\lambda^2\sum_{i=1}^N m_i^{(\varepsilon)}|\mathbf{u}_\mathbf{i}^{(\varepsilon)}|^2\right)^{1/2}.\end{aligned}$$

Hence (using (1.9)),

$$\frac{1}{2}\sum_{i,j}\langle C_\varepsilon^{ij}(\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}),\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}\rangle+\lambda^2\sum_{i=1}^N m_i^{(\varepsilon)}|\mathbf{u}_\mathbf{i}^{(\varepsilon)}|^2\leqslant\frac{4}{\lambda^2}\sum_{i=1}^N m_i^{(\varepsilon)}|\mathbf{a}_\mathbf{i}^{(\varepsilon)}|^2\leqslant\frac{4}{\lambda^2}C=const. \tag{3.3}$$

From this, we conclude that the sum $\sum_{i,j}\langle C_\varepsilon^{ij}(\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}),\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}\rangle$ is bounded uniformly in ε.

We now construct the linear spline $\mathbf{u}_\varepsilon(x)$ which corresponds to the discrete vector-function $\mathbf{u}_\mathbf{i}^{(\varepsilon)}$:

$$\mathbf{u}_\varepsilon(x)=\sum_{i=1}^N\mathbf{u}_\mathbf{i}^{(\varepsilon)}L_\varepsilon^i(x), \tag{3.4}$$

where $L_\varepsilon^i(x), i\in\overline{1,N}$, are constructed as follows. Fix a site $\mathbf{x}_\mathbf{i}^{(\varepsilon)}$ and define a linear function $L_\varepsilon^i(x)$ such that $L_\varepsilon^i(\mathbf{x}_\mathbf{i}^{(\varepsilon)})=1$. Consider all sites $\mathbf{x}_\mathbf{j}^{(\varepsilon)}$ that are connected to $\mathbf{x}_\mathbf{i}^{(\varepsilon)}$ by an edge of our triangulized network and set $L_\varepsilon^i(\mathbf{x}_\mathbf{j}^{(\varepsilon)})=0$ for all such neighboring sites. Then extend the function $L_\varepsilon^i(x)$ by linearity into all simplices adjacent to the site $\mathbf{x}_\mathbf{i}^{(\varepsilon)}$. Finally, set $L_\varepsilon^i(x)\equiv 0$ outside these simplices. Clearly, $L_\varepsilon^i(\mathbf{x}_\mathbf{k}^{(\varepsilon)})=\delta_{ik}$ and $L_\varepsilon^i(x)$ is linear. Next, define $\mathbf{u}_\varepsilon(x)$ by (3.4). Then,

$$C_1\cdot\|\mathbf{u}_\varepsilon(x)\|^2_{H^1(\Omega)}\leqslant\varepsilon{\sum_{i,j}}'|\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}|^2+\varepsilon^3\sum_i|\mathbf{u}_\mathbf{i}^{(\varepsilon)}|^2\leqslant C_2\cdot\|\mathbf{u}_\varepsilon(x)\|^2_{H^1(\Omega)},$$

where the summation $\sum'_{i,j}$ is taken over all pairs of sites connected by the edges of our network, and the constants C_1 and C_2 do not depend on ε.

We now prove the following discrete Korn's inequality. We formulate it here for the Laplace transform $\mathbf{u}_{\mathbf{i}}^{(\varepsilon)}(\lambda)$ defined above; however, the same proof works for any vector-function defined on the set $\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}$ and satisfying the boundary condition (1.8).

Theorem 1. *Suppose that the network satisfies the triangulization condition from Definition* 1, *and let* $\mathbf{u}_{\mathbf{i}}^{(\varepsilon)}$ *be Laplace transforms of solutions of* (1.6)–(1.8). *Then the following inequality holds*:

$$\|\mathbf{u}_\varepsilon(x)\|^2_{H^1(\Omega)} \leqslant C \cdot \sum_{i,j} \langle C_\varepsilon^{ij}(\mathbf{u}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{u}_{\mathbf{j}}^{(\varepsilon)}), \mathbf{u}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{u}_{\mathbf{j}}^{(\varepsilon)} \rangle, \tag{3.5}$$

where the constant $C = 4c_1^3/3k_1c_2$ *does not depend on* ε (*see* (1.1) *and* (1.5)).

Proof. Consider a simplex $P_{\alpha\varepsilon}$ of our triangulization and denote by $\mathbf{x}_\varepsilon^\alpha$ its center of mass. Since the spline $\mathbf{u}_\varepsilon(x)$ is a linear vector-function in the interior of this simplex, we can write

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{u}_\varepsilon(\mathbf{x}_\varepsilon^\alpha) + (\nabla \mathbf{u}_\varepsilon(\mathbf{x}_\varepsilon^\alpha), \mathbf{x} - \mathbf{x}_\varepsilon^\alpha), \quad \mathbf{x} \in P_{\alpha\varepsilon}.$$

Next we separate the symmetric and antisymmetric parts in the linear term:

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{u}_\varepsilon(\mathbf{x}_\varepsilon^\alpha) + \sum_{n,p=1}^{3} \{\varepsilon_{np}[\mathbf{u}_\varepsilon(\mathbf{x}_\varepsilon^\alpha)]\varphi^{\mathbf{np}}(\mathbf{x} - \mathbf{x}_\varepsilon^\alpha) + w_{np}[\mathbf{u}_\varepsilon(\mathbf{x}_\varepsilon^\alpha)]\psi^{\mathbf{np}}(\mathbf{x} - \mathbf{x}_\varepsilon^\alpha)\}, \quad \mathbf{x} \in P_{\alpha\varepsilon}, \tag{3.6}$$

where $\varepsilon_{np}[\mathbf{u}] = 1/2(\partial u_n/\partial x_p + \partial u_p/\partial x_n), w_{np}[\mathbf{u}] = 1/2(\partial u_n/\partial x_p - \partial u_p/\partial x_n)$. Denote by $\sum_{i,j}^\alpha$ the summation over all edges $\mathbf{x}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{x}_{\mathbf{j}}^{(\varepsilon)}$ of the simplex $P_{\alpha\varepsilon}$. Then, using Eqs. (3.4) and (3.6), we get

$$\begin{aligned}
\sum_{i,j}^{\alpha} \langle C_\varepsilon^{ij}(\mathbf{u}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{u}_{j}^{(\varepsilon)}), \mathbf{u}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{u}_{\mathbf{j}}^{(\varepsilon)} \rangle &= \sum_{i,j}^{\alpha} \langle C_\varepsilon^{ij}(\mathbf{u}_\varepsilon(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}) - \mathbf{u}_\varepsilon(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})), \mathbf{u}_\varepsilon(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}) - \mathbf{u}_\varepsilon(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}) \rangle \\
&= \sum_{i,j}^{\alpha} \Bigg\langle C_\varepsilon^{ij} \sum_{n,p=1}^{3} \{\varepsilon_{np}[\mathbf{u}_\varepsilon(\mathbf{x}_\varepsilon^\alpha)]\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{x}_{\mathbf{j}}^{(\varepsilon)}) + w_{np}[\mathbf{u}_\varepsilon(\mathbf{x}_\varepsilon^\alpha)]\psi^{\mathbf{np}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{x}_{\mathbf{j}}^{(\varepsilon)})\}, \\
&\quad \sum_{q,r=1}^{3} \{\varepsilon_{qr}[\mathbf{u}_\varepsilon(\mathbf{x}_\varepsilon^\alpha)]\varphi^{\mathbf{qr}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{x}_{\mathbf{j}}^{(\varepsilon)}) + w_{qr}[\mathbf{u}_\varepsilon(\mathbf{x}_\varepsilon^\alpha)]\psi^{\mathbf{qr}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{x}_{\mathbf{j}}^{(\varepsilon)})\} \Bigg\rangle \\
&\geqslant \frac{c}{\varepsilon} \sum_{n,p,q,r=1}^{3} \varepsilon_{np}[\mathbf{u}_\varepsilon(\mathbf{x}_\varepsilon^\alpha)]\varepsilon_{qr}[\mathbf{u}_\varepsilon(\mathbf{x}_\varepsilon^\alpha)] \sum_{i,j}^{\alpha} (x_{i_n}^{(\varepsilon)} - x_{j_n}^{(\varepsilon)})(x_{i_p}^{(\varepsilon)} - x_{j_p}^{(\varepsilon)}) \\
&\quad \times (x_{i_q}^{(\varepsilon)} - x_{j_q}^{(\varepsilon)})(x_{i_r}^{(\varepsilon)} - x_{j_r}^{(\varepsilon)}),
\end{aligned} \tag{3.7}$$

where the constant $c = k_1$ is independent of ε. Hereafter we denote all constants independent of ε by c, and the values $x_{i_n}^{(\varepsilon)}$ $(n \in \overline{1,3})$ are the coordinates of the vector $\mathbf{x}_{\mathbf{i}}^{(\varepsilon)} = (x_{i_1}^{(\varepsilon)}, x_{i_2}^{(\varepsilon)}, x_{i_3}^{(\varepsilon)})$. The last inequality in (3.7) follows from (1.1) and (1.3). Finally, we denote by X_{npqr}^ε the sum $\sum_{i,j}^\alpha (x_{i_n}^{(\varepsilon)} - x_{j_n}^{(\varepsilon)})(x_{i_p}^{(\varepsilon)} - x_{j_p}^{(\varepsilon)})(x_{i_q}^{(\varepsilon)} - x_{j_q}^{(\varepsilon)})(x_{i_r}^{(\varepsilon)} - x_{j_r}^{(\varepsilon)})$. Consider now a set of real numbers $\tau = \{\tau_{np}\}_{n,p=1}^{3}$ such that $\tau_{np} = \tau_{pn}, |\tau|^2 = \sum_{n,p=1}^{3}\tau_{np}^2 > 0$.Then

$$\sum_{n,p,q,r=1}^{3} X_{npqr}^{\varepsilon}\tau_{np}\tau_{qr} = \sum_{i,j}^{\alpha}\left(\sum_{n,p=1}^{3}(x_{i_n}^{(\varepsilon)} - x_{j_n}^{(\varepsilon)})(x_{i_p}^{(\varepsilon)} - x_{j_p}^{(\varepsilon)})\tau_{np}\right)^2 \geqslant 0. \tag{3.8}$$

We now show that this quadratic form is positive definite by a contradiction argument. Assume that there exists a non-zero set $\{\tau_{np}\}_{n,p=1}^{3}$ such that the form (3.8) is equal to zero on this set. Then

$$\sum_{n,p=1}^{3}(x_{i_n}^{(\varepsilon)} - x_{j_n}^{(\varepsilon)})(x_{i_p}^{(\varepsilon)} - x_{j_p}^{(\varepsilon)})\tau_{np} = 0 \tag{3.9}$$

for each pair of indices (i,j) that corresponds to the edge $\mathbf{x}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{x}_{\mathbf{j}}^{(\varepsilon)}$ of simplex $P_{\alpha\varepsilon}$. Equations (3.9) form a system of 6 linear equations for 6 unknowns $(\tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33})$. Simple but tedious calculations show that the absolute value of the determinant of this system is equal to $2^3 \cdot 6^4 \cdot |P_{\alpha\varepsilon}|^4$, where $|P_{\alpha\varepsilon}|$ is the volume of simplex $P_{\alpha\varepsilon}$. Due to the non-degeneracy condition in Definition 1, $|P_{\alpha\varepsilon}| \neq 0$. Thus the system (3.9) has only the trivial solution, which establishes a contradiction. Next we show that

$$\Phi_{\varepsilon}(t) = \sum_{i,j}^{\alpha}\left(\sum_{n,p=1}^{3}(x_{i_n}^{(\varepsilon)} - x_{j_n}^{(\varepsilon)})(x_{i_p}^{(\varepsilon)} - x_{j_p}^{(\varepsilon)})\tau_{np}\right)^2 \geqslant c(\varepsilon)|\tau|^2. \tag{3.10}$$

Indeed, if (3.10) does not hold, then for every $\delta > 0$ there exists a set of numbers $\{t_{np}\}_{n,p=1}^{3}$ such that $0 \leqslant \Phi_{\varepsilon}(\frac{\tau}{|\tau|}) < \delta$. This implies that $\Phi_{\varepsilon}(\frac{\tau}{|\tau|}) = 0$, which contradicts the fact that the system (3.9) has only the trivial solution, since $|\frac{\tau}{|\tau|}| = 1 \neq 0$. Taking into account (1.1), we conclude that $c(\varepsilon) = c_2\varepsilon^4, c_2 > 0$. Choosing $\tau_{np} = \varepsilon_{np}[\mathbf{u}_{\varepsilon}(\mathbf{x}_{\varepsilon}^{\alpha})]$ in (3.10) and using (3.7), we obtain

$$\sum_{i,j}^{\alpha}\langle C_{\varepsilon}^{ij}(\mathbf{u}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{u}_{\mathbf{j}}^{(\varepsilon)}), \mathbf{u}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{u}_{j}^{(\varepsilon)}\rangle \geqslant k_1 \cdot c_2 \sum_{n,p=1}^{3} \varepsilon_{np}^2[\mathbf{u}_{\varepsilon}(\mathbf{x}_{\varepsilon}^{\alpha})]\varepsilon^3.$$

Since the vector-function $\mathbf{u}_{\varepsilon}(x)$ is linear inside each $P_{\alpha\varepsilon}$, $\varepsilon_{np}[\mathbf{u}_{\varepsilon}(x)]$ is constant in $P_{\alpha\varepsilon}$ and thus

$$\sum_{i,j}^{\alpha}\langle C_{\varepsilon}^{ij}(\mathbf{u}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{u}_{\mathbf{j}}^{(\varepsilon)}), \mathbf{u}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{u}_{\mathbf{j}}^{(\varepsilon)}\rangle \geqslant c\int_{P_{\alpha\varepsilon}} \sum_{n,p=1}^{3} \varepsilon_{np}^2[\mathbf{u}_{\varepsilon}(x)]\,\mathrm{d}x,$$

where $c = 6k_1\, c_2/c_1^3$. Summing up over all simplices of the triangulization of the network in Ω, and taking into account that each edge $\mathbf{x}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{x}_{\mathbf{j}}^{(\varepsilon)}$ enters this sum no more than 4 times, we have

$$\begin{aligned}4\sum_{i,j}\langle C_{\varepsilon}^{ij}(\mathbf{u}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{u}_{\mathbf{j}}^{(\varepsilon)}), \mathbf{u}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{u}_{\mathbf{j}}^{(\varepsilon)}\rangle &\geqslant 4{\sum_{i,j}}'\langle C_{\varepsilon}^{ij}(\mathbf{u}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{u}_{\mathbf{j}}^{(\varepsilon)}), \mathbf{u}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{u}_{\mathbf{j}}^{(\varepsilon)}\rangle \\ &\geqslant c\int_{\Omega}\sum_{n,p=1}^{3}\varepsilon_{np}^2[\mathbf{u}_{\varepsilon}(x)]\,\mathrm{d}x.\end{aligned} \tag{3.11}$$

Since $\mathbf{u}_\varepsilon(\mathbf{x_i}^{(\varepsilon)}) = \mathbf{u_i}^{(\varepsilon)} = \mathbf{0}$ for $\mathbf{x_i}^{(\varepsilon)} \in \partial\Omega$, the spline $\mathbf{u}_\varepsilon(x) \in H_0^1(\Omega)$. Now we can use the standard continuum Korn's inequality

$$\|\mathbf{u}_\varepsilon(x)\|^2_{H_0^1(\Omega)} \leqslant 2\int_\Omega \sum_{n,p=1}^{3} \varepsilon^2_{np}[\mathbf{u}_\varepsilon(x)]\,\mathrm{d}x, \quad \mathbf{u}_\varepsilon(x) \in H_0^1(\Omega). \tag{3.12}$$

Inequalities (3.11) and (3.12) imply (3.5). Theorem 1 is proved. □

Finally, we establish the compactness of the family $\{\mathbf{u}_\varepsilon(x)\}$. Since $\mathbf{u}_\varepsilon(x) \in H_0^1(\Omega)$, inequalities (3.3), (3.5) imply that the sequence of splines $\mathbf{u}_\varepsilon(x)$ is bounded uniformly in ε in the space $H_0^1(\Omega)$:

$$\|\mathbf{u}_\varepsilon(x)\|^2_{H_0^1(\Omega)} \leqslant const.$$

Hence it is weakly compact, and due to the standard embedding theorem there exists a subsequence of the sequence $\mathbf{u}_\varepsilon(x)$ (which we denote for convenience $\mathbf{u}_\varepsilon(x)$) such that

$$\mathbf{u}_\varepsilon(x) \rightharpoonup \mathbf{u}(x) \text{ weakly in } H_0^1(\Omega), \quad \mathbf{u}_\varepsilon(x) \to \mathbf{u}(x) \text{ strongly in } L_2(\Omega), \quad \mathbf{u}(x) \in H_0^1(\Omega). \tag{3.13}$$

As we shall show in the next section, the limiting vector-function $\mathbf{u}(x)$ will be a solution of the homogenized problem.

4. Minimizer of the homogenized problem

In order to describe the homogenized problem, we introduce functions $\rho(\mathbf{x}) > 0$ and $\mathbf{a}(\mathbf{x})$ defined via the *Voronoi cell* U_i. Recall that for a given set of sites $\mathbf{x_i}$ in the domain Ω, the Voronoi tessellation is defined via the Voronoi cells of the points $\mathbf{x_i}$. The Voronoi cell U_i of a point $\mathbf{x_i}$ is defined as

$$U_i = \{\mathbf{x} \in \Omega : |\mathbf{x_i} - \mathbf{x}| \leqslant |\mathbf{x_j} - \mathbf{x}|,\ j \neq i\}.$$

It is known that the cells of the Voronoi tesselation are polyhedra, and thus we obtain a partition of the domain Ω into polyhedra U_i. If $\chi_{U_i}(x)$ is the characteristic function of U_i, we introduce the distributed density function $\overline{\rho_\varepsilon(\mathbf{x})} = \sum_{i=1}^N \frac{m_i^{(\varepsilon)}}{|U_i|}\chi_{U_i}(\mathbf{x})$, where $|U_i|$ is the volume of U_i. We suppose that the sequence of functions $\overline{\rho_\varepsilon(\mathbf{x})}$ weak-star converges in $L^\infty(\Omega)$ to a function $\rho(\mathbf{x}) > 0$ as $\varepsilon \to 0$:

$$\overline{\rho_\varepsilon(\mathbf{x})} \rightharpoonup \rho(\mathbf{x}) \quad (\textit{weak-star in } L^\infty(\Omega)). \tag{4.1}$$

Next, we introduce the distributed velocity vector-function $\overline{\mathbf{a}_\varepsilon(\mathbf{x})} = \sum_{i=1}^N \mathbf{a_i}^{(\varepsilon)}\chi_{U_i}(\mathbf{x})$ by assuming that the sequence of vector-functions $\overline{\mathbf{a}_\varepsilon(\mathbf{x})}$ converges strongly in $L_2(\Omega)$ to a vector-function $\mathbf{a}(\mathbf{x})$ as $\varepsilon \to 0$:

$$\overline{\mathbf{a}_\varepsilon(x)} \overset{L_2(\Omega)}{\to} \mathbf{a}(x). \tag{4.2}$$

We further assume that $\rho(\mathbf{x})$ and $\mathbf{a}(\mathbf{x})$ are smooth; otherwise, a standard approximation by smooth functions can be used. Note that the existence of limits (4.1)–(4.2) is a rather general restriction on the spatial distributions of the locations of point masses and their initial velocities. Since we do not consider any spatial periodicity, it is necessary to impose some conditions on these spatial distributions.

Theorem 2. *Suppose that* $\mathbf{u}(x)$ *is the limiting vector-function defined in* (3.13), *the array of particles satisfies the non-periodic triangulization condition in the sense of Definition* 1, *and the limits* (2.7), (4.1), (4.2) *exist. Then* $\mathbf{u}(x)$ *minimizes the limiting* (*homogenized*) *functional*

$$\Phi_0(\mathbf{v}) = \int_\Omega \left\{ \sum_{n,p,q,r=1}^{3} a_{npqr}(x)\varepsilon_{np}[\mathbf{v}]\varepsilon_{qr}[\mathbf{v}] + \lambda^2\rho(x)|\mathbf{v}(x)|^2 - 2\rho(x)\langle \mathbf{a}(x), \mathbf{v}(x)\rangle \right\} \mathrm{d}x \tag{4.3}$$

in the class of vector-functions from $H_0^1(\Omega)$.

We first briefly outline the scheme of the proof. Our goal is to establish the following inequality:

$$\Phi_0(\mathbf{u}) \leqslant \Phi_0(\mathbf{w}), \quad \forall \mathbf{w}(x) \in H_0^1(\Omega). \tag{4.4}$$

We do so first for smooth functions $\mathbf{w} \in C_0^2(\Omega)$, and then we use a standard smoothing procedure. Since we use the variational duality approach, the proof consists of two steps:

(i) *Establishing an upper bound, and*
(ii) *establishing a lower bound.*

In the first step, we prove the following inequality:

$$\overline{\lim_{\varepsilon\to 0}}\, \Phi_\varepsilon[\mathbf{u}_\varepsilon] \leqslant \Phi_0[\mathbf{w}], \quad \forall \mathbf{w} \in C_0^2(\Omega), \tag{4.5}$$

where the minimizer $\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon(\mathbf{x}_\mathbf{i}^{(\varepsilon)})$ is defined by (3.4). The key point of this step is the construction of a discrete vector-function $\mathbf{w}_{\varepsilon\mathbf{h}}$ for any $\mathbf{w}(x) \in C_0^2(\Omega)$. This function is called the *quasiminimizer* because if $\mathbf{w} = \mathbf{u}$ (where $\mathbf{u}$ is defined by (3.13)), then the resulting function $\mathbf{u}_{\varepsilon\mathbf{h}}$ provides the almost-minimum energy to the functional Φ_ε. This function is constructed by a *mesopartition* of the domain Ω into mesocubes of size h ($\varepsilon \ll h \ll 1$).

The fact that $\mathbf{u}_\varepsilon$ minimizes the functional Φ_ε among all discrete vector-functions implies that

$$\Phi_\varepsilon[\mathbf{u}_\varepsilon] \leqslant \Phi_\varepsilon[\mathbf{w}_{\varepsilon\mathbf{h}}].$$

The vector-function $\mathbf{w}_{\varepsilon\mathbf{h}}$ has the following two properties. First, the function

$$\overline{\mathbf{w}_{\varepsilon\mathbf{h}}} = \sum_{i=1}^{N} \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\chi_{U_i}(x)$$

approximates $\mathbf{w}(x)$ in the following sense:

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} \|\overline{\mathbf{w}_{\varepsilon\mathbf{h}}(x)} - \mathbf{w}(x)\|_{L_2(\Omega)} = 0.$$

Furthermore, the explicit construction of the vector-function $\mathbf{w}_{\varepsilon\mathbf{h}}$ allows us to pass to the limit in $\Phi_\varepsilon[\mathbf{w}_{\varepsilon\mathbf{h}}]$ and obtain the following inequality:

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\, \Phi_\varepsilon[\mathbf{w}_{\varepsilon\mathbf{h}}] \leqslant \Phi_0[\mathbf{w}], \tag{4.6}$$

where Φ_0 is the homogenized functional (4.3).

Since the construction of $\mathbf{w}_{\varepsilon\mathbf{h}}$ is the key point of the proof, we now explain the idea behind this construction. In each cube of the mesopartition, this discrete vector-function has the form

$$\mathbf{w}^{\mathbf{x}_\alpha}_{\varepsilon\mathbf{h}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) = \mathbf{w}(\mathbf{x}_\alpha) + \sum_{n,p=1}^{3} \varepsilon_{np}[\mathbf{w}(\mathbf{x}_\alpha)]\mathbf{v}^{\mathbf{np}}_\alpha(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) + \sum_{n,p=1}^{3} w_{np}[\mathbf{w}(\mathbf{x}_\alpha)]\psi^{\mathbf{np}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}_\alpha), \tag{4.7}$$

where the constant vectors $\mathbf{v}^{\mathbf{np}}_\alpha(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}})$ $(i \in \overline{1,p})$ minimize the functional $E^{\varepsilon(np)}_{K^{x_\alpha}_h}(\mathbf{v})$ (see (2.2) with $T = T^{np}$). To explain (4.7), we note that if $\mathbf{v}^{\mathbf{np}}_\alpha(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}})$ is replaced by $\varphi^{\mathbf{np}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}_\alpha)$, then it becomes the linear part of the Taylor expansion of $\mathbf{w}(x)$ in the mesocube $K^{x_\alpha}_h$ centered at point $\mathbf{x}_\alpha$.

The vector-function (4.7) possesses the following properties. First, it approximates (in some sense) the vector-function $\mathbf{w}(x)$ in the cube $K^{x_\alpha}_h$. Second, it almost minimizes the local energy functional (mesocharacteristic) (2.2) when $T_{jk} = \varepsilon_{jk}[\mathbf{w}(\mathbf{x}_\alpha)]$. Indeed, the first and the second terms in (4.7) do not contribute to the elastic energy. The latter property allows us to pass to the limit as $\varepsilon \to 0$ and compute the limiting functional Φ_0 via mesocharacteristics.

If $K^{x_\alpha}_h$ is a partition of the domain Ω, then a global quasiminimizer can be constructed, roughly speaking, by gluing together the quasiminimizers of (2.2) with the help of an appropriate partition of unity.

In the second step we establish the following inequality:

$$\Phi_0[\mathbf{u}] \leqslant \varliminf_{\varepsilon\to 0} \Phi_\varepsilon[\mathbf{u}_\varepsilon]. \tag{4.8}$$

Clearly, (4.8) and (4.5) imply (4.4).

Proof of Theorem 2.

(i) The upper bound.

Take any vector-function $\mathbf{w}(x) \in C^2_0(\Omega)$. Choose a mesoscale parameter h such that there exists a cover of the domain Ω by cubes $K^{x_\alpha}_h$ centered at points $\mathbf{x}_\alpha : \overline{\Omega} \subset \bigcup_{\alpha\in\Lambda} K^{x_\alpha}_h$. These points form a cubic lattice of period $h - h^{1+\frac{\gamma}{2}}, 0<\gamma<0$ such that the cubes overlap. Due to the overlap of the cubes, we can further select smaller cubes $K^{x_\alpha}_{h'}$ with the same center $\mathbf{x}_\alpha$ and edges of length $h' = h - 2h^{1+\frac{\gamma}{2}}$. It is well known that there exists a set of functions $\{\phi_\alpha(x) \in C^\infty_0(\Omega)\}_{\alpha\in\Lambda}$ (called a *partition of unity*) such that

$$\phi_\alpha(x) = \begin{cases} 1, x \in K^{x_\alpha}_h, \\ 0, x \notin K^{x_\alpha}_h, \end{cases} \quad 0 \leqslant \phi_\alpha(x) \leqslant 1, \quad |\nabla\phi_\alpha(x)| \leqslant \frac{c}{h^{1+\frac{\gamma}{2}}}, \quad \sum_{\alpha\in\Lambda} \phi_\alpha(x) \equiv 1, \quad x \in \overline{\Omega}.$$

We now construct a discrete vector-function (4.9)

$$\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) = \sum_{\alpha\in\Lambda} \left\{ \mathbf{w}(\mathbf{x}_\alpha) + \sum_{n,p=1}^{3} \varepsilon_{np}[\mathbf{w}(\mathbf{x}_\alpha)]\mathbf{v}^{\mathbf{np}}_\alpha(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) \right.$$
$$\left. + \sum_{n,p=1}^{3} w_{np}[\mathbf{w}(\mathbf{x}_\alpha)]\psi^{\mathbf{np}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}_\alpha) \right\} \cdot \phi_\alpha(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}), \tag{4.10}$$

where $\mathbf{v}^{\mathbf{np}}_\alpha(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}})$ are the minimizers of the functional $E^{\varepsilon(np)}_{K^{x_\alpha}_h}(\mathbf{v})$ defined by (2.2). Taking into account (2.6) and (2.7), we estimate the interaction energy in cube $K^{x_\alpha}_h$:

$$\sum_{i,j \;\; K^{x_\alpha}_h} \langle C^{ij}_\varepsilon(v^{np}_\alpha(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \mathbf{v}^{\mathbf{np}}_\alpha(\mathbf{x}^{(\varepsilon)}_{\mathbf{j}})), \mathbf{v}^{\mathbf{qr}}_\alpha(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \mathbf{v}^{\mathbf{qr}}_\alpha(\mathbf{x}^{(\varepsilon)}_{\mathbf{j}})\rangle = O(h^3),$$

$$\varepsilon^3 \sum_{i \;\; K^{x_\alpha}_h} \langle \mathbf{v}^{\mathbf{np}}_\alpha(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \varphi^{\mathbf{np}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}_\alpha), \mathbf{v}^{\mathbf{qr}}_\alpha(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \varphi^{\mathbf{qr}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}_\alpha)\rangle = O(h^{5+\gamma}). \tag{4.11}$$

Next, we estimate the energy in the overlapping parts $K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}$ as $h \to 0$. To this end, we subtract the energy of the particles in the smaller cube $K_{h'}^{x_\alpha}$ from the energy of the particles in the larger cube $K_h^{x_\alpha}$. This leads to an upper bound of the energy functional because some particles in the cube $K_{h'}^{x_\alpha}$ may interact with the particles in the cube $K_h^{x_\alpha}$ (i.e., the corresponding springs cross the boundary $\partial K_{h'}^{x_\alpha}$):

$$\begin{aligned}
&\sum_{i,j\;K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}} \langle C_\varepsilon^{ij}(\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})), \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\rangle \\
&\quad + h^{-2-\gamma}\varepsilon^3 \sum_{i\;K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}} \langle \mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\alpha), \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\alpha)\rangle \\
&\quad \leqslant \Bigg\{ \sum_{i,j\;K_h^{x_\alpha}} \langle C_\varepsilon^{ij}(\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})), \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\rangle \\
&\qquad + h^{-2-\gamma}\varepsilon^3 \sum_{i\;K_h^{x_\alpha}} \langle \mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\alpha), \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\alpha)\rangle \Bigg\} \\
&\qquad - \Bigg\{ \sum_{i,j\;K_{h'}^{x_\alpha}} \langle C_\varepsilon^{ij}(\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})), \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\rangle \\
&\qquad + (h')^{-2-\gamma}\varepsilon^3 \sum_{i\;K_{h'}^{x_\alpha}} \langle \mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\alpha), \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\alpha)\rangle \Bigg\} \\
&\qquad + ((h')^{-2-\gamma} - h^{-2-\gamma})\varepsilon^3 \sum_{i\;K_{h'}^{x_\alpha}} \langle \mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\alpha), \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) \\
&\qquad - \varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\alpha)\rangle\} \\
&\quad = a_{npqr}(x_\alpha, \varepsilon, h, \gamma) - a_{npqr}(x_\alpha, \varepsilon, h', \gamma) + O((h')^{5+\gamma})\left(\frac{1}{(h')^{2+\gamma}} - \frac{1}{h^{2+\gamma}}\right) \\
&\quad = O(h^3 - (h')^3) + O((h')^3)\left(1 - \left(\frac{h'}{h}\right)^{2+\gamma}\right) = O(h^3)\left(1 - \left(\frac{h'}{h}\right)^3\right) \\
&\qquad + O((h')^3)\left(1 - \left(\frac{h'}{h}\right)^{2+\gamma}\right) = O(h^3)(1 - ((1 - 2h^{\frac{\gamma}{2}})^3) \\
&\qquad + O((h')^3)(1 - (1 - 2h^{\frac{\gamma}{2}})^{2+\gamma}) \overset{h\to 0}{\leqslant} O(h^3)h^{\frac{\gamma}{2}} = o(h^3).
\end{aligned}$$

In the first equality we used (2.7) and (4.11); in the second equality we used (2.7). Thus

$$\begin{aligned}
&\sum_{i,j\;K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}} \langle C_\varepsilon^{ij}(\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})), \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}\rangle = o(h^3), \\
&\varepsilon^3 \sum_{i\;K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}} \langle \mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\alpha), \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\alpha)\rangle = o(h^{5+\gamma}). \qquad (4.12)
\end{aligned}$$

Since $\mathbf{w} \in C_0^2(\Omega)$, it can be written in the form

$$\mathbf{w}(x) = \mathbf{w}(x_\alpha) + \sum_{n,p=1}^{3} \{\varepsilon_{np}[\mathbf{w}(x_\alpha)]\varphi^{\mathbf{np}}(x - x_\alpha) + w_{np}[\mathbf{u}_\varepsilon(x_\varepsilon^\alpha)]\psi^{\mathbf{np}}(x - x_\alpha)\} + \mathbf{g}_\alpha(x), \tag{4.13}$$

where $\mathbf{g}_\alpha(x) = O(h^2)$.

Substituting (4.13) into (4.10) we obtain

$$\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) = \sum_{\alpha\in\Lambda} \left\{ \mathbf{w}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) + \sum_{n,p=1}^{3} \varepsilon_{np}[\mathbf{w}(x_\alpha)] \cdot (\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\alpha)) \right.$$
$$\left. -\mathbf{g}_\alpha(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) \right\} \cdot \phi_\alpha(\mathbf{x}_\mathbf{i}^{(\varepsilon)}). \tag{4.14}$$

This allows us to evaluate $\Phi_\varepsilon(\mathbf{w}_{\varepsilon\mathbf{h}})$:

$$\Phi_\varepsilon(\mathbf{w}_{\varepsilon\mathbf{h}}) = \frac{1}{2}\sum_{i,j} \langle C_\varepsilon^{ij}(\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})), \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\rangle$$
$$+ \lambda^2 \sum_{i=1}^{N} m_i^{(\varepsilon)} |\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})|^2 - 2\sum_{i=1}^{N} m_i^{(\varepsilon)} \langle \mathbf{a}_\mathbf{i}^{(\varepsilon)}, \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\rangle. \tag{4.15}$$

We now use the estimates (4.11) and (4.12) in order to obtain the upper bound (4.6).

Consider the first term of (4.15):

$$\frac{1}{2}\sum_{i,j} \langle C_\varepsilon^{ij}(\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})), \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\rangle$$
$$= \frac{1}{2}\sum_{\alpha\in\Lambda} \sum_{i,j \; K_{h'}^{x_\alpha}} \langle C_\varepsilon^{ij}(\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})), \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\rangle + A_{\varepsilon h}$$
$$= H_{\varepsilon h} + A_{\varepsilon h}, \tag{4.16}$$

where $A_{\varepsilon h}$ contains terms that correspond to neighboring particles $\mathbf{x}_\mathbf{i}^{(\varepsilon)} \in K_{h'}^{x_\alpha}, \mathbf{x}_\mathbf{j}^{(\varepsilon)} \in K_h^{x_\alpha} \backslash K_{h'}^{x_\alpha}$ or $\mathbf{x}_\mathbf{i}^{(\varepsilon)}, \mathbf{x}_\mathbf{j}^{(\varepsilon)} \in K_h^{x_\alpha} \backslash K_{h'}^{x_\alpha}$. Substituting (4.10) in $H_{\varepsilon h}$, we get

$$\frac{1}{2}\sum_{\alpha\in\Lambda} \sum_{i,j \; K_{h'}^{x_\alpha}} \left\langle C_\varepsilon^{ij} \left\{ \sum_{n,p=1}^{3} \varepsilon_{np}[\mathbf{w}(x_\alpha)] \cdot (\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})) \right.\right.$$
$$\left. + \sum_{n,p=1}^{3} w_{np}[\mathbf{w}(x_\alpha)] \cdot \psi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\mathbf{j}^{(\varepsilon)}) \right\},$$
$$\left. \sum_{q,r=1}^{3} \varepsilon_{qr}[\mathbf{w}(x_\alpha)] \cdot (\mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})) + \sum_{q,r=1}^{3} w_{qr}[\mathbf{w}(x_\alpha)] \cdot \psi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\mathbf{j}^{(\varepsilon)}) \right\rangle$$

$$
\begin{aligned}
&= \sum_{\alpha\in\Lambda}\sum_{n,p,q,r=1}^{3} \varepsilon_{np}[\mathbf{w}(x_\alpha)]\cdot\varepsilon_{qr}[\mathbf{w}(x_\alpha)]\\
&\quad\cdot\left(\frac{1}{2}\sum_{i,j\;K^{x_\alpha}_{h'}} \langle C^{ij}_{\varepsilon}(\mathbf{v}^{\mathbf{np}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \mathbf{v}^{\mathbf{np}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{j}})), \mathbf{v}^{\mathbf{qr}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \mathbf{v}^{\mathbf{qr}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{j}})\rangle\right)\\
&\quad+\frac{1}{2}\sum_{\alpha\in\Lambda}\sum_{n,p,q,r=1}^{3} \varepsilon_{np}[\mathbf{w}(x_\alpha)]\cdot w_{qr}[\mathbf{w}(x_\alpha)]\\
&\quad\cdot\sum_{i,j\;K^{x_\alpha}_{h'}} \langle C^{ij}_{\varepsilon}(\mathbf{v}^{\mathbf{np}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \mathbf{v}^{\mathbf{np}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{j}})), \psi^{\mathbf{qr}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}^{(\varepsilon)}_{\mathbf{j}})\rangle\\
&\quad+\frac{1}{2}\sum_{\alpha\in\Lambda}\sum_{n,p,q,r=1}^{3} w_{np}[\mathbf{w}(x_\alpha)]\cdot \varepsilon_{qr}[\mathbf{w}(x_\alpha)]\\
&\quad\cdot\sum_{i,j\;K^{x_\alpha}_{h'}} \langle C^{ij}_{\varepsilon}\psi^{\mathbf{np}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}^{(\varepsilon)}_{\mathbf{j}}), \mathbf{v}^{\mathbf{qr}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \mathbf{v}^{\mathbf{qr}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{j}})\rangle\\
&\quad+\frac{1}{2}\sum_{\alpha\in\Lambda}\sum_{n,p,q,r=1}^{3} w_{np}[\mathbf{w}(x_\alpha)]\cdot w_{qr}[\mathbf{w}(x_\alpha)]\\
&\quad\cdot\sum_{i,j\;K^{x_\alpha}_{h'}} \langle C^{ij}_{\varepsilon}\psi^{\mathbf{np}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}^{(\varepsilon)}_{\mathbf{j}}), \psi^{\mathbf{qr}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}^{(\varepsilon)}_{\mathbf{j}})\rangle.
\end{aligned}
$$

Taking into account (1.4) and the definition of $\psi^{\mathbf{np}}(x)$ (see (2.1)), we see that

$$
\begin{aligned}
&\langle C^{ij}_{\varepsilon}(\mathbf{v}^{\mathbf{np}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \mathbf{v}^{\mathbf{np}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{j}})), \psi^{\mathbf{qr}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}^{(\varepsilon)}_{\mathbf{j}})\rangle\\
&\quad= \langle C^{ij}_{\varepsilon}\psi^{\mathbf{np}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}^{(\varepsilon)}_{\mathbf{j}}), \mathbf{v}^{\mathbf{qr}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \mathbf{v}^{\mathbf{qr}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{j}})\rangle\\
&\quad= \langle C^{ij}_{\varepsilon}\psi^{\mathbf{np}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}^{(\varepsilon)}_{\mathbf{j}}), \psi^{\mathbf{qr}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}^{(\varepsilon)}_{\mathbf{j}})\rangle = 0.
\end{aligned}
$$

The expression $H_{\varepsilon h}$ in (4.16) can then be bounded from above by

$$
\begin{aligned}
&\sum_{\alpha\in\Lambda}\sum_{n,p,q,r=1}^{3} \varepsilon_{np}[\mathbf{w}(x_\alpha)]\cdot\varepsilon_{qr}[\mathbf{w}(x_\alpha)]\cdot\left\{\frac{1}{2}\sum_{i,j\;K^{x_\alpha}_{h'}} \langle C^{ij}_{\varepsilon}(\mathbf{v}^{\mathbf{np}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \mathbf{v}^{\mathbf{np}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{j}})), \mathbf{v}^{\mathbf{qr}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}})\right.\\
&\quad\left. - \mathbf{v}^{\mathbf{qr}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{j}})\rangle + h^{-2-\gamma}\varepsilon^{3}\sum_{i\;K^{x_\alpha}_{h'}} \langle \mathbf{v}^{\mathbf{np}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \varphi^{\mathbf{np}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}} - \mathbf{x}_{\alpha}), \mathbf{v}^{\mathbf{qr}}_{\alpha}(\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}) - \varphi^{\mathbf{qr}}(\mathbf{x}^{(\varepsilon)}_{\mathbf{j}} - \mathbf{x}_{\alpha})\rangle\right\}
\end{aligned}
$$

which, due to (2.6) and (3.11), gives in the limit as $\varepsilon \to 0$ and $h \to 0$

$$
\int_{\Omega}\sum_{n,p,q,r=1}^{3} \varepsilon_{np}[\mathbf{w}(x)]\cdot\varepsilon_{qr}[\mathbf{w}(x)]\cdot a_{npqr}(x)\,\mathrm{d}x. \tag{4.17}
$$

We now prove that $A_{\varepsilon h}$ vanishes as $\varepsilon \to 0$ and $h \to 0$. To this end we consider the case when the particles $\mathbf{x}^{(\varepsilon)}_{\mathbf{i}}$ and $\mathbf{x}^{(\varepsilon)}_{\mathbf{j}}$ are located in $K^{x_\alpha}_{h}\backslash K^{x_\alpha}_{h'}$. Then, making use of (4.9) and (4.14), we have

$$
\begin{aligned}
&\sum_{\alpha\in\Lambda}\sum_{\substack{i,j\\ K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}} \langle C_\varepsilon^{ij}(\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})), \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\rangle \\
&\quad = \sum_{\alpha\in\Lambda}\sum_{\substack{i,j\\ K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}} \Bigg\langle C_\varepsilon^{ij}\Bigg[\mathbf{w}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}) \\
&\quad + \sum_{\beta\in\Lambda}\sum_{n,p=1}^{3} \varepsilon_{np}[\mathbf{w}(x_\beta)]\cdot\{(\mathbf{v}_\beta^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\beta) \\
&\quad - \mathbf{g}_\beta(\mathbf{x}_\mathbf{i}^{(\varepsilon)}))\cdot\phi_\beta(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - (\mathbf{v}_\beta^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)} - \mathbf{x}_\beta) - \mathbf{g}_\beta(\mathbf{x}_\mathbf{j}^{(\varepsilon)}))\cdot\phi_\beta(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\}\Bigg], \\
&\quad \mathbf{w}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}) + \sum_{\gamma\in\Lambda}\sum_{n,p=1}^{3} \varepsilon_{np}[\mathbf{w}(x_\gamma)]\cdot\{(\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\gamma) \\
&\quad - \mathbf{g}_\gamma(\mathbf{x}_\mathbf{i}^{(\varepsilon)}))\cdot\phi_\gamma(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - (\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)} - \mathbf{x}_\gamma) - \mathbf{g}_\gamma(\mathbf{x}_\mathbf{j}^{(\varepsilon)}))\cdot\phi_\gamma(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\}\Bigg\rangle. \qquad (4.18)
\end{aligned}
$$

Next, using (1.4) and the fact that for a smooth vector-function $\mathbf{w}(x)$ the inequality $\|\mathbf{w}(x) - \mathbf{w}(y)\|_{\mathbb{R}_3} \leqslant c\|x - y\|_{\mathbb{R}_3}$ holds, we get

$$
\begin{aligned}
&\left|\sum_{\alpha\in\Lambda}\sum_{\substack{i,j\\ K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}} \langle C_\varepsilon^{ij}(\mathbf{w}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})), \mathbf{w}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\rangle\right| \\
&\quad \leqslant c\sum_{\alpha\in\Lambda}\sum_{\substack{i,j\\ K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}} k^{ij}\varepsilon\cdot|\mathbf{w}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})|^2 \\
&\quad \leqslant c\sum_{\alpha\in\Lambda}\sum_{\substack{i,j\\ K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}} \varepsilon^3 \leqslant c\sum_{\alpha\in\Lambda}\frac{h^2(h-h')}{|\Omega|}\cdot N\cdot\varepsilon^3 \\
&\quad \leqslant c\sum_{\alpha\in\Lambda} h^{3+\frac{\gamma}{2}} \leqslant c\frac{|\Omega|}{h^3}\cdot h^{3+\frac{\gamma}{2}} \leqslant ch^{\frac{\gamma}{2}} \xrightarrow{h\to 0} 0.
\end{aligned}
$$

Straightforward calculations show that

$$
\begin{aligned}
&(\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\gamma))\cdot\phi_\gamma(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - (\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)} - \mathbf{x}_\gamma))\cdot\phi_\gamma(\mathbf{x}_\mathbf{j}^{(\varepsilon)}) \\
&\quad = (\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\mathbf{j}^{(\varepsilon)}))\cdot\phi_\gamma(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) \\
&\qquad + (\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)} - \mathbf{x}_\gamma))\cdot(\phi_\gamma(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \phi_\gamma(\mathbf{x}_\mathbf{j}^{(\varepsilon)})).
\end{aligned}
$$

Next, we use this equality to estimate some of the terms in (4.18):

$$
\left|\sum_{\alpha\in\Lambda}\sum_{\substack{i,j\\ K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}} \Bigg\langle C_\varepsilon^{ij}(\mathbf{w}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{w}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})), \sum_{\gamma\in\Lambda}^{\alpha}\sum_{n,p=1}^{3} \varepsilon_{np}[\mathbf{w}(x_\gamma)]\cdot(\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\right.
$$

$$
\left. -\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{x}_{\mathbf{j}}^{(\varepsilon)})) \cdot \phi_\gamma(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}) \right\rangle \Bigg| \leqslant c \sum_{\alpha \in \Lambda} \sum_{i,j} \sum_{K_h^{x_\alpha} \backslash K_{h'}^{x_\alpha}} \sum_{n,p=1}^{3} \{ \langle C_\varepsilon^{ij}(\mathbf{w}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}) - \mathbf{w}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})),
$$

$$
\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}) - \mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}) \rangle + k^{ij} \varepsilon^2 \cdot |\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{x}_{\mathbf{j}}^{(\varepsilon)})| \}
$$

$$
\leqslant c \Bigg\{ \left(\sum_{\alpha \in \Lambda} \sum_{i,j} \sum_{K_h^{x_\alpha} \backslash K_{h'}^{x_\alpha}} \sum_{n,p=1}^{3} \langle C_\varepsilon^{ij}(\mathbf{w}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}) - \mathbf{w}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})), \mathbf{w}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}) - \mathbf{w}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}) \rangle \right)^{1/2}
$$

$$
\cdot \left(\sum_{\alpha \in \Lambda} \sum_{i,j} \sum_{K_h^{x_\alpha} \backslash K_{h'}^{x_\alpha}} \sum_{n,p=1}^{3} \langle C_\varepsilon^{ij}(\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}) - \mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})), \mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}) - \mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}) \rangle \right)^{1/2}
$$

$$
+ \sum_{\alpha \in \Lambda} \sum_{i,j} \sum_{K_h^{x_\alpha} \backslash K_{h'}^{x_\alpha}} k^{ij} \varepsilon^3 \Bigg\} \leqslant c \left\{ h^{\frac{\gamma}{4}} \cdot \left(\sum_{\alpha \in \Lambda} \overline{o}(h^3) \right)^{1/2} + \sum_{\alpha \in \Lambda} \sum_{i,j} \sum_{K_h^{x_\alpha} \backslash K_{h'}^{x_\alpha}} k^{ij} \varepsilon^3 \right\}
$$

$$
\leqslant c \left\{ h^{\frac{\gamma}{4}} \cdot \left(\frac{|\Omega|}{h^3} \cdot \overline{o}(h^3) \right)^{1/2} + h^{\frac{\gamma}{2}} \right\} \xrightarrow{h \to 0} 0.
$$

Here we use the fact that the symbol $\sum_{\gamma \in \Lambda}^{\alpha}$ stands for the summation over the cubes that contain the slab $K_h^{x_\alpha} \backslash K_{h'}^{x_\alpha}$. The number of such cubes is bounded by a constant c which does not depend on h. In the first inequality we used (1.4); in the second we used (4.12). Next, we estimate the remaining terms in (4.18) using the Cauchy–Schwartz inequality, (1.4), (4.9) and (4.12):

$$
\Bigg| \sum_{\alpha \in \Lambda} \sum_{i,j} \sum_{K_h^{x_\alpha} \backslash K_{h'}^{x_\alpha}} \left\langle C_\varepsilon^{ij}(\mathbf{w}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}) - \mathbf{w}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})), \sum_{\gamma \in \Lambda}^{\alpha} \sum_{n,p=1}^{3} \varepsilon_{np}[\mathbf{w}(x_\gamma)] \cdot (\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)} - \mathbf{x}_\gamma)) \right.
$$

$$
\left. \cdot (\phi_\gamma(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}) - \phi_\gamma(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})) \right\rangle \Bigg| \leqslant \sum_{i,j} \sum_{K_h^{x_\alpha} \backslash K_{h'}^{x_\alpha}} k^{ij} \varepsilon^2 \sum_{n,p=1}^{3} |\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}) - \varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)} - \mathbf{x}_\alpha)| \cdot \frac{|\mathbf{x}_{\mathbf{i}}^{(\varepsilon)} - \mathbf{x}_{\mathbf{j}}^{(\varepsilon)}|}{h^{1+\frac{\gamma}{2}}}
$$

$$
\leqslant \frac{c}{h^{1+\frac{\gamma}{2}}} \sum_{\alpha \in \Lambda} \left(\sum_{i,j} \sum_{K_h^{x_\alpha} \backslash K_{h'}^{x_\alpha}} k^{ij} \varepsilon^3 \right)^{1/2} \cdot \left(\sum_{i,j} \sum_{K_h^{x_\alpha} \backslash K_{h'}^{x_\alpha}} k^{ij} \varepsilon^3 \cdot \sum_{n,p=1}^{3} |\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}) \right.
$$

$$
\left. - \varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)} - \mathbf{x}_\alpha)|^2 \right)^{1/2} \leqslant \frac{c}{h^{1+\frac{\gamma}{2}}} \sum_{\alpha \in \Lambda} h^{\frac{3}{2}+\frac{\gamma}{4}} \cdot \overline{o}(h^{\frac{5}{2}+\frac{\gamma}{2}})
$$

$$
= c \sum_{\alpha \in \Lambda} \overline{o}(h^{3+\frac{\gamma}{4}}) \leqslant c \frac{|\Omega|}{h^3} \cdot h^{3+\frac{\gamma}{4}} \xrightarrow{h \to 0} 0.
$$

Similarly,

$$\Bigg|\sum_{\alpha\in\Lambda}\sum_{i,j}\,{}_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}\Bigg\langle C_\varepsilon^{ij}\Bigg(\sum_{\beta\in\Lambda}^{\alpha}\sum_{n,p=1}^{3}\varepsilon_{np}[\mathbf{w}(x_\beta)]\cdot(\mathbf{v}_\beta^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})-\mathbf{v}_\beta^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}))\cdot\phi_\beta(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\Bigg),$$

$$\sum_{\gamma\in\Lambda}^{\alpha}\sum_{q,r=1}^{3}\varepsilon_{qr}[\mathbf{w}(x_\gamma)]\cdot(\mathbf{v}_\gamma^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})-\varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}-\mathbf{x}_\gamma))\cdot(\phi_\gamma(\mathbf{x}_\mathbf{i}^{(\varepsilon)})-\phi_\gamma(\mathbf{x}_\mathbf{j}^{(\varepsilon)}))\Bigg\rangle\Bigg|$$

$$\leqslant c\sum_{\alpha\in\Lambda}\sum_{i,j}\,{}_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}\Bigg\langle C_\varepsilon^{ij}(\sum_{n,p=1}^{3}(\mathbf{v}_\beta^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})-\mathbf{v}_\beta^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})),$$

$$\sum_{\gamma\in\Lambda}^{\alpha}\sum_{q,r=1}^{3}(\mathbf{v}_\gamma^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})-\varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}-\mathbf{x}_\gamma))\Bigg\rangle\cdot\frac{\varepsilon}{h^{1+\frac{\gamma}{2}}}$$

$$\leqslant\frac{c}{h^{1+\frac{\gamma}{2}}}\left(\sum_{\alpha\in\Lambda}\sum_{i,j}\,{}_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}\sum_{n,p=1}^{3}\langle C_\varepsilon^{ij}(\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})-\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})),\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})-\mathbf{v}_\gamma^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\rangle\right)^{1/2}$$

$$\cdot\left(\sum_{\alpha\in\Lambda}\sum_{i,j}\,{}_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}\varepsilon^3\sum_{q,r=1}^{3}|\mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})-\varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}-\mathbf{x}_\alpha)|^2\right)^{1/2}$$

$$\leqslant\frac{c}{h^{1+\frac{\gamma}{2}}}\left(\sum_{\alpha\in\Lambda}\overline{o}(h^3)\right)^{1/2}\cdot\left(\sum_{\alpha\in\Lambda}\overline{o}(h^{5+\gamma})\right)^{1/2}$$

$$\leqslant\frac{c}{h^{1+\frac{\gamma}{2}}}\cdot\left(\frac{\overline{o}(h^3)}{h^3}\right)^{1/2}\cdot\left(\frac{\overline{o}(h^{5+\gamma})}{h^3}\right)^{1/2}=c\cdot\left(\frac{\overline{o}(h^3)}{h^3}\right)^{1/2}\cdot\left(\frac{\overline{o}(h^{5+\gamma})}{h^{5+\gamma}}\right)^{1/2}\xrightarrow{h\to 0}0$$

and

$$\Bigg|\sum_{\alpha\in\Lambda}\sum_{i,j}\,{}_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}\Bigg\langle C_\varepsilon^{ij}\Bigg(\sum_{\beta\in\Lambda}\sum_{n,p=1}^{3}\varepsilon_{np}[\mathbf{w}(x_\beta)]\cdot(\mathbf{v}_\beta^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})-\varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}-\mathbf{x}_\beta))\cdot(\phi_\beta(\mathbf{x}_\mathbf{i}^{(\varepsilon)})$$

$$-\phi_\beta(\mathbf{x}_\mathbf{j}^{(\varepsilon)}))\Bigg),\sum_{\gamma\in\Lambda}\sum_{q,r=1}^{3}\varepsilon_{qr}[\mathbf{w}(x_\gamma)]\cdot(\mathbf{v}_\gamma^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})-\varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}-\mathbf{x}_\gamma))\cdot(\phi_\gamma(\mathbf{x}_\mathbf{i}^{(\varepsilon)})-\phi_\gamma(\mathbf{x}_\mathbf{j}^{(\varepsilon)}))\Bigg\rangle\Bigg|$$

$$\leqslant c\sum_{\alpha\in\Lambda}\sum_{i,j}\,{}_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}\varepsilon\cdot\sum_{\beta\in\Lambda}\sum_{n,p=1}^{3}|\mathbf{v}_\beta^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})-\varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}-\mathbf{x}_\beta)|$$

$$\cdot\sum_{\gamma\in\Lambda}\sum_{q,r=1}^{3}|\mathbf{v}_\gamma^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})-\varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}-\mathbf{x}_\gamma)|\cdot\frac{\varepsilon^2}{h^{2+\gamma}}$$

$$\leqslant \frac{c}{h^{2+\gamma}}\left(\sum_{\alpha\in\Lambda}\sum_{i,j}\,\sum_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}} \varepsilon^3\cdot\sum_{n,p=1}^{3}|\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})-\varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}-\mathbf{x}_\alpha)|^2\right)^{1/2}$$

$$\cdot\left(\sum_{\alpha\in\Lambda}\sum_{i,j}\,\sum_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}} \varepsilon^3\cdot\sum_{q,r=1}^{3}|\mathbf{v}_\alpha^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})-\varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)}-\mathbf{x}_\alpha)|^2\right)^{1/2}$$

$$\leqslant \frac{c}{h^{2+\gamma}}\left[\sum_{\alpha\in\Lambda}\bar{o}(h^{5+\gamma})\right]\leqslant\frac{c}{h^{2+\gamma}}\cdot\frac{|\Omega|}{h^3}\cdot\bar{o}(h^{5+\gamma})\leqslant c\cdot\frac{\bar{o}(h^{5+\gamma})}{h^{5+\gamma}}\xrightarrow{h\to 0}0. \tag{4.19}$$

We now rewrite the terms in (4.18) which contain the vector-function $\mathbf{g}_\alpha(x)$:

$$\begin{aligned}\mathbf{g}_\alpha(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\phi_\alpha(\mathbf{x}_\mathbf{j}^{(\varepsilon)})-\mathbf{g}_\alpha(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\phi_\alpha(\mathbf{x}_\mathbf{j}^{(\varepsilon)}) &= (\mathbf{g}_\alpha(\mathbf{x}_\mathbf{j}^{(\varepsilon)})-\mathbf{g}_\alpha(\mathbf{x}_\mathbf{i}^{(\varepsilon)}))\cdot\phi_\alpha(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\\ &\quad + g_\alpha(x_j^{(\varepsilon)})\cdot(\phi_\alpha(\mathbf{x}_\mathbf{j}^{(\varepsilon)})-\phi_\alpha(\mathbf{x}_\mathbf{j}^{(\varepsilon)})).\end{aligned}$$

Then, analogous to the derivation of (4.19), we get

$$\lim_{h\to 0}\lim_{\varepsilon\to 0}\sum_{\alpha\in\Lambda}\sum_{i,j}\,\sum_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}\langle C_\varepsilon^{ij}(\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})-\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})),\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})-\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{j}^{(\varepsilon)})\rangle=0.$$

The case when $\mathbf{x}_\mathbf{i}^{(\varepsilon)}\in K_{h'}^{x_\alpha}$ and $\mathbf{x}_\mathbf{j}^{(\varepsilon)}\in K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}$ can be considered similarly. Thus we have shown that the second term in the RHS of (4.18) vanishes:

$$\lim_{h\to 0}\lim_{\varepsilon\to 0}A_{\varepsilon h}=0. \tag{4.20}$$

We now show that the vector-functions $\overline{\mathbf{w}_{\varepsilon\mathbf{h}}(x)}=\sum_{i=1}^{N}\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\cdot\chi_{U_i}(x)$ converge to the vector-function $\mathbf{w}(\mathbf{x})$ in $L_2(\Omega)$. Indeed,

$$\|\overline{\mathbf{w}_{\varepsilon\mathbf{h}}(x)}-\mathbf{w}(x)\|_{L_2(\Omega)}^2=\left\|\sum_{i=1}^{N}\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\cdot\chi_{U_i}(x)-\mathbf{w}(x)\right\|_{L_2(\Omega)}^2$$

$$=\left\|\sum_{i=1}^{N}\left[\mathbf{w}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})+\sum_{\alpha\in\Lambda}\left\{\sum_{n,p=1}^{3}\varepsilon_{np}[\mathbf{w}(\mathbf{x}_\alpha)]\cdot(\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\right.\right.\right.$$

$$\left.\left.\left.-\varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}-\mathbf{x}_\alpha))-\mathbf{g}_\alpha(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\right\}\cdot\phi_\alpha(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\right]\cdot\chi_{U_i}(x)-\mathbf{w}(x)\right\|_{L_2(\Omega)}^2$$

$$\leqslant 2\left\|\sum_{i=1}^{N}\mathbf{w}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\cdot\chi_{U_i}(x)-\mathbf{w}(x)\right\|_{L_2(\Omega)}^2+2\left\|\sum_{i=1}^{N}\left[\sum_{\alpha\in\Lambda}\left\{\sum_{n,p=1}^{3}\varepsilon_{np}[\mathbf{w}(\mathbf{x}_\alpha)]\cdot(\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\right.\right.\right.$$

$$\left.\left.\left.-\varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}-\mathbf{x}_\alpha))-\mathbf{g}_\alpha(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\right\}\cdot\phi_\alpha(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\right]\cdot\chi_{U_i}(x)\right\|_{L_2(\Omega)}^2$$

$$
\begin{aligned}
&\leqslant 2\sum_{i=1}^{N}\|\mathbf{w}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)})-\mathbf{w}(x)\|_{L_2(U_i)}^2+2\Bigg\|\sum_{\alpha\in\Lambda}\sum_{i\;K_h^{x\alpha}}\Bigg\{\sum_{n,p=1}^{3}\varepsilon_{np}[\mathbf{w}(\mathbf{x}_\alpha)]\cdot(\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)})\\
&\quad-\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}-\mathbf{x}_\alpha))-\mathbf{g}_\alpha(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)})\Bigg\}\cdot\chi_{U_i}(x)\Bigg\|_{L_2(\Omega)}^2\leqslant 2c\sum_{i=1}^{N}\max_{\mathbf{x}\in U_i}|\mathbf{x}-\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}|^2\cdot|U_i|\\
&\quad+2\sum_{j=1}^{N}\Bigg\|\sum_{\alpha\in\Lambda}\sum_{i\;K_h^{x\alpha}}\Bigg\{\sum_{n,p=1}^{3}\varepsilon_{np}[\mathbf{w}(x_\alpha)]\cdot(\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)})-\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)}-\mathbf{x}_\alpha))-\mathbf{g}_\alpha(\mathbf{x}_{\mathbf{i}}^{(\varepsilon)})\Bigg\}\\
&\quad\cdot\chi_{U_i}(x)\Bigg\|_{L_2(U_j)}^2\leqslant\frac{2c}{\varepsilon^3}\cdot\varepsilon^2\cdot\varepsilon^3+2c\sum_{j=1}^{N}\Bigg\|\sum_{n,p=1}^{3}\varepsilon_{np}[\mathbf{w}(\mathbf{x}_\alpha)]\\
&\quad\cdot(\mathbf{v}_{\alpha_j}^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})-\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}-\mathbf{x}_{\alpha_j}))-\mathbf{g}_{\alpha_j}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})\Bigg\|_{L_2(U_j)}^2\\
&\leqslant 2c\cdot\varepsilon^2+2c\sum_{j=1}^{N}\Bigg\{\sum_{n,p=1}^{3}\|\mathbf{v}_{\alpha_j}^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})-\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}-\mathbf{x}_{\alpha_\mathbf{j}})\|_{L_2(U_j)}^2+\|\mathbf{g}_{\alpha_\mathbf{j}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})\|_{L_2(U_j)}^2\Bigg\},\qquad(4.21)
\end{aligned}
$$

where α_j is the index of a cube which contains the jth particle. Clearly, the term $2c\varepsilon^2$ vanishes as $\varepsilon\to 0$. We next show that the second term in the RHS of the last inequality in (4.21) also vanishes as $\varepsilon\to 0$:

$$
\begin{aligned}
&\sum_{j=1}^{N}\Bigg\{\sum_{n,p=1}^{3}\|\mathbf{v}_{\alpha_j}^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})-\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}-\mathbf{x}_{\alpha_\mathbf{j}})\|_{L_2(U_j)}^2+\|\mathbf{g}_{\alpha_\mathbf{j}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})\|_{L_2(U_j)}^2\Bigg\}\\
&\quad=\sum_{j=1}^{N}\Bigg\{\sum_{n,p=1}^{3}|\mathbf{v}_{\alpha_\mathbf{j}}^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})-\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}-\mathbf{x}_{\alpha_\mathbf{j}})|^2+|\mathbf{g}_{\alpha_\mathbf{j}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})|^2\Bigg\}\cdot|U_i|\\
&\quad\leqslant c\sum_{j=1}^{N}\Bigg\{\sum_{n,p=1}^{3}|\mathbf{v}_{\alpha_\mathbf{j}}^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})-\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}-\mathbf{x}_{\alpha_\mathbf{j}})|^2+O(h^4)\Bigg\}\cdot\varepsilon^3\\
&\quad\leqslant c\sum_{n,p=1}^{3}\varepsilon^3\sum_{j=1}^{N}|\mathbf{v}_{\alpha_\mathbf{j}}^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})-\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}-\mathbf{x}_{\alpha_\mathbf{j}})|^2+c\cdot N\cdot O(h^4)\cdot\varepsilon^3.
\end{aligned}
$$

Since $N=\varepsilon^{-3}$, the term $c\cdot N\cdot O(h^4)\cdot\varepsilon^3$ vanishes as $h\to 0$. Next we observe that (4.11) implies

$$
\begin{aligned}
\varepsilon^3\sum_{j=1}^{N}|\mathbf{v}_{\alpha_\mathbf{j}}^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})-\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}-\mathbf{x}_{\alpha_\mathbf{j}})|^2&\leqslant\varepsilon^3\sum_{\alpha\in\Lambda}\sum_{j\;K_h^{x\alpha}}|\mathbf{v}_\alpha^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)})-\varphi^{\mathbf{np}}(\mathbf{x}_{\mathbf{j}}^{(\varepsilon)}-\mathbf{x}_\alpha)|^2\\
&=\sum_{\alpha\in\Lambda}O(h^{5+\gamma})\leqslant c\frac{|\Omega|}{h^3}\cdot O(h^{5+\gamma})\xrightarrow{h\to 0}0.
\end{aligned}
$$

Thus we have shown that

$$
\lim_{h\to 0}\lim_{\varepsilon\to 0}\|\overline{\mathbf{w}_{\varepsilon\mathbf{h}}(x)}-\mathbf{w}(x)\|_{L_2(\Omega)}=0. \qquad (4.22)
$$

We now pass to the limit as $\varepsilon \to 0$ and $h \to 0$ in the second term of (4.15). We have

$$\sum_{i=1}^{N} m_i^{(\varepsilon)}\langle \mathbf{a}_\mathbf{i}^{(\varepsilon)}, \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\rangle = \sum_{i=1}^{N} \overline{\rho_\varepsilon(\mathbf{x}_\mathbf{i}^{(\varepsilon)})}\langle \overline{\mathbf{a}_\varepsilon(\mathbf{x}_\mathbf{i}^{(\varepsilon)})}, \overline{\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})}\rangle \cdot |U_i| = \int_\Omega \overline{\rho_\varepsilon(x)}\langle \overline{\mathbf{a}_\varepsilon(x)}, \overline{\mathbf{w}_{\varepsilon\mathbf{h}}(x)}\rangle \, \mathrm{d}x.$$

Now using (4.1), (4.2), (4.22) and the last equality, we obtain

$$\lim_{h\to 0}\lim_{\varepsilon\to 0}\sum_{i=1}^{N} m_i^{(\varepsilon)}\langle \mathbf{a}_\mathbf{i}^{(\varepsilon)}, \mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})\rangle = \int_\Omega \rho(x)\langle \mathbf{a}(x), \mathbf{w}(x)\rangle \, \mathrm{d}x. \tag{4.23}$$

Analogously,

$$\lim_{h\to 0}\lim_{\varepsilon\to 0}\sum_{i=1}^{N} m_i^{(\varepsilon)}|\mathbf{w}_{\varepsilon\mathbf{h}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})|^2 = \int_\Omega \rho(x)|\mathbf{w}(x)|^2 \, \mathrm{d}x. \tag{4.24}$$

Combining (4.15)–(4.17), (4.20), (4.23) and (4.24) we obtain the upper bound (4.6).

(ii) We now establish the lower bound (4.8).

Assume for simplicity that the vector-function $\mathbf{u}(x)$ defined in (3.13) belongs to the class $C_0^2(\Omega)$. We partition the domain Ω by cubes $K_h^{x_\alpha}$ whose centers form a cubic lattice of period h, and thus the cubes do not overlap:

$$\overline{\Omega} \subset \bigcup_{\alpha\in\Lambda} K_h^{x_\alpha}, \quad K_h^{x_\alpha} \cap K_h^{x_\beta} = \oslash, \quad \alpha \neq \beta.$$

Note that these cubes partition the domain Ω, as opposed to covering it as in (i).

We now construct another quasiminimizer $\mathbf{w}_\varepsilon^\alpha(x)$ in each cube $K_h^{x_\alpha}$ which almost minimizes the interaction (first) term in (3.2) as follows:

$$\mathbf{w}_\varepsilon^\alpha(x) = \mathbf{u}_\varepsilon(x) - \mathbf{u}(x_\alpha) - \sum_{n,p=1}^{3} w_{np}[\mathbf{u}(x_\alpha)]\psi^{\mathbf{np}}(x - x_\alpha).$$

Then

$$\min_{\mathbf{v}} E_{K_h^{x_\alpha}}^{\varepsilon}(\mathbf{v}) = \sum_{n,p,q,r=1}^{3} a_{npqr}(x_\alpha, \varepsilon, h, \gamma)T_{np}T_{qr} \leqslant E_{K_h^{x_\alpha}}^{\varepsilon}(\mathbf{w}_\varepsilon^\alpha).$$

Choosing $T_{np} = \varepsilon_{np}[\mathbf{u}(x_\alpha)]$, we get

$$\begin{aligned}
&\sum_{n,p,q,r=1}^{3} a_{npqr}(x_\alpha, \varepsilon, h, \gamma)\varepsilon_{np}[\mathbf{u}(x_\alpha)]\varepsilon_{qr}[\mathbf{u}(x_\alpha)] \\
&\quad \leqslant \frac{1}{2}\sum_{i,j\ K_h^{x_\alpha}} \langle C_\varepsilon^{ij}(\mathbf{w}_\varepsilon^\alpha(x_i^{(\varepsilon)}) - \mathbf{w}_\varepsilon^\alpha(x_j^{(\varepsilon)})), \mathbf{w}_\varepsilon^\alpha(x_i^{(\varepsilon)}) - \mathbf{w}_\varepsilon^\alpha(x_j^{(\varepsilon)})\rangle \\
&\qquad + h^{-2-\gamma}\varepsilon^3\sum_{i\ K_h^{x_\alpha}} \left| \mathbf{w}_\varepsilon^\alpha(x_i^{(\varepsilon)}) - \sum_{n,p=1}^{3} \varepsilon_{np}[\mathbf{u}(x_\alpha)]\varphi^{\mathbf{np}}(x_i^{(\varepsilon)} - x_\alpha) \right|^2.
\end{aligned} \tag{4.25}$$

Using (1.3) and the obvious equality

$$\mathbf{w}_\varepsilon^\alpha(x_i^{(\varepsilon)}) - \mathbf{w}_\varepsilon^\alpha(x_j^{(\varepsilon)}) = \mathbf{u}_\varepsilon(x_i^{(\varepsilon)}) - \mathbf{u}_\varepsilon(x_j^{(\varepsilon)}) - \sum_{n,p=1}^{3} w_{np}[\mathbf{u}(x_\alpha)]\psi^{\mathbf{np}}(x_i^{(\varepsilon)} - \mathbf{x}_\mathbf{j}^{(\varepsilon)}),$$

we obtain

$$\langle C_\varepsilon^{ij}(\mathbf{w}_\varepsilon^\alpha(x_i^{(\varepsilon)}) - \mathbf{w}_\varepsilon^\alpha(x_j^{(\varepsilon)})), \mathbf{w}_\varepsilon^\alpha(x_i^{(\varepsilon)}) - \mathbf{w}_\varepsilon^\alpha(x_j^{(\varepsilon)})\rangle = \langle C_\varepsilon^{ij}(\mathbf{u}_\varepsilon(x_i^{(\varepsilon)}) - \mathbf{u}_\varepsilon(x_j^{(\varepsilon)})), \mathbf{u}_\varepsilon(x_i^{(\varepsilon)}) - \mathbf{u}_\varepsilon(x_j^{(\varepsilon)})\rangle.$$

We now evaluate the second term of the RHS of inequality (4.25):

$$\begin{aligned}
&\left|\mathbf{w}_\varepsilon^\alpha(x_i^{(\varepsilon)}) - \sum_{n,p=1}^{3} \varepsilon_{np}[\mathbf{u}(x_\alpha)]\varphi^{\mathbf{np}}(x_i^{(\varepsilon)} - x_\alpha)\right|^2 \\
&\quad = \Bigg|\mathbf{u}_\varepsilon(x_i^{(\varepsilon)}) - \mathbf{u}(x_\alpha) - \sum_{n,p=1}^{3} w_{np}[\mathbf{u}(x_\alpha)]\psi^{\mathbf{np}}(x_i^{(\varepsilon)} - x_\alpha) \\
&\qquad - \sum_{n,p=1}^{3} \varepsilon_{np}[\mathbf{u}(x_\alpha)]\varphi^{\mathbf{np}}(x_i^{(\varepsilon)} - x_\alpha)\Bigg|^2 \\
&\quad \leqslant 2|\mathbf{u}_\varepsilon(x_i^{(\varepsilon)}) - \mathbf{u}(x_i^{(\varepsilon)})|^2 + 2\Bigg|\mathbf{u}(x_i^{(\varepsilon)}) - \mathbf{u}(x_\alpha) \\
&\qquad - \sum_{n,p=1}^{3} \varepsilon_{np}[\mathbf{u}(x_\alpha)]\varphi^{\mathbf{np}}(x_i^{(\varepsilon)} - x_\alpha) - \sum_{n,p=1}^{3} w_{np}[\mathbf{u}(x_\alpha)]\psi^{\mathbf{np}}(x_i^{(\varepsilon)} - x_\alpha)\Bigg|^2 \\
&\quad = 2|\mathbf{u}_\varepsilon(x_i^{(\varepsilon)}) - \mathbf{u}(x_i^{(\varepsilon)})|^2 + O(h^4).
\end{aligned}$$

Summing up inequality (4.25) over all cubes of our partition and passing to the limit as $\varepsilon \to 0$ and $h \to 0$, we obtain

$$\begin{aligned}
&\lim_{h\to 0}\underline{\lim}_{\varepsilon\to 0}\sum_{\alpha\in\Lambda}\sum_{n,p,q,r=1}^{3} a_{npqr}(x_\alpha,\varepsilon,h,\gamma)\varepsilon_{np}[\mathbf{u}(x_\alpha)]\varepsilon_{qr}[\mathbf{u}(x_\alpha)] \\
&\quad = \int_\Omega \sum_{n,p,q,r=1}^{3} a_{npqr}(x)\varepsilon_{np}[\mathbf{u}(x)]\varepsilon_{qr}[\mathbf{u}(x)]\,\mathrm{d}x \\
&\quad \leqslant \lim_{h\to 0}\underline{\lim}_{\varepsilon\to 0}\Bigg\{\frac{1}{2}\sum_{i,j}\langle C_\varepsilon^{ij}(\mathbf{u}_\varepsilon(x_i^{(\varepsilon)}) - \mathbf{u}_\varepsilon(x_j^{(\varepsilon)})), \mathbf{u}_\varepsilon(x_i^{(\varepsilon)}) - \mathbf{u}_\varepsilon(x_j^{(\varepsilon)})\rangle \\
&\qquad + 2h^{-2-\gamma}\varepsilon^3\cdot\sum_{i=1}^{N}|\mathbf{u}_\varepsilon(x_i^{(\varepsilon)}) - \mathbf{u}(x_i^{(\varepsilon)})|^2 + h^{-2-\gamma}\varepsilon^3\cdot\sum_{i=1}^{N}O(h^4)\Bigg\}.
\end{aligned}$$

Obviously, the last term is of order $h^{-2-\gamma}\cdot O(h^4)$, and it vanishes in the limit as $\varepsilon \to 0$, since $0<\gamma<2$. The second term is equal to zero in the limit as $\varepsilon \to 0$, due to (3.13) and the following lemma:

Lemma 1. *If the sequence of splines* $\mathbf{u}_\varepsilon(\mathbf{x})$ *defined by* (3.4) *is bounded uniformly on* ε *in* $H^1(\Omega)$, *then the sequence of the piecewise-constant vector-functions* $\overline{\mathbf{u}_\varepsilon}(\mathbf{x}) = \sum_{i=1}^{N}\mathbf{u}_\mathbf{i}^{(\varepsilon)}\cdot\chi_{U_i}(\mathbf{x})$ *converges in* $L_2(\Omega)$ *to the vector-function* $\mathbf{u}(\mathbf{x})$ *defined by* (3.13).

Proof of this lemma is presented in Appendix A.

Using Lemma 1, we now obtain

$$
\begin{aligned}
\varepsilon^3 \sum_{i=1}^{N} |\mathbf{u}_\varepsilon(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{u}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})|^2 &= \varepsilon^3 \sum_{i=1}^{N} |\mathbf{u}_\varepsilon(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \underline{\mathbf{u}_\varepsilon}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})|^2 \\
&\leqslant c \sum_{i=1}^{N} |\overline{\mathbf{u}_\varepsilon}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \underline{\mathbf{u}_\varepsilon}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})|^2 \cdot |U_i| \\
&= c \sum_{i=1}^{N} \| \overline{\mathbf{u}_\varepsilon} - \underline{\mathbf{u}_\varepsilon} \|^2_{L_2(U_i)} = c \| \overline{\mathbf{u}_\varepsilon} - \underline{\mathbf{u}_\varepsilon} \|^2_{L_2(\Omega)} \leqslant 2c(\| \overline{\mathbf{u}_\varepsilon} - \mathbf{u} \|^2_{L_2(\Omega)} + \| \mathbf{u} - \underline{\mathbf{u}_\varepsilon} \|^2_{L_2(\Omega)}),
\end{aligned}
$$

where $\underline{\mathbf{u}_\varepsilon}(\mathbf{x}) = \sum_{i=1}^{N} \mathbf{u}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) \cdot \chi_{U_i}(\mathbf{x})$. Clearly $\| \overline{\mathbf{u}_\varepsilon} - \mathbf{u} \|^2_{L_2(\Omega)} \to 0$ as $\varepsilon \to 0$, due to Lemma 1. For the term $\| \mathbf{u} - \underline{\mathbf{u}_\varepsilon} \|^2_{L_2(\Omega)}$, we have

$$
\begin{aligned}
\sum_{i=1}^{N} \| \mathbf{u} - \underline{\mathbf{u}_\varepsilon} \|^2_{L_2(U_i)} &= \sum_{i=1}^{N} \int_{U_i} |\mathbf{u}(x) - \mathbf{u}(\mathbf{x}_\mathbf{i}^{(\varepsilon)})|^2 \, \mathrm{d}x \\
&\leqslant c \sum_{i=1}^{N} \max_{\mathbf{x} \in U_i} |\mathbf{x} - \mathbf{x}_\mathbf{i}^{(\varepsilon)}|^2 \cdot |U_i| \leqslant c \sum_{i=1}^{N} \varepsilon^5 = c\varepsilon^2 \overset{\varepsilon \to 0}{\to} 0.
\end{aligned}
$$

Thus we have shown that

$$
\begin{aligned}
&\int_\Omega \sum_{n,p,q,r=1}^{3} a_{npqr}(x) \varepsilon_{np}[\mathbf{u}(x)] \varepsilon_{qr}[\mathbf{u}(x)] \, \mathrm{d}x \\
&\quad \leqslant \varliminf_{\varepsilon \to 0} \frac{1}{2} \sum_{i,j} \langle C_\varepsilon^{ij}(\mathbf{u}_\varepsilon(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{u}_\varepsilon(\mathbf{x}_\mathbf{j}^{(\varepsilon)})), \mathbf{u}_\varepsilon(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) - \mathbf{u}_\varepsilon(\mathbf{x}_\mathbf{j}^{(\varepsilon)}) \rangle.
\end{aligned} \tag{4.26}
$$

We now estimate the second and the third terms of functional Φ_ε. We have

$$
\lambda^2 \sum_{i=1}^{N} m_i^{(\varepsilon)} |\mathbf{u}_\mathbf{i}^{(\varepsilon)}|^2 - 2 \sum_{i=1}^{N} m_i^{(\varepsilon)} \langle \mathbf{a}_\mathbf{i}^{(\varepsilon)}, \mathbf{u}_\mathbf{i}^{(\varepsilon)} \rangle = \int_\Omega [\lambda^2 \overline{\rho_\varepsilon}(x) |\overline{\mathbf{u}_\varepsilon}(x)|^2 - 2\overline{\rho_\varepsilon}(x) \langle \overline{\mathbf{a}_\varepsilon}(x), \overline{\mathbf{u}_\varepsilon}(x) \rangle] \, \mathrm{d}x.
$$

Then, taking into consideration (4.1),(4.2) and using Lemma 1, we conclude that

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \left\{ \lambda^2 \sum_{i=1}^{N} m_i^{(\varepsilon)} |\mathbf{u}_\mathbf{i}^{(\varepsilon)}|^2 - 2 \sum_{i=1}^{N} m_i^{(\varepsilon)} \langle \mathbf{a}_\mathbf{i}^{(\varepsilon)}, \mathbf{u}_\mathbf{i}^{(\varepsilon)} \rangle \right\} \\
&\quad = \int_\Omega [\lambda^2 \rho(x) |\mathbf{u}(x)|^2 - 2\rho(x) \langle \mathbf{a}(x), \overline{\mathbf{u}}(x) \rangle] \, \mathrm{d}x.
\end{aligned} \tag{4.27}
$$

Combining (4.26) and (4.27), we get

$$
\Phi_0[\mathbf{u}] \leqslant \varliminf_{\varepsilon \to 0} \Phi_\varepsilon[\mathbf{u}_\varepsilon], \tag{4.28}
$$

provided that the limiting vector-function $\mathbf{u}(x)$ is smooth ($C_0^2(\Omega)$).

If $\mathbf{u}(x) \in H_0^1(\Omega)$, we use a standard approximation procedure: since the domain Ω has a C^1-boundary, for any vector-function $\mathbf{u}(x) \in H_0^1(\Omega)$ there exists a sequence of vector-functions $\mathbf{u}_\delta(x) \in C_0^2(\Omega)$ (Mizohata, 1973) such that

$$
\lim_{\delta \to 0} \| \mathbf{u}_\delta - \mathbf{u} \|_{H^1(\Omega)} = 0. \tag{4.29}
$$

We now use the following:

Lemma 2. *For any vector-function* $\mathbf{u}_\delta(x) \in C_0^2(\Omega)$, *there exists a sequence of vector-functions* $\mathbf{u}_{\varepsilon\delta}(x) \in H_0^1(\Omega)$ *of the form* (3.4) *such that*

$$(1)\ \lim_{\varepsilon\to 0} \|\mathbf{u}_{\varepsilon\delta} - \mathbf{u}_\delta\|_{L_2(\Omega)} = 0; \quad (2)\ \lim_{\delta\to 0} \|\mathbf{u}_{\varepsilon\delta} - \mathbf{u}_\varepsilon\|_{H^1(\Omega)} = 0; \quad (3)\ \lim_{\varepsilon\to 0} \|\bar{\mathbf{u}}_{\varepsilon\delta} - \mathbf{u}_\delta\|_{L_2(\Omega)} = 0.$$

Proof of this lemma is presented in Appendix B.

It follows from (4.28) and Lemma 2 that

$$\Phi_0[\mathbf{u}_\delta] \leqslant \varliminf_{\varepsilon\to 0} \Phi_\varepsilon[\mathbf{u}_{\varepsilon\delta}].$$

Let $\delta \to 0$. Then, taking into account (4.29) and (2) from Lemma 2, we obtain the lower bound (4.8), since the functionals $\Phi_0[\mathbf{u}]$ and $\Phi_\varepsilon[\mathbf{u}_\varepsilon]$ are continuous in the space $H^1(\Omega)$. Finally, we combine (4.6) and (4.8) to obtain

$$\Phi_0[\mathbf{u}] \leqslant \varliminf_{\varepsilon\to 0} \Phi_\varepsilon[\mathbf{u}_\varepsilon] \leqslant \lim_{h\to 0} \varlimsup_{\varepsilon\to 0} \Phi_\varepsilon[\mathbf{w}_{\varepsilon\mathbf{h}}] \leqslant \Phi_0[\mathbf{w}], \quad \forall \mathbf{w} \in C_0^2(\Omega).$$

Using the smoothing approximation (4.29), we conclude that

$$\Phi_0[\mathbf{u}] \leqslant \Phi_0[\mathbf{w}], \quad \forall \mathbf{w} \in H_0^1(\Omega).$$

5. The homogenization theorem

Clearly, the limiting vector-function $\mathbf{u}(x)$ defined in (3.13) satisfies the Euler–Lagrange equations

$$\begin{cases} -\sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_q}\{a_{npqr}(x)\varepsilon_{np}[\mathbf{u}(x)]\}\mathbf{e_r} + \lambda^2\rho(x)\mathbf{u}(x) = \rho(x)\mathbf{a}(x), & x \in \Omega, \\ \mathbf{u}(x) = \mathbf{0}, \quad x \in \partial\Omega. \end{cases} \tag{5.1}$$

It is not difficult to show that this problem has a unique solution. The consequence of this is the fact that the whole sequence $\mathbf{u}_\varepsilon(x)$ converges to the uniquely defined vector-function $\mathbf{u}(x)$ in $L_2(\Omega)$ when $\lambda > 0$.

By taking the inverse Laplace transform we show that $\mathbf{u}(\mathbf{x}, t) = L^{-1}(\mathbf{u}(\mathbf{x}, \lambda))$ satisfies the following initial boundary value problem:

$$\begin{cases} \rho(\mathbf{x})\frac{\partial^2\mathbf{u}(\mathbf{x}, t)}{\partial t^2} - \sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_q} a_{npqr}(\mathbf{x})\varepsilon_{np}[\mathbf{u}(\mathbf{x}, t)]\mathbf{e_r} = 0, & \mathbf{x} \in \Omega,\ t > 0, \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad \mathbf{x} \in \partial\Omega,\ t \geqslant 0, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{0}, \left.\frac{\partial\mathbf{u}(\mathbf{x}, t)}{\partial t}\right|_{t=0} = a(x), \quad \mathbf{x} \in \Omega. \end{cases} \tag{5.2}$$

Note that the functions $\mathbf{u}_\varepsilon(\mathbf{x}, \lambda)$ and $\mathbf{u}(\mathbf{x}, \lambda)$ were defined for real λ only. In order to apply L^{-1} we need to extend these functions into $Re\,\lambda > 0$ and verify that the extended functions have the appropriate rate of decay as $|\lambda| \to \infty$ (see Appendix C for details). Thus we prove the following:

Theorem 3. *Let* $\mathbf{u}_\varepsilon$ *be a solution of the problem* (1.6)–(1.9). *Suppose that the array of the network* (*particles* $\mathbf{x}_\mathbf{i}^{(\varepsilon)}$) *satisfies the non-periodic triangulization condition in the sense of* Definition 1, *and the limits* (2.7), (4.1), (4.2) *exist. Then the solution* $\mathbf{u}_\varepsilon(\mathbf{x}_\mathbf{i}^{(\varepsilon)}, t) = \mathbf{u}_\mathbf{i}^{(\varepsilon)}(t)\,(i \in \overline{1,N})$ *of the discrete problem* (1.6)–(1.8) *converges to the solution* $\mathbf{u}(\mathbf{x}, t)$ *of the homogenized continuum problem* (5.2) *in the following sense*: *the linear interpolation of the solution of the discrete problem* $\mathbf{u}_\varepsilon(\mathbf{x}, t) = \sum_{i=1}^{N} \mathbf{u}_\mathbf{i}^{(\varepsilon)}(t) \cdot L_\varepsilon^i(\mathbf{x})$ (*see* (3.4))*converges weakly in* $L_2(\Omega \times [0, T])$ (*for any* $T > 0$) *to the vector-function* $\mathbf{u}(\mathbf{x}, t)$.

6. Explicit formulae for the elastic moduli of a periodic array of particles

We now show the existence of the limit (2.7) for a particular example of a periodic cubic lattice. This periodic example recovers previously known formulae (obtained by the Cauchy–Born rule), but in addition it clearly states the limits of validity of these formulae which were not obtained in previous derivations by the Cauchy–Born rule. As the recent analysis by Friesecke and Theil shows, this rule may fail and thus a rigorous mathematical derivation of the limits of validity is necessary.

We consider a cubic lattice in which each vertex is connected by a spring to its nearest neighbors *NN* (the edges of the unit periodicity cube), to its next nearest neighbors *NNN* (the diagonals of the faces of the cube) and to its next-to-next nearest neighbors *NNNN* (the diagonals of the cube). For simplicity, we assume that the elastic constants of all these springs are the same. It is straightforward to carry out our calculation for a more realistic case when there are three different spring constants (macroelasticity parameters) for *NN*, *NNN* and *NNNN*. Our basic periodicity cell is depicted on Fig. 1, where it is shown that each vertex is connected to $3^3 - 1 = 26$ vertices in the lattice.

On this figure a fixed point $\mathbf{x}_\mathbf{i} = (0, 0, 0)$ is shown as a large dark ball and all its neighbors $\mathbf{x}_\mathbf{j}$ are shown as smaller balls. The coordinates of these neighbors are $x_j^k = -\varepsilon$, 0 or ε.

We prove the following:

Theorem 4. *For the cubic lattice described above* (*see also Fig.* 1), *the elastic moduli in the homogenized equation* (5.2) *are constants and are given by the following formulae*:

$$a_{nnnn} = k\left(1 + \sqrt{2} + \tfrac{4}{9}\sqrt{3}\right), \quad a_{nnpp} = a_{npnp} = k\left(\tfrac{1}{2}\sqrt{2} + \tfrac{4}{9}\sqrt{3}\right), \quad n, p \in \overline{1, 3},$$

$a_{npqr} = 0$ *in all other cases*; $k = k^{ij}$ (*see* (1.5)).

Remark. If we introduce the notations $\lambda = a_{nnnn}$, $\mu = a_{nnpp} = a_{npnp}$, then the elasticity equations (5.2) have the form

$$\begin{cases} \rho(x)\dfrac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2} - (\lambda - 3\mu)\displaystyle\sum_{r=1}^{3} \dfrac{\partial^2 u_r(\mathbf{x}, t)}{\partial x_r^2}\mathbf{e}_\mathbf{r} - \mu\Delta\mathbf{u}(\mathbf{x}, t) - 2\mu\nabla div\, \mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad \mathbf{x} \in \Omega, \\ \quad t > 0, \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad \mathbf{x} \in \partial\Omega, \quad t \geqslant 0, \\ \mathbf{u}(\mathbf{x}, 0) = 0, \left.\dfrac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t}\right|_{t=0} = \mathbf{a}(x), \quad \mathbf{x} \in \Omega. \end{cases}$$

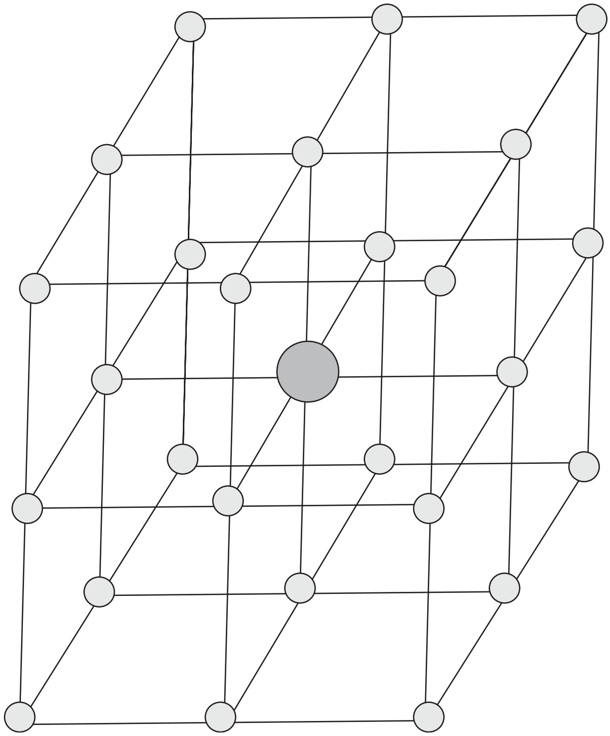

Fig. 1. The basic periodicity cell.

Proof of Theorem 4. We compute the coefficients $a_{npqr}(\mathbf{y})$ using formulae (2.6)–(2.7).

Taking into consideration (1.3) and the definition of $\varphi^{np}(x)$, we see that for particles which are not located near the faces of the cube K_h^y, the vector $\mathbf{w}_\mathbf{i}^{\mathbf{np}}$ coincides with $\varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{y})$ (see (2.3)). Thus for an arbitrary particle $\mathbf{x}_i^{(\varepsilon)}$ we look for a solution of the Eq. (2.3) of the form

$$\mathbf{w}_\mathbf{i}^{\mathbf{np}} = \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{y}) + \mathbf{v}_\mathbf{i}, \tag{6.1}$$

where $\mathbf{v}_\mathbf{i} = \mathbf{0}$ for internal particles. We call a particle $\mathbf{x}_i^{(\varepsilon)}$ internal if all 26 particles connected to $\mathbf{x}_i^{(\varepsilon)}$ lie inside the cube K_h^y. For a particle $\mathbf{x}_i^{(\varepsilon)} \in K_h^y$ which is not internal (hereafter a *boundary particle*), $\mathbf{v}_\mathbf{i} \neq \mathbf{0}$. Our strategy is to substitute (6.1) into (2.6) and show that contribution of $\mathbf{v}_\mathbf{i}$ vanishes in the limit (2.7). Thus the calculation of $a_{npqr}(\mathbf{y})$ amounts to establishing the following relation:

$$a_{npqr}(\mathbf{y}) = \lim_{h\to 0}\lim_{\varepsilon\to 0} \frac{\frac{1}{2}\sum_{i,j\,K_h^y} \langle C_\varepsilon^{ij} \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\mathbf{j}^{(\varepsilon)}), \varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\mathbf{j}^{(\varepsilon)})\rangle}{h^3}. \tag{6.2}$$

Substituting (6.1) into (2.6), we obtain

$$a_{npqr}(\mathbf{y}) = \lim_{h\to 0}\lim_{\varepsilon\to 0} \frac{\frac{1}{2}\sum_{i,j\,K_h^y} \langle C_\varepsilon^{ij} \varphi^{\mathbf{np}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\mathbf{j}^{(\varepsilon)}), \varphi^{\mathbf{qr}}(\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_\mathbf{j}^{(\varepsilon)})\rangle}{h^3}$$

$$
\begin{aligned}
&+\frac{1}{2}\lim_{h\to 0}\lim_{\varepsilon\to 0}\left\{\sum_{i,j\ K_h^y}\langle C_\varepsilon^{ij}(\mathbf{v_i}-\mathbf{v_j}),\varphi^{\mathbf{qr}}(\mathbf{x_i}^{(\varepsilon)}-\mathbf{x_j}^{(\varepsilon)})\rangle\right.\\
&+\sum_{i,j\ K_h^y}\langle C_\varepsilon^{ij}\varphi^{\mathbf{np}}(\mathbf{x_i}^{(\varepsilon)}-\mathbf{x_j}^{(\varepsilon)}),\mathbf{v_i}-\mathbf{v_j}\rangle\\
&\left.+\sum_{i,j\ K_h^y}\langle C_\varepsilon^{ij}(\mathbf{v_i}-\mathbf{v_j}),\mathbf{v_i}-\mathbf{v_j}\rangle+h^{-2-\gamma}\varepsilon^3\sum_{i\ K_h^y}|\mathbf{v_i}|^2\right\}\cdot\frac{1}{h^3}.
\end{aligned}
\tag{6.3}
$$

We next show that

$$
\begin{aligned}
&\left\{\sum_{i,j\ K_h^y}\langle C_\varepsilon^{ij}(\mathbf{v_i}-\mathbf{v_j}),\varphi^{\mathbf{qr}}(\mathbf{x_i}^{(\varepsilon)}-\mathbf{x_j}^{(\varepsilon)})\rangle+\sum_{i,j\ K_h^y}\langle C_\varepsilon^{ij}\varphi^{\mathbf{np}}(\mathbf{x_i}^{(\varepsilon)}-\mathbf{x_j}^{(\varepsilon)}),\mathbf{v_i}-\mathbf{v_j}\rangle\right.\\
&\left.+\sum_{i,j\ K_h^y}\langle C_\varepsilon^{ij}(\mathbf{v_i}-\mathbf{v_j}),\mathbf{v_i}-\mathbf{v_j}\rangle+h^{-2-\gamma}\varepsilon^3\sum_{i\ K_h^y}|\mathbf{v_i}|^2\right\}\cdot\frac{1}{h^3}\leqslant ch^{\frac{\gamma}{2}}
\end{aligned}
\tag{6.4}
$$

(the first term in (6.3) is a finite number).

We first estimate the following quantity:

$$
A=\sum_{i,j\ K_h^y}\langle C_\varepsilon^{ij}(\mathbf{v_i}-\mathbf{v_j}),\mathbf{v_i}-\mathbf{v_j}\rangle+h^{-2-\gamma}\varepsilon^3\sum_{i\ K_h^y}|\mathbf{v_i}|^2\leqslant c\cdot h\cdot\varepsilon\cdot\left(\sum_{i\ K_h^y}{}^{'}|\mathbf{v_i}|^2\right)^{1/2},
\tag{6.5}
$$

where $\sum'_{i K_h^y}$ stands for the summation over the boundary particles. Using the inequality $E^{\varepsilon(np)}_{K_h^y}(\mathbf{w_i^{np}})\leqslant E^{\varepsilon(np)}_{K_h^y}(\varphi^{\mathbf{np}}(\mathbf{x_i}^{(\varepsilon)}-\mathbf{y}))$ together with (2.2) and (6.1), we obtain

$$
A\leqslant\left|\sum_{i,j\ K_h^y}\langle C_\varepsilon^{ij}\varphi^{\mathbf{np}}(\mathbf{x_i}^{(\varepsilon)}-\mathbf{x_j}^{(\varepsilon)}),\mathbf{v_i}-\mathbf{v_j}\rangle\right|+\left|\sum_{i,j\ K_h^y}\langle C_\varepsilon^{ij}(\mathbf{v_i}-\mathbf{v_j}),\varphi^{\mathbf{np}}(\mathbf{x_i}^{(\varepsilon)}-\mathbf{x_j}^{(\varepsilon)})\rangle\right|.
\tag{6.6}
$$

Then

$$
\begin{aligned}
A&\leqslant 2\left|\sum_{i,j\ K_h^y}\langle C_\varepsilon^{ij}\varphi^{\mathbf{np}}(\mathbf{x_i}^{(\varepsilon)}-\mathbf{x_j}^{(\varepsilon)}),\mathbf{v_i}-\mathbf{v_j}\rangle\right|\\
&=4\left|\sum_{i,j\ K_h^y}{}^{'}\langle C_\varepsilon^{ij}\varphi^{\mathbf{np}}(\mathbf{x_i}^{(\varepsilon)}-\mathbf{x_j}^{(\varepsilon)}),\mathbf{v_i}\rangle\right|\leqslant 4c\cdot\varepsilon^2\cdot\sum_{i\ K_h^y}{}^{'}|\mathbf{v_i}|,
\end{aligned}
\tag{6.7}
$$

where the first inequality follows from the symmetry of the operators C_ε^{ij}, and the equality follows from the fact that $C_\varepsilon^{ij}=C_\varepsilon^{ji}$ and $\mathbf{v_i}=\mathbf{0}$ for the internal particles. Finally, the second inequality in (6.7) follows from (1.3). Next,

$$
\sum_{i\ K_h^y}{}^{'}|\mathbf{v_i}|\leqslant\left(\sum_{i\ K_h^y}{}^{'}1\right)^{1/2}\cdot\left(\sum_{i\ K_h^y}{}^{'}|\mathbf{v_i}|^2\right)^{1/2}\leqslant c\cdot\frac{h}{\varepsilon}\cdot\left(\sum_{i\ K_h^y}{}^{'}|\mathbf{v_i}|^2\right)^{1/2}.
$$

Thus, we establish (6.5).

Second, we prove the estimate

$$\varepsilon^2 \cdot \sum_{i\ K_h^z}{}' |\mathbf{v}_\mathbf{i}|^2 \leqslant A \cdot h^{1+\frac{\gamma}{2}}. \tag{6.8}$$

We begin with the obvious equality

$$|\mathbf{v}_\mathbf{i}|^2 = |\mathbf{v}_{\mathbf{k}_\mathbf{i}}|^2 + \sum_{l=1}^{i-k_i} (|\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}}|^2 - |\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}-\mathbf{1}}|^2), \tag{6.9}$$

where $\mathbf{x}_\mathbf{i}^{(\varepsilon)}$ is a particle located near a cube's face F_h^y, and the vector $\mathbf{x}_\mathbf{i}^{(\varepsilon)} - \mathbf{x}_{\mathbf{k}_\mathbf{i}}^{(\varepsilon)}$ and the vectors $\mathbf{x}_{\mathbf{k}_\mathbf{i}+\mathbf{l}}^{(\varepsilon)} - \mathbf{x}_{\mathbf{k}_\mathbf{i}+\mathbf{l}-\mathbf{1}}^{(\varepsilon)}$ ($l = \overline{1, i-k_i}$) are perpendicular to this face.

Now we evaluate $|\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}}|^2 - |\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}-\mathbf{1}}|^2$ using Young's inequality:

$$\begin{aligned} |\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}}|^2 - |\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}-\mathbf{1}}|^2 &= (|\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}}| - |\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}-\mathbf{1}}|) \cdot (|\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}}| + |\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}-\mathbf{1}}|) \\ &\leqslant \mu(|\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}}| - |\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}-\mathbf{1}}|)^2 + \frac{1}{4\mu}(|\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}}| + |\mathbf{v}_{\mathbf{k}_\mathbf{i}+\mathbf{l}-\mathbf{1}}|)^2, \end{aligned} \tag{6.10}$$

where μ is any strictly positive number.

Let $\mathbf{x}_{n_i}^{(\varepsilon)} - \mathbf{x}_i^{(\varepsilon)}$ be the segment perpendicular to the face F_h^y so that $\mathbf{x}_{n_i}^{(\varepsilon)}$ and $\mathbf{x}_{n_i}^{(\varepsilon)}$ lie near the face F_h^y and the opposite face, respectively. The distance between $\mathbf{x}_i^{(\varepsilon)}$ and $\mathbf{x}_{n_i}^{(\varepsilon)}$ is thus of order h.

Recall that $\mathbf{x}_{k_i}^{(\varepsilon)}$ is an internal particle lying on the segment $\mathbf{x}_{n_i}^{(\varepsilon)} - \mathbf{x}_i^{(\varepsilon)}$. Note that in order to sum up over all particles in the cube K_h^y, we first fix a boundary particle $\mathbf{x}_i^{(\varepsilon)}$ and sum over all particles lying in the segment $\mathbf{x}_{n_i}^{(\varepsilon)} - \mathbf{x}_i^{(\varepsilon)}$, and then sum over all boundary particles $\mathbf{x}_i^{(\varepsilon)}$ lying near the face F_h^y (that is, summing over all segments perpendicular to F_h^y). Making such summations in (6.9) and applying (6.10), we get

$$\frac{h}{\varepsilon} \sum_{i\ K_h^y}{}' |\mathbf{v}_\mathbf{i}|^2 \leqslant c \left\{ \sum_{i\ K_h^y} |\mathbf{v}_\mathbf{i}|^2 + \frac{h}{\varepsilon} \cdot \mu \cdot \sum_{i,j\ K_h^y} |\mathbf{v}_\mathbf{i} - \mathbf{v}_\mathbf{j}|^2 + \frac{h}{\varepsilon} \cdot \frac{1}{2\mu} \cdot \sum_{i\ K_h^y} |\mathbf{v}_\mathbf{i}|^2 \right\}, \tag{6.11}$$

where the constant c does not depend on ε or h. This implies that

$$\varepsilon^2 \sum_{i\ K_h^y}{}' |\mathbf{v}_\mathbf{i}|^2 \leqslant c \left\{ \frac{\varepsilon^3}{h} \sum_{i\ K_h^y} |\mathbf{v}_\mathbf{i}|^2 + \varepsilon^2 \cdot \mu \cdot \sum_{i,j\ K_h^y} |\mathbf{v}_\mathbf{i} - \mathbf{v}_\mathbf{j}|^2 + \varepsilon^2 \cdot \frac{1}{\mu} \cdot \sum_{i\ K_h^y} |\mathbf{v}_\mathbf{i}|^2 \right\}. \tag{6.12}$$

We now use the standard Korn's inequality for vector-functions from $H^1(\Omega)$ (see, for example, Duvaut and Lions (1972)):

$$\|\mathbf{v}\|^2_{H^1(\Omega)} \leqslant c \left\{ \int_\Omega \sum_{n,p=1}^{3} \varepsilon_{np}^2[\mathbf{v}(x)]\,\mathrm{d}x + \int_\Omega |\mathbf{v}(x)|^2\,\mathrm{d}x \right\}. \tag{6.13}$$

Following the derivation of (3.5) with (6.13) in place of (3.12), we obtain

$$c\left(\varepsilon \sum_{i,j}{}' |\mathbf{v}_\mathbf{i} - \mathbf{v}_\mathbf{j}|^2 + \varepsilon^3 \sum_i |\mathbf{v}_\mathbf{i}|^2 \right) \leqslant \sum_{i,j} \langle C_\varepsilon^{ij}(\mathbf{v}_\mathbf{i} - \mathbf{v}_\mathbf{j}), \mathbf{v}_\mathbf{i} - \mathbf{v}_\mathbf{j} \rangle + \varepsilon^3 \sum_i |\mathbf{v}_\mathbf{i}|^2. \tag{6.14}$$

Now we choose $\mu = (h^{1+\frac{\gamma}{2}})/\varepsilon$ in (6.10). Since $h<1$, we obtain

$$\begin{aligned}\varepsilon^2 \sideset{}{'}\sum_{i\;K_h^y} |\mathbf{v_i}|^2 &\leqslant c\left\{\frac{\varepsilon^3}{h^{1+\frac{\gamma}{2}}}\sum_{i\;K_h^y}|\mathbf{v_i}|^2 + h^{1+\frac{\gamma}{2}}\sum_{i,j}\langle C_\varepsilon^{ij}(\mathbf{v_i}-\mathbf{v_j}), \mathbf{v_i}-\mathbf{v_j}\rangle\right.\\ &\qquad\left. +h^{1+\frac{\gamma}{2}}\varepsilon^3\sum_{i\;K_h^y}|\mathbf{v_i}|^2 + \frac{\varepsilon^3}{h^{1+\frac{\gamma}{2}}}\sum_{i\;K_h^y}|\mathbf{v_i}|^2\right\}\\ &\leqslant c\left\{\sum_{i,j}\langle C_\varepsilon^{ij}(\mathbf{v_i}-\mathbf{v_j}), \mathbf{v_i}-\mathbf{v_j}\rangle + h^{-2-\gamma}\varepsilon^3\sum_{i\;K_h^y}|\mathbf{v_i}|^2\right\}\cdot h^{1+\frac{\gamma}{2}} = c\cdot A\cdot h^{1+\frac{\gamma}{2}},\end{aligned}$$

which establishes (6.8).

Combining (6.8) and (6.5), we get

$$A \leqslant c\cdot h\cdot\left(\varepsilon^2\sideset{}{'}\sum_{i\;K_h^y}|\mathbf{v_i}|^2\right)^{1/2} \leqslant c\cdot h\cdot\sqrt{A}\cdot h^{\frac{1+\frac{\gamma}{2}}{2}},$$

which implies that

$$A \leqslant ch^{3+\frac{\gamma}{2}}, \quad h\left(\varepsilon^2\sideset{}{'}\sum_{i\;K_h^y}|\mathbf{v_i}|^2\right)^{1/2} \leqslant ch^{3+\frac{\gamma}{2}}. \tag{6.15}$$

Now (6.4) follows from (6.7) and (6.16). Clearly, (6.4) and (6.3) imply (6.2).

To complete the proof of Theorem 4, we fix all values $k^{ij} = k$ and use (1.3) to obtain

$$\sum_j^i \langle C_\varepsilon^{ij}\varphi^{\mathbf{np}}(\mathbf{x_i}^{(\varepsilon)} - \mathbf{x_j}^{(\varepsilon)}), \varphi^{\mathbf{qr}}(\mathbf{x_i}^{(\varepsilon)} - \mathbf{x_j}^{(\varepsilon)})\rangle = 0,$$

$$\sum_j^i \langle C_\varepsilon^{ij}\varphi^{\mathbf{nn}}(\mathbf{x_i}^{(\varepsilon)} - \mathbf{x_j}^{(\varepsilon)}), \varphi^{\mathbf{nn}}(\mathbf{x_i}^{(\varepsilon)} - \mathbf{x_j}^{(\varepsilon)})\rangle = 2k\varepsilon^3\left(1+\sqrt{2}+\frac{4\sqrt{3}}{9}\right),$$

$$\sum_j^i \langle C_\varepsilon^{ij}\varphi^{\mathbf{np}}(\mathbf{x_i}^{(\varepsilon)} - \mathbf{x_j}^{(\varepsilon)}), \varphi^{\mathbf{np}}(\mathbf{x_i}^{(\varepsilon)} - \mathbf{x_j}^{(\varepsilon)})\rangle = k\varepsilon^3\left(\sqrt{2}+\frac{8\sqrt{3}}{9}\right),$$

$$\sum_j^i \langle C_\varepsilon^{ij}\varphi^{\mathbf{nn}}(\mathbf{x_i}^{(\varepsilon)} - \mathbf{x_j}^{(\varepsilon)}), \varphi^{\mathbf{pp}}(\mathbf{x_i}^{(\varepsilon)} - \mathbf{x_j}^{(\varepsilon)})\rangle = k\varepsilon^3\left(\sqrt{2}+\frac{8\sqrt{3}}{9}\right),$$

where $\sum_j^i$ means the summation over all particles $\mathbf{x_j}^{(\varepsilon)}$ connected with the particle $\mathbf{x_i}^{(\varepsilon)}$. Then

$$a_{npqr}(\mathbf{y}) = 0,$$

$$\begin{aligned} a_{nnnn}(\mathbf{y}) &= \lim_{h\to 0}\lim_{\varepsilon\to 0}\frac{\frac{1}{2}\sum_i \sum_{K_h^y} 2k\varepsilon^3\left(1+\sqrt{2}+\frac{4\sqrt{3}}{9}\right)}{h^3} \\ &= \lim_{h\to 0}\lim_{\varepsilon\to 0}\frac{\left(\frac{h-O(\varepsilon)}{\varepsilon}\right)^3\cdot k\varepsilon^3\left(1+\sqrt{2}+\frac{4\sqrt{3}}{9}\right)}{h^3} \\ &= \lim_{h\to 0}\lim_{\varepsilon\to 0}\frac{\frac{h^3}{\varepsilon^3}\cdot k\varepsilon^3\left(1+\sqrt{2}+\frac{4\sqrt{3}}{9}\right)}{h^3} = k\left(1+\sqrt{2}+\frac{4}{9}\sqrt{3}\right). \end{aligned} \tag{6.16}$$

Note that in (6.16) we estimated the number of ε-cells in the mesocube K_h^y as $((h-O(\varepsilon))/\varepsilon)^3$, which is asymptotically equivalent to h^3/ε^3. Clearly, in this calculation we only need such asymptotic estimates rather than the exact number of such cells.

Analogously

$$a_{nnpp}(\mathbf{y}) = a_{npnp}(\mathbf{y}) = k\left(\frac{1}{2}\sqrt{2}+\frac{4}{9}\sqrt{3}\right).$$

So the limits (2.7) exist and the coefficients $a_{npqr}(\mathbf{y})$ are constants as claimed. The theorem is proved.

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Appendix A. Proof of Lemma 1

Lemma 1. *If the sequence of splines* $\mathbf{u}_\varepsilon(x)$, *defined by* (3.4), *is bounded uniformly in* ε *in* $H^1(\Omega)$, *then the sequence of piecewise-constant vector-functions* $\overline{\mathbf{u}_\varepsilon}(\mathbf{x}) = \sum_{i=1}^N \mathbf{u}_\mathbf{i}^{(\varepsilon)}\chi_{U_i}(\mathbf{x})$ *converges to the vector-function* $\mathbf{u}(\mathbf{x})$ *defined by* (3.13).

Proof. It is sufficient to show that $\|\mathbf{u}_\varepsilon - \overline{\mathbf{u}_\varepsilon}\|_{L_2(\Omega)} \overset{\varepsilon\to 0}{\to} 0$. From the (3.4), any coordinate of the spline $\mathbf{u}_\varepsilon(x) = (u_\varepsilon^{(1)}(\mathbf{x}), u_\varepsilon^{(2)}(\mathbf{x}), u_\varepsilon^{(3)}(\mathbf{x}))$ can be written in the form

$$u_\varepsilon^{(k)}(\mathbf{x}) = \int_{\mathbf{x}_\mathbf{i}^{(\varepsilon)}}^{\mathbf{x}} \langle \nabla u_\varepsilon^{(k)}(\mathbf{r}), \mathrm{d}\mathbf{r}\rangle + u_\varepsilon^{(k)}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}), \quad \mathbf{x}\in U_i, \ \ k\in\overline{1,3},$$

inside of each Voronoi cell, where the integration is taken along the segment $\mathbf{x}-\mathbf{x}_\mathbf{i}^{(\varepsilon)}$. Then

$$\|u_\varepsilon^{(k)} - \overline{u_\varepsilon}^{(k)}\|_{L_2(\Omega)}^2 = \sum_{i=1}^N \int_{U_i}\left|\int_{\mathbf{x}_\mathbf{i}^{(\varepsilon)}}^{\mathbf{x}} \langle \nabla u_\varepsilon^{(k)}(\mathbf{r}), \mathrm{d}\mathbf{r}\rangle\right|^2 \mathrm{d}\mathbf{x}.$$

Since the segment $\mathbf{x}-\mathbf{x}_\mathbf{i}^{(\varepsilon)}$ can be parameterized as

$$\mathbf{r}(t,\mathbf{x}) = (1-t)\mathbf{x}_\mathbf{i}^{(\varepsilon)} + t\mathbf{x}, \quad t\in[0,1],$$

the integral along this segment can be written and bounded as follows:

$$\int_{\mathbf{x}_\mathbf{i}^{(\varepsilon)}}^{\mathbf{x}} \langle \nabla u_\varepsilon^{(k)}(\mathbf{r}), \mathrm{d}\mathbf{r} \rangle = \int_0^1 \langle \nabla_r u_\varepsilon^{(k)}(\mathbf{r}(t,\mathbf{x})), \mathbf{x} - \mathbf{x}_\mathbf{i}^{(\varepsilon)} \rangle \,\mathrm{d}t \leqslant c \cdot \varepsilon \int_0^1 |\nabla_\mathbf{r} u_\varepsilon^{(k)}(\mathbf{r}(t,\mathbf{x}))| \,\mathrm{d}t.$$

From this we get

$$\left| \int_{\mathbf{x}_\mathbf{i}^{(\varepsilon)}}^{\mathbf{x}} \langle \nabla u_\varepsilon^{(k)}(\mathbf{r}), \mathrm{d}\mathbf{r} \rangle \right|^2 \leqslant c \left| \varepsilon \int_0^1 |\nabla_\mathbf{r} u_\varepsilon^{(k)}(\mathbf{r}(t,\mathbf{x}))| \,\mathrm{d}t \right|^2 \leqslant c\varepsilon^2 \int_0^1 |\nabla_\mathbf{r} u_\varepsilon^{(k)}(\mathbf{r}(t,\mathbf{x}))|^2 \,\mathrm{d}t.$$

In the last inequality we used the Cauchy–Schwartz inequality. Next, we integrate over Voronoi cell U_i and sum up over all cells:

$$\begin{aligned}\sum_{i=1}^N \int_{U_i} \left[c\varepsilon^2 \int_0^1 |\nabla_\mathbf{r} u_\varepsilon^{(k)}(\mathbf{r}(t,\mathbf{x}))|^2 \,\mathrm{d}t \right] \mathrm{d}x &= \sum_{i=1}^N c\varepsilon^2 \int_0^1 \left[\int_{U_i} |\nabla_\mathbf{r} u_\varepsilon^{(k)}(\mathbf{r}(t,\mathbf{x}))|^2 \,\mathrm{d}x \right] \mathrm{d}t \\ &= \sum_{i=1}^N c\varepsilon^2 \int_0^1 \left[\frac{1}{t^3} \int_{(1-t)\mathbf{x}_\mathbf{i}^{(\varepsilon)} + tU_i} |\nabla_\mathbf{r} u_\varepsilon^{(k)}(\mathbf{r})|^2 \,\mathrm{d}\mathbf{r} \right] \mathrm{d}t.\end{aligned}$$

In the last equality we used the Fubini theorem to change the order of integration. Since $\nabla_\mathbf{r} u_\varepsilon^{(k)}(\mathbf{r})$ is constant inside of simplices $P_{\alpha\varepsilon}$ and the integration is taken over the Voronoi cell rescaled by factor t, we obtain the following inequality:

$$\int_{(1-t)\mathbf{x}_\mathbf{i}^{(\varepsilon)} + tU_i} |\nabla_\mathbf{r} u_\varepsilon^{(k)}(\mathbf{r})|^2 \,\mathrm{d}\mathbf{r} = t^3 \int_{U_i} |\nabla_r u_\varepsilon^{(k)}(\mathbf{r})|^2 \,\mathrm{d}\mathbf{r}.$$

Hence,

$$\begin{aligned}\sum_{i=1}^N c\varepsilon^2 \int_0^1 \left[\frac{1}{t^3} \int_{(1-t)\mathbf{x}_\mathbf{i}^{(\varepsilon)} + tU_i} |\nabla_\mathbf{r} u_\varepsilon^{(k)}(\mathbf{r})|^2 \,\mathrm{d}\mathbf{r} \right] \mathrm{d}t &= c\varepsilon^2 \sum_{i=1}^N \int_{U_i} |\nabla_\mathbf{r} u_\varepsilon^{(k)}(\mathbf{r})|^2 \,\mathrm{d}\mathbf{r} \\ &= c\varepsilon^2 \int_\Omega |\nabla_\mathbf{r} u_\varepsilon^{(k)}(\mathbf{r})|^2 \,\mathrm{d}\mathbf{r} \leqslant c\varepsilon^2 \|u_\varepsilon^{(k)}\|^2_{H_0^1(\Omega)} \\ &\leqslant c\varepsilon^2 \xrightarrow{\varepsilon \to 0} 0.\end{aligned}$$

The lemma is proved. □

Appendix B. Proof of Lemma 2

Lemma 2. *For any vector-function* $\mathbf{u}_\delta(x) \in C_0^2(\Omega)$ (*see* (4.29)), *there exists a sequence of vector-functions* $\mathbf{u}_{\varepsilon\delta}(x) \in H_0^1(\Omega)$ *of the form* (3.4) *such that*

$$(1)\ \lim_{\varepsilon \to 0} \|\mathbf{u}_{\varepsilon\delta} - \mathbf{u}_\delta\|_{L_2(\Omega)} = 0; \quad (2)\ \lim_{\delta \to 0} \|\mathbf{u}_{\varepsilon\delta} - \mathbf{u}_\varepsilon\|_{H^1(\Omega)} = 0; \quad (3)\ \lim_{\varepsilon \to 0} \|\overline{\mathbf{u}_{\varepsilon\delta}} - \mathbf{u}_\delta\|_{L_2(\Omega)} = 0.$$

Proof. Clearly, for any vector-function $\mathbf{w}(x) \in C_0^2(\Omega)$ there exists a sequence of splines

$$\widehat{\mathbf{w}}_\varepsilon(x) = \sum_{i=1}^N \mathbf{w}(\mathbf{x}_\mathbf{i}^{(\varepsilon)}) \cdot L_\varepsilon^i(x)$$

such that

$$\lim_{\varepsilon\to 0}\|\widehat{\mathbf{w}}_\varepsilon-\mathbf{w}\|_{H^1(\Omega)}=0. \tag{B.1}$$

From this, it follows that

$$\|\widehat{\mathbf{w}}_\varepsilon\|_{H^1(\Omega)}\leqslant 2\|\mathbf{w}\|_{H^1(\Omega)}$$

for $\varepsilon<\widehat{\varepsilon}(\mathbf{w})$. Next, using (4.29), we see that for any vector-function $\mathbf{w}(x)\in H_0^1(\Omega)$ there exists a sequence of vector-functions $\widehat{\mathbf{w}}_\varepsilon(x)\in H_0^1(\Omega)$ of the form (3.4) such that

$$\text{(a)}\ \lim_{\varepsilon\to 0}\|\widehat{\mathbf{w}}_\varepsilon-\mathbf{w}\|_{H^1(\Omega)}=0;\quad \text{(b)}\ \|\widehat{\mathbf{w}}_\varepsilon\|_{H^1(\Omega)}\leqslant 2\|\mathbf{w}\|_{H^1(\Omega)}$$

for $\varepsilon<\widehat{\varepsilon}(\mathbf{w})$. So for every vector-function $\mathbf{w}_\delta=\mathbf{u}_\delta-\mathbf{u}$ there exists a sequence of vector-functions $\widehat{\mathbf{w}_{\varepsilon\delta}}$ with properties (a) and (b). Now set $\mathbf{u}_{\varepsilon\delta}=\mathbf{u}_\varepsilon+\widehat{\mathbf{w}_{\varepsilon\delta}}$. Then

$$\begin{aligned}\lim_{\varepsilon\to 0}\|\mathbf{u}_{\varepsilon\delta}-\mathbf{u}_\delta\|_{L_2(\Omega)}&=\lim_{\varepsilon\to 0}\|\mathbf{u}_\varepsilon-\mathbf{u}+\widehat{\mathbf{w}_{\varepsilon\delta}}-\mathbf{w}_\delta\|_{L_2(\Omega)}\\&\leqslant\lim_{\varepsilon\to 0}\|\mathbf{u}_\varepsilon-\mathbf{u}\|_{L_2(\Omega)}+\lim_{\varepsilon\to 0}\|\widehat{\mathbf{w}_{\varepsilon\delta}}-\mathbf{w}_\delta\|_{L_2(\Omega)}=0,\end{aligned}$$

$$\lim_{\delta\to 0}\|\mathbf{u}_{\varepsilon\delta}-\mathbf{u}_\varepsilon\|_{H^1(\Omega)}=\lim_{\delta\to 0}\|\widehat{\mathbf{w}_{\varepsilon\delta}}\|_{H^1(\Omega)}\leqslant 2\cdot\lim_{\delta\to 0}\|\mathbf{w}_\delta\|_{H^1(\Omega)}=2\cdot\lim_{\delta\to 0}\|\mathbf{u}_\delta-\mathbf{u}\|_{H^1(\Omega)}=0.$$

Thus, properties (1) and (2) hold. Property (3) follows from Lemma 1. Lemma 2 is proved. □

Appendix C. Inverse Laplace transform

Return to problem (3.1):

$$\lambda^2\mathbf{u}_\varepsilon+\frac{1}{m_\varepsilon}\nabla_{\mathbf{u}_\varepsilon}H(\mathbf{u}_\varepsilon)=\mathbf{a}_\varepsilon,$$

where

$$\frac{1}{m_\varepsilon}\nabla_{\mathbf{u}_\varepsilon}H(\mathbf{u}_\varepsilon)=\sum_{i=1}^{N}\sum_{k=1}^{3}\frac{1}{m_i^{(\varepsilon)}}\sum_{j}^{i}\langle C_\varepsilon^{ij}(\mathbf{u}_\mathbf{i}^{(\varepsilon)}-\mathbf{u}_\mathbf{j}^{(\varepsilon)}),\mathbf{e}_\mathbf{k}\rangle\mathbf{e}_{3(\mathbf{i}-\mathbf{1})+\mathbf{k}}.$$

Here

$$\mathbf{u}_\varepsilon=(\mathbf{u}_\mathbf{1}^{(\varepsilon)},\ldots,\mathbf{u}_\mathbf{N}^{(\varepsilon)}),\quad \mathbf{a}_\varepsilon=(\mathbf{a}_\mathbf{1}^{(\varepsilon)},\ldots,\mathbf{a}_\mathbf{N}^{(\varepsilon)}),\quad \mathbf{u}_\mathbf{i}^{(\varepsilon)}=(u_i^{1(\varepsilon)},u_i^{2(\varepsilon)},u_i^{3(\varepsilon)}),$$

$$\mathbf{a}_\mathbf{i}^{(\varepsilon)}=(a_i^{1(\varepsilon)},a_i^{2(\varepsilon)},a_i^{3(\varepsilon)})$$

and the vectors $\mathbf{e}_{3(\mathbf{i}-\mathbf{1})+\mathbf{k}}(i\in\overline{1,N},\ k\in\overline{1,3})$ form an orthonormal basis in $\mathbb{R}^{3N}$.

As we have shown, the solution $\mathbf{u}_\varepsilon(\mathbf{x},\lambda)$ of this problem converges to the solution $\mathbf{u}(\mathbf{x},\lambda)$ of problem (5.1) in $L_2(\Omega)$ for fixed $\lambda>0$.

Define now two operators which map $\mathbb{C}^{3N}$ into $\mathbb{C}^{3N}$ and $H_0^1(\Omega)\cap H^2(\Omega)$ into $L_2(\Omega)$, respectively:

$$A\mathbf{u}_\varepsilon=\frac{1}{m_\varepsilon}\nabla_{\mathbf{u}_\varepsilon}H(\mathbf{u}_\varepsilon),\quad B\mathbf{u}=-\frac{1}{\rho(x)}\cdot\sum_{n,p,q,r=1}^{3}\frac{\partial}{\partial x_q}\{a_{npqr}(x)\varepsilon_{np}[\mathbf{u}(x)]\}\mathbf{e}_\mathbf{r}.$$

It is easy to see that operator A and the Friedrich's extension of operator B are both self-adjoint and non-negative if we set

$$(A\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon) = \sum_{i=1}^{N} m_i^{(\varepsilon)} \langle (A\mathbf{u}_\varepsilon)_i, \mathbf{v}_\mathbf{i}^{(\varepsilon)} \rangle_{\mathbb{C}^3}, \quad (B\mathbf{u}, \mathbf{v}) = \int_\Omega \rho(x) \langle B\mathbf{u}(x), \mathbf{v}(x) \rangle_{\mathbb{C}^3} \, \mathrm{d}x.$$

Rewrite Eqs. (3.1) and (5.1) in the form

$$C\mathbf{u} - (-\lambda^2)\mathbf{u} = \mathbf{f}, \tag{C.1}$$

where C is a self-adjoint operator. Next, since the eigenvalues of a non-negative operator are non-negative, we see that for any RHS in (C.1) ($\mathbf{a}_\varepsilon \in \mathbb{C}^{3N}, \mathbf{a}(\mathbf{x}) \in L_2(\Omega)$), there exist unique solutions of these problems for $Re\,\lambda > 0$ which are analytic in λ (Ahiezer and Glazman, 1963). Denote them as before by $\mathbf{u}_\varepsilon(\mathbf{x}, \lambda)$ and $\mathbf{u}(\mathbf{x}, \lambda)$, respectively. It is easy to show (since the operators A and B are non-negative) that in the domain $Re\,\lambda > \lambda_1 > 0$

$$\|\mathbf{u}_\varepsilon\|_{L_2(\Omega)} \leqslant \frac{c}{|\lambda|}, \quad \|\mathbf{u}\|_{L_2(\Omega)} \leqslant \frac{c}{|\lambda|}. \tag{C.2}$$

Moreover, the sequence of vector-functions $\mathbf{u}_\varepsilon(\mathbf{x}, \lambda)$ (which are analytic in the domain $Re\,\lambda > \lambda_1$) is compact on this domain and converges in $L_2(\Omega)$ to the analytic vector-function $\mathbf{u}(\mathbf{x}, \lambda)$ for any real $\lambda \geqslant 2\lambda_1$. Note that the limiting point $2\lambda_1$ of the set $\lambda \geqslant 2\lambda_1$ belongs to $Re\,\lambda > \lambda_1$, which allows us to apply the Vitaly theorem (Marcushevich, 1978). It follows from this theorem that the sequence $\mathbf{u}_\varepsilon(\mathbf{x}, \lambda)$ converges uniformly in λ on the whole domain $Re\,\lambda > \lambda_1$ to an analytic vector-function $\mathbf{u}_1(\mathbf{x}, \lambda)$. Moreover, $\mathbf{u}_1(\mathbf{x}, \lambda) = \mathbf{u}(\mathbf{x}, \lambda)$ for real $\lambda \geqslant 2\lambda_1$. Then, due to the uniqueness theorem for analytic functions, the vector-functions $\mathbf{u}_1(\mathbf{x}, \lambda)$ and $\mathbf{u}(\mathbf{x}, \lambda)$ coincide on the whole domain $Re\,\lambda > \lambda_1$. Thus, we show that

$$\lim_{\varepsilon \to 0} \|\mathbf{u}_\varepsilon(\mathbf{x}, \lambda) - \mathbf{u}(\mathbf{x}, \lambda)\|_{L_2(\Omega)} = 0, \tag{C.3}$$

uniformly in λ for $Re\,\lambda > \lambda_1$.

We now introduce the notation $\lambda = s + i\sigma$.

Since the solutions $\mathbf{u}_\varepsilon(\mathbf{x}, \lambda)$ and $\mathbf{u}(\mathbf{x}, \lambda)$ of the problems (3.1) and (5.1) are the Laplace transforms of the solutions $\mathbf{u}_\varepsilon(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$ of the problems (1.6)–(1.8) and (5.2), respectively, we have

$$\mathbf{u}_\varepsilon(\mathbf{x}, t) = \frac{1}{2\pi i} \int_{2\lambda_1 - i\infty}^{2\lambda_1 + i\infty} \mathrm{e}^{\lambda t} \mathbf{u}_\varepsilon(\mathbf{x}, \lambda) \, \mathrm{d}\lambda, \quad \mathbf{u}(\mathbf{x}, t) = \frac{1}{2\pi i} \int_{2\lambda_1 - i\infty}^{2\lambda_1 + i\infty} \mathrm{e}^{\lambda t} \mathbf{u}(\mathbf{x}, \lambda) \, \mathrm{d}\lambda. \tag{C.4}$$

Let $\psi(\mathbf{x})$ be an arbitrary vector-function of the class $L_2(\Omega)$, and let $\varphi(t)$ be an arbitrary function of the class $C_0^1(0, T)$. We now show that

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \langle \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t), \psi(\mathbf{x})\varphi(t) \rangle \, \mathrm{d}x \, \mathrm{d}t = 0, \quad \forall T > 0. \tag{C.5}$$

Indeed, using (C.2), (C.4) and the condition $\varphi(t) \in C_0^1(0, T)$ (from this condition it follows that $\int_0^T \mathrm{e}^{\lambda t} \varphi(t) \, \mathrm{d}t = O(\frac{1}{|\lambda|})$), we have

$$\lim_{\varepsilon \to 0} \left| \int_0^T \int_\Omega \langle \mathbf{u}_\varepsilon(\mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t), \psi(\mathbf{x})\varphi(t) \rangle \, \mathrm{d}x \, \mathrm{d}t \right|$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0}\left|\int_0^T \int_\Omega \left\langle \frac{1}{2\pi i}\int_{2\lambda_1-i\infty}^{2\lambda_1+i\infty} \mathrm{e}^{\lambda t}(\mathbf{u}_\varepsilon(\mathbf{x},\lambda)-\mathbf{u}(\mathbf{x},\lambda))\,\mathrm{d}\lambda, \psi(\mathbf{x})\varphi(t)\right\rangle \mathrm{d}x\,\mathrm{d}t\right| \\
&= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\left|\int_{2\lambda_1-i\infty}^{2\lambda_1+i\infty} \mathrm{d}\lambda \int_\Omega \langle(\mathbf{u}_\varepsilon(\mathbf{x},\lambda)-\mathbf{u}(\mathbf{x},\lambda)), \psi(\mathbf{x})\rangle\,\mathrm{d}x \cdot \int_0^T \mathrm{e}^{\lambda t}\varphi(t)\,\mathrm{d}t\right| \\
&\leqslant c\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} \|\mathbf{u}_\varepsilon(\mathbf{x},2\lambda_1+i\sigma)-\mathbf{u}(\mathbf{x},2\lambda_1+i\sigma)\|_{L_2(\Omega)}\,\frac{\mathrm{d}\sigma}{\sigma} \\
&= c\lim_{\varepsilon\to 0}\left(\int_{-p}^{p} \|\mathbf{u}_\varepsilon(\mathbf{x},2\lambda_1+i\sigma)-\mathbf{u}(\mathbf{x},2\lambda_1+i\sigma)\|_{L_2(\Omega)}\frac{\mathrm{d}\sigma}{\sigma}\right. \\
&\quad + \int_{-\infty}^{-p} \|\mathbf{u}_\varepsilon(\mathbf{x},2\lambda_1+i\sigma)-\mathbf{u}(\mathbf{x},2\lambda_1+i\sigma)\|_{L_2(\Omega)}\frac{\mathrm{d}\sigma}{\sigma} \\
&\quad \left. + \int_{p}^{+\infty} \|\mathbf{u}_\varepsilon(\mathbf{x},2\lambda_1+i\sigma)-\mathbf{u}(\mathbf{x},2\lambda_1+i\sigma)\|_{L_2(\Omega)}\frac{\mathrm{d}\sigma}{\sigma}\right).
\end{aligned}
\tag{C.6}
$$

Due to (C.2), the last two integrals in σ in (C.6) are uniformly small in ε for sufficiently large p. The first integral in σ vanishes as $\varepsilon \to 0$ due to (C.3). Thus (C.5) follows. Next, since the set of vector-functions $\{\sum_i \psi_{\mathbf{i}}(\mathbf{x})\varphi_i(t), \psi_{\mathbf{i}}(\mathbf{x}) \in L_2(\Omega), \varphi_i(t) \in C_0^1(0,T)\}$ is dense in $L_2(\Omega \times [0,T])$, Theorem 3 follows. Due to standard properties of the Laplace transform and (5.1), we conclude that the vector-function $\mathbf{u}(\mathbf{x},t)$ is a solution of the following initial boundary value problem:

$$
\begin{cases}
\rho(\mathbf{x})\dfrac{\partial^2 \mathbf{u}(\mathbf{x},t)}{\partial t^2} - \displaystyle\sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_q} a_{npqr}(x)\varepsilon_{np}[\mathbf{u}(\mathbf{x},t)]\mathbf{e}_\mathbf{r} = \mathbf{0}, \quad \mathbf{x} \in \Omega, \quad t > 0, \\
\mathbf{u}(\mathbf{x},t) = \mathbf{0}, \quad \mathbf{x} \in \partial\Omega, \quad t \geqslant 0, \\
\mathbf{u}(\mathbf{x},0) = \mathbf{0}, \left.\dfrac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t}\right|_{t=0} = \mathbf{a}(\mathbf{x}), \quad \mathbf{x} \in \Omega.
\end{cases}
\tag{C.7}
$$

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