SUPERCONDUCTIVITY, INCLUDING HIGH-TEMPERATURE SUPERCONDUCTIVITY

Higgs mechanism in superconducting structures

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(Submitted May 27, 2016; revised August 1, 2016)
Fiz. Nizk. Temp. **43**, 401–419 (March 2017)

It is shown that in equilibrium superconducting structures with s-wave pairing, the unique part of the phase of the complex ordering parameter $\langle\psi_\uparrow\psi_\downarrow\rangle$ transforms into the longitudinal component of the vector potential as in the Abelian Higgs model of relativistic field theory. This analysis is based on a microscopic Hamiltonian of the system in the presence of an external static magnetic field and infinitely small Cooper pair sources. Impurities and nonsuperconducting barriers are assumed to be present, and the quantum nature of the induced electromagnetic field is taken into account. Quantization of the latter is done under the condition $A_0=0$ (A_0 is the scalar potential) that the invariance with respect to time-independent gauge transformations is not broken. Exact relations determining the quasi-averages $\langle\psi_\uparrow\psi_\downarrow\rangle$ are established. These relations play a key role in the new derivation of the mean-field equations discussed in this article. A new physical treatment of the Josephson effect (without a "phase difference") is proposed on the basis of these results and some of its consequences are discussed. *Published by AIP Publishing.*
[http://dx.doi.org/10.1063/1.4979958]

1. Introduction

The concept[1–3] of "soft" generation of gauge fields by spontaneous violation of local gauge symmetry (the Higgs mechanism) is one of the cornerstones of modern particle theory.[4–6] Although the fundamental ideas[1–3] have been definitively confirmed only recently in high energy physics (with the experimental discovery of the Higgs boson[7,8]), it is well known[4–6,9] that the Higgs mechanism is actually realized at lower energies, in particular, in the phenomenon of superconductivity.[10–14] (According to the definition of Ref. 6, "a superconductor is simply a material in which electromagnetic gauge invariance is spontaneously broken.")

As an example, the Meissner effect, in which an external static magnetic field does not penetrate into the depth of a bulk superconductor, is usually invoked.[4–6,9] This can be interpreted as the "acquisition of mass by a photon." This is not the only analogy with high energy physics. In particular, there has recently been a report[15] of the experimental detection of a low-energy analog of the Higgs boson in superconductors which had been predicted theoretically.[16,17] (It should, however, be kept in mind that in superconductors there is no fundamental Higgs field, and spontaneous violation of electromagnetic symmetry takes place dynamically through the formation of a Cooper pair condensate.[12–14])

We want to bring attention to another analog which has not been noticed before in the literature: the transformation of the unique part of the phase of the complex superconducting ordering parameter (the "Goldstone" field of the model of Ref. 1) into a longitudinal component of the static magnetic field (the electromagnetic field in the model of Ref. 1). In order make it easy to grasp the main point of this article, we recall the key assumptions of the Abelian model,[1] which are of direct concern for the subject of our discussion.

The Lagrangian of the classical Higgs model of a complex scalar field $\Psi=\Psi_1+i\Psi_2$ (charge q) interacting with an electric field $A_\mu=(A_0,-\mathbf{A})$, is given by

$$\begin{aligned}\mathcal{L}_H &= \mathcal{L}_H\left[\Psi^*,\Psi,A_\mu\right]\\ &\equiv \int d^3\mathbf{r}\Big[\left[\partial_\mu - iqA_\mu(\mathbf{r}t)\right],\Psi^*(\mathbf{r}t)[\partial^\mu + iqA^\mu(\mathbf{r}t)]\Psi(\mathbf{r}t)\\ &\quad -M^2|\Psi(\mathbf{r}t)|^2-|\lambda||\Psi(\mathbf{r}t)|^4-\frac{1}{16\pi}F_{\mu\nu}(\mathbf{r}t)F^{\mu\nu}(\mathbf{r}t)\Big],\end{aligned} \tag{1}$$

where M^2 is a parameter (positive or negative) and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the electromagnetic field tensor. (In this section we use the metric and four-dimensional notation of Landau[18] but take $\hbar=c=1$.) The Lagrangian (1) is invariant with respect to the local gauge transformation

$$\begin{aligned}&A_\mu\to A_\mu-\partial_\mu\chi,\\ &\Psi\to\Psi e^{iq\chi},\quad \Psi^*\to\Psi^* e^{-iq\chi},\quad \chi=\chi(\mathbf{r}t).\end{aligned} \tag{2}$$

When $M^2<0$, however, the ground state of the system

$$A_\mu\equiv 0,\quad \Psi_0=\sqrt{\frac{-M^2}{2|\lambda|}}e^{i\phi_0},\quad \phi_0=\text{const}\in[0,2\pi), \tag{3}$$

does not have this property. In order to understand the consequence of violating the local gauge symmetry, it is convenient to proceed to a polar representation of the field Ψ. Selecting a ground state ("vacuum") from the condition $\phi_0=0$, we write

$$\Psi(\mathbf{r}t)=[|\Psi_0|+\rho(\mathbf{r}t)]e^{i\tilde{\phi}(rt)}, \tag{4}$$

where $\tilde{\phi}/q$ has the significant of the "Goldstone" field. Substituting Eq. (4) in Eq. (1) and transforming to the unitary gauge[5,6]

$$A_\mu \to A_\mu - \frac{1}{q}\partial_\mu\tilde{\phi}, \tag{5}$$

make it possible to avoid the Goldstone field. Assuming that the "physical" fields ρ and A_μ are small quantities and expanding the Lagrangian (1) to terms of second order, we find the corresponding equations of motion

$$\partial_\mu\partial^\mu\rho + m_H^2\rho = 0, \quad m_H^2 \equiv 4|\lambda||\Psi_0|^2, \tag{6}$$

$$\partial_\nu F^{\nu\mu} = m_A^2 A^\mu, \quad m_A^2 \equiv 8\pi q^2|\Psi_0|^2. \tag{7}$$

Equation (6) describes the massive Higgs boson (mass m_H) and Eq. (7) represents the Proca equation for a massive vector field (mass m_A).[5]

Solving Eq. (7) for A_μ requires the additional condition

$$\partial_\mu A^\mu = 0, \tag{8}$$

which is known from classical electrodynamics as the "Lorentz gauge."[18] The condition (8) makes it possible to eliminate the scalar potential A_0, which is not a dynamic variable, from the discussion. Thus, for spontaneous breaking of the local gauge symmetry, the vector field $\mathbf{A}$, in addition to the two independent transverse components ($\mathbf{A}_\perp$) of the free field, acquires a longitudinal component ($\mathbf{A}_\parallel$) owing to the phase Ψ (the Goldstone phase $\tilde{\phi}/q$).

In the case of a time-independent field $\mathbf{A} = \mathbf{A}(\mathbf{r})$, Eq. (7) and the condition (8) convert to the following:

$$\nabla \times \nabla \times \mathbf{A} = -m_A^2\mathbf{A}, \tag{9}$$

$$\nabla\mathbf{A} = \mathbf{0}. \tag{10}$$

Equation (9) is formally the same as the fundamental theory of the phenomenological theory of London[12,13] (with the identification $m_A^2 \to 1/\lambda_L^2$, where λ_L is the London penetration depth). Under this condition (10), which denotes the vanishing of the longitudinal component of the vector potential, the conservation law for the superconducting current has a simple consequence, which has the form $\mathbf{j} = -\mathbf{A}/(4\pi\lambda_L^2)$ in the London theory.

Although Eq. (9) correctly reflects the existence of the Meissner effect, on the whole the London theory does not provide an adequate description of other aspects of the physics of superconductivity.

The more refined phenomenological Ginzburg-Landau theory,[19] which holds for temperatures T close to the superconducting transition temperature T_c, begins with the free energy functional $\mathcal{F}_{GL} = \mathcal{F}_{GL}[\Psi^*, \Psi, \mathbf{A}]$, where the complex field Ψ has the significance of a superconducting order parameter. The functional $\mathcal{F}_{GL}$ can be obtained formally from the Lagrangian (1) for the case of time-independent fields ($\Psi = \Psi(\mathbf{r})$ and $\mathbf{A} = \mathbf{A}(\mathbf{r})$). Setting $q = 2e$ (e is the electron charge) and defining the new constants

$$a \equiv (T - T_c)|\alpha| = \frac{M^2}{4m}, \quad b = \frac{|\lambda|}{8m^2},$$

where m is the "electron mass," we obtain

$$\mathcal{F}_{GL}[\Psi^*, \Psi, \mathbf{A}] = -\mathcal{L}_H\left[\frac{\Psi^*}{2\sqrt{m}}, \frac{\Psi}{2\sqrt{m}}, \mathbf{A}\right] = \int_V d^3\mathbf{r}\left[\frac{1}{4m}|\nabla - 2e\mathbf{A}|^2 + a|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{(\nabla\times\mathbf{A})^2}{8\pi}\right]. \tag{11}$$

For a given temperature $T < T_c$, the functional (11) is minimized by the equilibrium values of the fields Ψ ($|\Psi| \neq 0$) and $\mathbf{A}$, which satisfy the nonlinear Ginzburg-Landau equation and the Maxwell equation, respectively. Then the superconducting current is given by

$$\mathbf{j} = -\frac{ie}{2m}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{2e^2}{m}|\Psi|^2\mathbf{A}. \tag{12}$$

As in the case of the Higgs model (1), in Eqs. (11) and (12) it is convenient to proceed to the polar representation $\Psi(\mathbf{r}) = |\Psi(\mathbf{r})|e^{i\phi(\mathbf{r})}$. In a singly-coupled superconducting phase $\phi = \phi(\mathbf{r})$, there is a unique function of coordinates ($\phi = \tilde{\phi}$), so it can be avoided by using the static analogy of the unitary gauge (5)

$$\mathbf{A} \to \mathbf{A} - \frac{1}{2e}\nabla\tilde{\phi}. \tag{13}$$

Now, however, as opposed to the London model (9), the current conservation law $\nabla\mathbf{j} = 0$ does not assume that part of the vector potential vanishes, since $|\Psi(\mathbf{r})|$ depends on the coordinates. In other words, a transformation $-\frac{1}{2e}\nabla\tilde{\phi} \to \mathbf{A}_\parallel$ must take place.

It should be said that the nonphysicality of the single-valued part of the phase of the order parameter in the Ginzburg-Landau theory was noted long ago.[20] But the fundamental question was left unanswered: where did the variable $\tilde{\phi}$ "disappear" to after the gauge transformation (13)? Our arguments (by analogy with the model of Ref. 1) require a strict foundation in terms of a rigorous microscopic theory, given the dynamic character of the breaking of electromagnetic symmetry. We strongly emphasize that the validity of using the transformation (13) can only be proved by taking into account the quantum nature of the electromagnetic field induced in a superconductor, since the procedure for quantizing that field depends substantially on the choice of gauge condition. In addition, the phenomenological Ginzburg-Landau theory is not capable of describing the most interesting (in terms of the subject of our article) case of superconducting structures with a Josephson coupling[14,21] and is clearly not applicable for temperatures $T \ll T_c$. The microscopic theory developed in the following sections is free of these shortcomings.

2. Initial microscopic model

The starting point for our discussion is the microscopic Hamiltonian in the Heisenberg representation (see the conclusion to Appendix A)

$$\mathcal{H} = \mathcal{H}_e + \mathcal{H}_b + \mathcal{H}_{\rm imp} + \mathcal{H}_{BCS} + \mathcal{H}_{\rm em}, \tag{14}$$

$$\mathcal{H}_e = \int_V d^3\mathbf{r}\left[\frac{1}{2m}\sum_\alpha \left[\nabla + ie\mathbf{A}(\mathbf{r}t) + ie\mathbf{A}_e(\mathbf{r})\right]\psi_\alpha^+(\mathbf{r}t)\right.$$
$$\left.\times\left[\nabla - ie\mathbf{A}(\mathbf{r}t) - ie\mathbf{A}_e(\mathbf{r})\right]\psi_\alpha(\mathbf{r}t),\right.$$
$$\mathcal{H}_b = \int_V d^3\mathbf{r}\sum_{\alpha,\beta}\psi_\alpha^+(\mathbf{r}t)\left[\hat{U}^b\right]_{\alpha\beta}(\mathbf{r})\psi_\beta(\mathbf{r}t),$$
$$\mathcal{H}_{\rm imp} = \int_V d^3\mathbf{r}\sum_{\alpha,\beta}\psi_\alpha^+(\mathbf{r}t)\left[\hat{U}^{\rm imp}\right]_{\alpha\beta}(\mathbf{r})\psi_\beta(\mathbf{r}t),$$
$$\mathcal{H}_{BCS} = -\frac{1}{2}\int_V d^3\mathbf{r}|g(\mathbf{r})|\sum_\alpha \psi_\alpha^+(\mathbf{r}t)\psi_{-\alpha}^+(\mathbf{r}t)\psi_{-\alpha}(\mathbf{r}t)\psi_\alpha(\mathbf{r}t),$$
$$\mathcal{H}_{em} = \frac{1}{8\pi}\int d^3r\left[\mathbf{E}^2(\mathbf{r}t) + (\nabla\times\mathbf{A})^2(\mathbf{r}t)\right]$$
$$\mathbf{E}(\mathbf{r}t) = -\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}t), \quad \nabla\times\mathbf{A}(\mathbf{r}t) = \mathbf{H}(\mathbf{r}t). \tag{15}$$

Here $\mathcal{H}_0$ is the Hamiltonian of a system of noninteracting electrons in an external magnetic field $\mathbf{H}_e$

$$\nabla\times\mathbf{A}_e = \mathbf{H}_e, \quad \nabla\mathbf{A}_e = 0, \tag{16}$$

where e and m are the charge and mass of the electron and V is the volume of the superconducting structure. The field operators for creation (ψ_α^+) and annihilation (ψ_α) of electrons with spin $\alpha = \uparrow, \downarrow$ obey the usual anticommutation relations

$$\left[\psi_\alpha(\mathbf{r}t), \psi_\beta^+(\mathbf{r}'t)\right]_+ = \delta_{\alpha\beta}\delta^3(\mathbf{r}-\mathbf{r}'),$$
$$\left[\psi_\alpha(\mathbf{r}t), \psi_\beta(\mathbf{r}'t)\right]_+ = \left[\psi_\alpha^+(\mathbf{r}t), \psi_\beta^+(\mathbf{r}'t)\right]_+ = 0,$$

where $[A,B]_+ = AB + BA$, and $\delta_{\alpha\beta}$ is the Kronecker symbol.

The contributions $\mathcal{H}_b$ and $\mathcal{H}_{\rm imp}$ describe the possible presence of nonsuperconducting layers (barriers)[22] and frozen-in impurities,[23] respectively. If the corresponding potentials $\hat{U}^b$ and $\hat{U}^{\rm imp}$ are non-exchange potentials, their dependence on the spin indices reduces to $\delta_{\alpha\beta}$. This assumes that the matrix elements $[\hat{U}^b]_{\alpha\beta} = U^b{}_{\alpha\beta}(\mathbf{r})$ and $[\hat{U}^{\rm imp}]_{\alpha\beta} = U^{\rm imp}{}_{\alpha\beta}(\mathbf{r})$ are smooth functions of the coordinates; see the example in Fig. 1.

The term $\mathcal{H}_{BCS}$ describes the effective electron-electron attraction leading to pairing.[11,14] The interaction parameter $g = g(\mathbf{r})$, is a smooth function of position, is negative in the superconducting layers, and equals zero in the nonsuperconducting barriers (see Fig. 2).

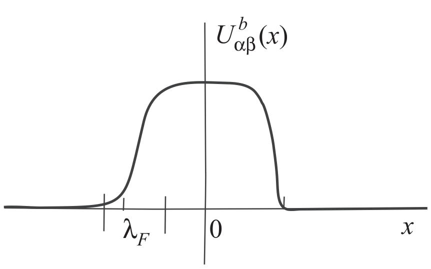

Fig. 1. Spatial dependence of the matrix element $[\hat{U}^b]_{\alpha\beta}$. Here the nonsuperconducting barrier is assumed to be homogeneous along the y and z axes and λ_F is the Fermi length ($\lambda_F \sim 1/PF$). (See Ref. 24 for a method for analytically constructing functions of the type $U^b_{\alpha\beta} = U^b_{\alpha\beta}(x)$.)

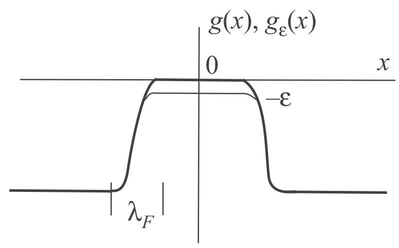

Fig. 2. Spatial dependence of the electron-electron interaction parameter ($(g \le 0)$) and the auxiliary function $g_\varepsilon(g_{\varepsilon\to 0} \to g)$, used in Section 5.

The induced electric field $\mathbf{A} = \mathbf{A}(\mathbf{r}t)$ corresponds to the Hamiltonian $\mathcal{H}_{\rm em}$. According to the explanation in the Introduction, the field $\mathbf{A}$ must be regarded as quantized. As the form of $\mathcal{H}_{\rm em}$ implies, when $\mathbf{A}$ is quantized we assume a gauge $A_0 = 0$, the use of which requires an explanation. In fact, in nonrelativistic solid-state physics problems, a transverse (Coulomb) gauge $\nabla\mathbf{A} = 0$, is more customary.[25] A transverse gauge, however, completely breaks the invariance with respect to local gauge transformations, so is entirely unsuitable for the purposes of this article. On the other hand, the gauge $A_0 = 0$ leaves the Hamiltonian invariant with respect to the time-independent local gauge transformations

$$\mathbf{A}(\mathbf{r}t) \to \mathbf{A}(\mathbf{r}t) + \nabla\chi(\mathbf{r}),$$
$$\psi_\alpha(\mathbf{r}t) \to e^{ie\chi(\mathbf{r})}\psi_\alpha(\mathbf{r}t), \quad \psi_\alpha^+(\mathbf{r}t) \to \psi_\alpha^+(\mathbf{r}t)e^{-ie\chi(\mathbf{r})}. \tag{17}$$

In addition, despite claims in the literature,[26] the gauge $A_0 = 0$ does not by any means lead to "loss" of Gauss' law (see our explanation in Appendix B).

In the gauge $A_0 = 0$, the commutation relations for the electromagnetic field operators have the following form:

$$\left[\frac{\partial A_i}{\partial t}(\mathbf{r}t), A_j(\mathbf{r}'t)\right] = -i4\pi\delta_{ij}\delta(\mathbf{r}-\mathbf{r}'),$$
$$\left[A_i(\mathbf{r}t), A_j(\mathbf{r}'t)\right] = \left[\frac{\partial A_i}{\partial t}(\mathbf{r}t), \frac{\partial A_j}{\partial t}(\mathbf{r}'t)\right] = 0, \quad i,j = x,y,z. \tag{18}$$

Naturally, it is assumed that the electromagnetic field operators commute with the electron operators ψ_α and ψ_α^+. If the induced electromagnetic field falls off fast enough toward spatial infinity, the following expansion holds for the operator $\mathbf{A}$:[27]

$$\mathbf{A} = \mathbf{A}_\perp + \mathbf{A}_\parallel, \tag{19}$$

where

$$\nabla\mathbf{A}_\perp = 0, \quad \nabla\mathbf{A}_\parallel = 0, \quad \int d^3\mathbf{r}\mathbf{A}_\perp(\mathbf{r}t)\mathbf{A}_\parallel(\mathbf{r}t) = 0,$$
$$\mathbf{A}_\perp(\mathbf{r}t) = \frac{1}{4\pi}\nabla_\mathbf{r}\times\int d^3\mathbf{r}'\frac{\nabla\times\mathbf{A}(\mathbf{r}'t)}{|\mathbf{r}-\mathbf{r}'|},$$
$$\mathbf{A}_\parallel(\mathbf{r}t) = -\frac{1}{4\pi}\nabla_\mathbf{r}\int d^3\mathbf{r}'\frac{\nabla\mathbf{A}(\mathbf{r}'t)}{|\mathbf{r}-\mathbf{r}'|}.$$

Given the commutativity of the transverse ($\perp$) and longitudinal ($\|$) components of the operators, the commutation relations (18) imply that

$$\left[\frac{\partial A_{\perp i}}{\partial t}(\mathbf{r}t),A_{\perp j}(\mathbf{r}'t)\right]=i4\pi\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')+i\frac{\partial}{\partial r_i}\frac{\partial}{\partial r'_j}\frac{1}{|\mathbf{r}-\mathbf{r}'|},$$
$$\left[A_{\perp i}(\mathbf{r}t),A_{\perp j}(\mathbf{r}'t)\right]=\left[\frac{\partial A_{\perp i}}{\partial t}(\mathbf{r}),\frac{\partial A_{\perp j}}{\partial t}(\mathbf{r}')\right]=0,$$
$$\left[\frac{\partial A_{\parallel i}}{\partial t}(\mathbf{r}t),A_{\parallel j}(\mathbf{r}'t)\right]=-i\frac{\partial}{\partial r_i}\frac{\partial}{\partial r'_j}\frac{1}{|\mathbf{r}-\mathbf{r}'|},$$
$$\left[A_{\parallel i}(\mathbf{r}t),A_{\parallel j}(\mathbf{r}'t)\right]=\left[\frac{\partial A_{\parallel i}}{\partial t}(\mathbf{r}t),\frac{\partial A_{\parallel j}}{\partial t}(\mathbf{r}'t)\right]=0,$$
$$\left[A_{\parallel i}(\mathbf{r}t),A_{\perp j}(\mathbf{r}'t)\right]=\left[\frac{\partial A_{\parallel i}}{\partial t}(\mathbf{r}t),\frac{\partial A_{\perp j}}{\partial t}(\mathbf{r}'t)\right]$$
$$=\left[\frac{\partial A_{\parallel i}}{\partial t}(\mathbf{r}t),A_{\perp j}(\mathbf{r}'t)\right]\quad i,j=x,y,z. \tag{20}$$

Since the Hamiltonian $\mathcal{H}$ contains equal numbers of electron creation and annihilation operators, there is an obvious integral of motion—the total number $\mathcal{N}$ of electrons, with

$$[\mathcal{N},\mathcal{H}]=0,\quad \mathcal{N}=\int_V d^3\mathbf{r}\,n_e(\mathbf{r}t),$$
$$n_e(\mathbf{r}t)=\sum_\alpha \psi^+_\alpha(\mathbf{r}t)\psi_\alpha(\mathbf{r}t), \tag{21}$$

where $n_e(\mathbf{r}t)$ is the number density of the electrons. The operator $\mathcal{N}$ generates the global gauge transformations

$$U_0=U_0(\chi_0)\equiv \mathrm{e}^{-ie\chi_0\mathcal{N}},\quad \chi_0=\mathrm{const}:$$
$$U_0(\chi_0)\psi_\alpha(\mathbf{r}t)U_0^{-1}(\chi_0)=\mathrm{e}^{-ie\chi_0}\psi_\alpha(\mathbf{r}t),$$
$$U_0(\chi_0)\psi^+_\alpha(\mathbf{r}t)U_0^{-1}(\chi_0)=\mathrm{e}^{-ie\chi_0}\psi^+_\alpha(\mathbf{r}t),$$
$$U_0(\chi_0)\mathbf{A}(\mathbf{r}t)U_0^{-1}(\chi_0)=\mathbf{A}(\mathbf{r}t), \tag{22}$$

which leave $\mathcal{H}$ invariant. Using the equation of motion for the operators in the Heisenberg representation, we obtain the law of charge conservation

$$\frac{e\partial n_e(\mathbf{r}t)}{\partial t}+\nabla\mathbf{j}(\mathbf{r}t)=0, \tag{23}$$

where

$$\mathbf{j}=\mathbf{j}(\mathbf{r}t)\equiv\sum_\alpha\left[\frac{ie}{2m}\left[\psi^+_\alpha(\mathbf{r}t)\nabla\psi_\alpha(\mathbf{r}t)-\left[\nabla\psi^+_\alpha(\mathbf{r}t)\right]\psi_\alpha(\mathbf{r}t)\right]\right.$$
$$\left.-\frac{e^2}{m}\left[\mathbf{A}(\mathbf{r}t)\mathbf{A}_e(\mathbf{r})\right]\psi^+_\alpha(\mathbf{r}t)\psi_\alpha(\mathbf{r}t)\right], \tag{24}$$

is the current operator and the Maxwell equations are

$$\nabla\times\mathbf{H}=4\pi\mathbf{j}+\frac{\partial\mathbf{E}}{\partial t}, \tag{25}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{H}}{\partial t}. \tag{26}$$

The conservation law (23) and the Maxwell equation (25) yield

$$[\mathcal{H},\nabla\mathbf{E}-4\pi e n_e]=0. \tag{27}$$

This condition, along with the condition

$$[\mathcal{N},\nabla\mathbf{E}-4\pi e n_e]=0, \tag{28}$$

guarantees the presence of a common system of eigenvectors for the operators $\mathcal{H}$, $\mathcal{N}$, and $\nabla\mathbf{E}-4\pi e n_e$. For the physically realizable states, we have

$$\mathcal{N}|N,E_{k,N},n_i\rangle=N|N,E_{k,N},n_i\rangle,$$
$$\mathcal{H}|N,E_{k,N},n_i\rangle=E_{k,N}|N,E_{k,N},n_i\rangle,$$
$$(\nabla\mathbf{E}-4\pi e n_e)|N,E_{k,N},n_i\rangle=-4\pi e n_i|N,E_{k,N},n_i\rangle,\quad n_i=n_i(\mathbf{r}). \tag{29}$$

The third of Eq. (29) (the coupling equation) represents Gauss' law, where n_i is a specified density of the distribution of ionic charge. The dynamic invariant $\nabla\mathbf{E}-4\pi e n_e$ generates the local gauge transformation (17) with a c-number function $\chi=\chi(\mathbf{r})$

$$U_1=U_1[\chi]=U_\mathbf{A}[\chi]U_\psi[\chi]=U_\psi[\chi]U_\mathbf{A}[\chi],$$
$$U_\mathbf{A}[\chi]\equiv\exp\left[\frac{i}{4\pi}\int d^3\mathbf{r}\chi(\mathbf{r})\nabla\mathbf{E}(\mathbf{r}t)\right],$$
$$U_\psi[\chi]\equiv\exp\left[-ie\int_V d^3\mathbf{r}_e\chi(\mathbf{r})n_e(\mathbf{r}t)\right]:$$
$$U_1[\chi]\mathbf{A}(\mathbf{r}t)U_1^{-1}[\chi]=\mathbf{A}(\mathbf{r}t)+\nabla\chi(\mathbf{r}),$$
$$U_1[\chi]\psi_\alpha(\mathbf{r}t)U_1^{-1}[\chi]=\mathrm{e}^{-ie\chi(\mathbf{r})}\psi_\alpha(\mathbf{r}t),$$
$$U_1[\chi]\psi^+_\alpha(\mathbf{r}t)U_1^{-1}[\chi]=\mathrm{e}^{-ie\chi(\mathbf{r})}\psi^+_\alpha(\mathbf{r}t), \tag{30}$$

where the function $\chi=\chi(\mathbf{r})$ is specified over all space and is of class C_2 (continuous, with all partial derivatives up to the second order, inclusive) and falls off as $|\mathbf{r}|\to\infty$. The mathematical conditions formulated here ensure uniqueness of the function $\chi=\chi(\mathbf{r})$, which can be written as the condition

$$\frac{1}{2\pi}\oint_\Gamma(\nabla\chi\cdot d\mathbf{l})=0, \tag{31}$$

for an arbitrary closed contour Γ.

If the system is in the normal state ($T>T_c$), in order to obtain the full set of equations for the observed quantities we shall take the average ($\langle\ldots\rangle$) over the grand canonical ensemble

$$\langle\ldots\rangle\equiv\mathrm{Tr}(\ldots\rho),\quad \rho=\frac{1}{Z}\mathrm{e}^{-\frac{\mathcal{H}-\mu\mathcal{N}}{T}},$$
$$Z=\mathrm{Tr}\exp\left(-\frac{\mathcal{H}-\mu\mathcal{N}}{T}\right)=\sum_{k,N}\exp\left(-\frac{E_{k,N}-\mu N}{T}\right), \tag{32}$$

where μ is the chemical potential, Tr is the trace in the effective space for all the operators, and Z is the grand statistical sum. Using the identity for operators in the Heisenberg representation

$$i\left\langle\frac{dO}{dt}\right\rangle=\langle[O,\mathcal{H}]\rangle=\mathrm{Tr}\left(\left[\mathcal{H},\rho_{(0)}\right]O\right)=0,$$

we quickly find

$$\nabla\times\langle\mathbf{H}\rangle=\frac{4\pi}{c}\langle\mathbf{j}\rangle,\quad \nabla\times\langle\mathbf{H}\rangle=0,\quad \nabla\times\langle\mathbf{j}\rangle=0,$$
$$\nabla\times\langle\mathbf{E}\rangle=0,\quad \langle\mathbf{E}\rangle=0, \tag{33}$$

where $\langle\mathbf{H}\rangle=\langle\mathbf{H}(\mathbf{r})\rangle$ and $\langle\mathbf{j}\rangle=\langle\mathbf{j}(\mathbf{r})\rangle$. For consistency of this last equation with Gauss' law

$$\nabla\langle \mathbf{E}\rangle - 4\pi e\langle n_e\rangle = -4\pi e n_i, \tag{34}$$

it is necessary to satisfy the condition of electrical neutrality

$$\langle n_e\rangle = n_i. \tag{35}$$

Another useful equation for the following is:

$$\langle \mathbf{j}\rangle \equiv \mathrm{Tr}(\mathbf{j}\rho) = -\mathrm{Tr}\left(\frac{\delta \mathcal{H}_e}{\delta \mathbf{A}_e}\rho\right) = \frac{T}{Z}\frac{\delta Z}{\delta \mathbf{A}_e} = -\frac{\delta\Omega}{\delta \mathbf{A}_e}, \tag{36}$$

where $\Omega = -T\ln Z$ is the thermodynamic potential ($\Omega = \Omega(T, P, \mu; \mathbf{H}_e)$).

Since in the normal state, Ohm's law $\langle \mathbf{j}\rangle = \sigma\langle \mathbf{E}\rangle$ must be satisfied, we also have $\langle \mathbf{H}\rangle = 0$. Thus, by using the gauge transformation (30) it is always possible to satisfy the condition $\langle \mathbf{A}\rangle = 0$.

It is well known that the superconducting state ($T < T_c$) is characterized by the appearance of an anomalous mean $\langle\psi_\downarrow\psi_\uparrow\rangle$ ($\langle\psi_\downarrow\psi_\uparrow\rangle^* = \langle\psi_\uparrow^+\psi_\downarrow^+\rangle$) (of the superconducting order parameter). Recall[28] that the anomalous mean cannot always be calculated by averaging over the grand canonical ensemble ρ. In fact, because of the invariance of the Hamiltonian $\mathcal{H}$ with respect to the gauge transformations (22) and (30) with arbitrary χ_0 and χ, we have

$$\begin{aligned}
\mathrm{Tr}\left[\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rho\right] &= \mathrm{Tr}\left[U_0(\chi_0)\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})U_0^{-1}(\chi_0)U_0(\chi_0)\rho U_0^{-1}(\chi_0)\right]\\
&= \mathrm{Tr}\left[U_0(\chi_0)\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})U_0^{-1}(\chi_0)\rho\right] = \exp(i2e\chi_0)\mathrm{Tr}\left[\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rho\right] = 0;\\
\mathrm{Tr}\left[\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rho\right] &= \mathrm{Tr}\left[U_1[\chi]\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})U_1^{-1}[\chi]U_1[\chi]\rho U_1^{-1}[\chi]\right] = \mathrm{Tr}\left[U_1[\chi]\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})U_1^{-1}[\chi]\rho\right]\\
&= \exp(i2e\chi(\mathbf{r}))\mathrm{Tr}\left[\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rho\right] = 0.
\end{aligned} \tag{37}$$

As Bogolyubov explained,[28] for a correct determination of similar anomalous averages (or quasiaverages in the terminology of Ref. 28), it is necessary to break the corresponding continuum symmetry by introducing infinitely small "sources" in the Hamiltonian. This approach was later greatly generalized to quantum field theory and was widely used in studies of various mechanisms for and types of spontaneous symmetry breaking.[6] (We note that in the theory of ferromagnetism, the "sources" have a real physical sense of an infinitely small external magnetic field.[29])

3. Hamiltonian with Cooper pair sources in the quasi-average

Since the founding paper[28] only dealt with the case of a structurally uniform superconductor without electromagnetic interactions, the use of the Hamiltonian (14) requires a generalization of the definition of quasi-average, i.e.,

$$\begin{aligned}
&\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle = \langle\psi_\uparrow^+(\mathbf{r})\psi_\downarrow^+(\mathbf{r})\rangle^* \equiv \lim_{\|\eta\|\to 0}\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle_\eta,\\
&\langle\ldots\rangle_\eta \equiv \mathrm{Tr}\left(\ldots\rho_\eta[\mathbf{A}_t]\right),\quad \rho_\eta[\mathbf{A}_t] = \frac{1}{Z_\eta}\exp\left(-\frac{\mathcal{H}_\eta[\mathbf{A}_t]-\mu\mathcal{N}}{T}\right),\quad Z_\eta = \mathrm{Tr}\,\exp\left(-\frac{\mathcal{H}_\eta[\mathbf{A}_t]-\mu\mathcal{N}}{T}\right),\\
&\mathcal{H}_\eta[\mathbf{A}_t] = \mathcal{H}[\mathbf{A}_t] + \frac{1}{2}\int_V d^3\mathbf{r}\left[\eta(\mathbf{r})\psi_\uparrow^+(\mathbf{r})\psi_\downarrow^+(\mathbf{r}) + \eta^*(\mathbf{r})\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\right],\quad \mathbf{A}_t \equiv \mathbf{A}+\mathbf{A}_e;\\
&\left[\mathcal{H}_\eta,\mathcal{N}\right]\neq 0,\quad \left[\mathcal{H}_\eta,\nabla\mathbf{E}\text{–}4\pi e n_e\right]\neq 0,
\end{aligned} \tag{38}$$

where the "source" of Cooper pairs $\eta = \eta(\mathbf{r})$ is a smooth complex function of position and $\|\eta\| = \min_{\mathbf{r}\in V}|\eta(\mathbf{r})|$. Note that the limiting transition $\|\eta\|\to 0$ in Eq. (38) is completed on going to the thermodynamic limit.

In addition, in determining the quantum statistical operator ρ_η, for convenience in the following we explicitly indicate the functional dependence on the operator of the full vector potential $\mathbf{A}_t \equiv \mathbf{A}+\mathbf{A}_e$. This operator appears in the definition of $\mathcal{H}_e$ (see Eq. (15)). Only this term, the original Hamiltonian $\mathcal{H}$, is subjected to a local gauge transformation.

The definition (38) can be assigned a somewhat different form if the field operators ψ_α are expanded in terms of complete system of normalized eigenfunctions of the operator $\mathcal{H}_e+\mathcal{H}_b+\mathcal{H}_{\mathrm{imp}}$ (on initially setting $\mathbf{A} = 0$)

$$\psi_\alpha(\mathbf{r}) = \sum_n c_{n\alpha}u_n(\mathbf{r}), \tag{39}$$

where $c_{n\alpha}$ is the electron annihilation operator in state $n\alpha$. We have

$$\begin{aligned}
\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle &= \sum_{n,n'}u_n(\mathbf{r})u_{n'}(\mathbf{r})\langle c_{n\downarrow}c_{n'\uparrow}\rangle\\
&\equiv \lim_{\|\eta\|\to 0}\sum_{n,n'}u_n(\mathbf{r})u_{n'}(\mathbf{r})\langle c_{n\downarrow}c_{n'\uparrow}\rangle_\eta.
\end{aligned} \tag{40}$$

This clarifies our requirement of smooth potentials in the definition of the operators $\mathcal{H}_b$ and $\mathcal{H}_{\rm imp}$: in a polar representation for the quasi-average

$$\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle = |\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle|e^{i\phi(\mathbf{r})}, \tag{41}$$

it ensures that the phases $\phi = \phi(\mathbf{r})$ belong to class C_2.

Although the complex function $\eta=\eta(\mathbf{r})$ in the standard sense is arbitrary (see Section 3), it cannot violate any symmetries of the Hamiltonian, except the gauge symmetries. For example, if $\mathcal{H}_b = \mathcal{H}_{\rm imp} = 0$, $g(\mathbf{r}) = $ const, and $\mathbf{A} = A_e = 0$, the electron momentum $\mathbf{k}$ will be a "good" quantum number and the "Cooper pair source" is equal to the complex constant $\eta = |\eta|e^{i\phi_0}$, where $|\eta|$ and ϕ_0 are independent of position. Thus,

$$\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle = \langle\psi_\downarrow(0)\psi_\uparrow(0)\rangle = \lim_{|\eta|\to 0}\sum_{\mathbf{k}}\langle c_{\mathbf{k}\downarrow}c_{-\mathbf{k}\uparrow}\rangle_\eta.$$

Since the quasi-average $\langle\psi_\downarrow\psi_\uparrow\rangle$ is given by the limiting value of the mean $\langle\psi_\downarrow\psi_\uparrow\rangle_\eta$, it is useful to begin by studying the properties of the latter. First of all, we establish the transformation laws for $\langle\psi_\downarrow\psi_\uparrow\rangle_\eta$ under the unitary transformations $\rho_\eta \to U_0\rho_\eta U_0^{-1}$ and $\rho_\eta \to U_\psi\rho_\eta U_\psi^{-1}$ (see the definitions (22) and (30))

$$\begin{aligned}
\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle_\eta &= \mathrm{Tr}\left[\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rho_\eta[\mathbf{A}_t]\right] = \mathrm{Tr}\left[U_0(\chi_0)\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})U_0^{-1}(\chi_0)U_0(\chi_0)\rho_\eta[\mathbf{A}_t]U_0^{-1}(\chi_0)\right]\\
&= \mathrm{Tr}\left[U_0(\chi_0)\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})U_0^{-1}(\chi_0)\rho_{\eta\exp(-i2e\chi_0)}[\mathbf{A}_t]\right] = \exp(i2e\chi_0)\mathrm{Tr}\left[\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rho_{\eta\exp(-i2e\chi_0)}[\mathbf{A}_t]\right];\\
\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle_\eta &= \mathrm{Tr}\left[U_\psi(\chi)\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})U_\psi^{-1}[\chi]U_\psi(\chi)\rho_\eta[\mathbf{A}_t]U_\psi^{-1}(\chi_1)\right]\\
&= \mathrm{Tr}\left[U_\psi(\chi)\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})U_\psi^{-1}[\chi]\rho_{\eta\exp(-i2e\chi_0)}[\mathbf{A}_t-\nabla_\chi]\right] = \exp(i2e\chi(\mathbf{r}))\mathrm{Tr}\left[\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rho_{\eta\exp(-i2e\chi_0)}[\mathbf{A}_t-\nabla_\chi]\right].
\end{aligned} \tag{42}$$

We write the source in a polar representation: $\eta = |\eta|e^{i\theta}$, where $|\eta|=\eta(\mathbf{r})$ and $\theta=\theta(\mathbf{r})$. Using the analog of Eq. (41) for $\langle\psi_\downarrow\psi_\uparrow\rangle_\eta$ and Eq. (42), it is easy to establish the following fundamental equality: $\phi(\mathbf{r})=\theta(\mathbf{r})$. Therefore, the phase $\langle\psi_\downarrow\psi_\uparrow\rangle_\eta$ is fully determined by the phase η of the source. This quickly yields the important physical result.

We define the "thermodynamic potential in the presence of sources,"

$$\Omega_\eta \equiv -T\ln Z_\eta, \tag{43}$$

and calculate the variational derivative

$$\frac{\delta\Omega_\eta}{\delta\phi(\mathbf{r})} = \int_V d^3\mathbf{r}'\langle\psi_\uparrow^+(\mathbf{r}')\psi_\downarrow^+(\mathbf{r}')\rangle_\eta\frac{\delta\eta(\mathbf{r}')}{\delta\phi(\mathbf{r}')} + \int_V d^3\mathbf{r}'\langle\psi_\downarrow(\mathbf{r}')\psi_\uparrow(\mathbf{r}')\rangle_\eta\frac{\delta\eta^*(\mathbf{r}')}{\delta\phi(\mathbf{r})} = 0.$$

On the other hand, Eq. (42) implies that

$$\Omega_\eta[\eta,\eta^*,\mathbf{A}_e] = \Omega_{|\eta|}\left[|\eta|,\mathbf{A}_e - \frac{1}{2e}\nabla\phi\right]. \tag{44}$$

Using the analog of Eq. (36), we find

$$\frac{\delta\Omega_\eta}{\delta\phi} = -\frac{1}{2e}\nabla\frac{\delta\Omega_{|\eta|}}{\delta\mathbf{A}_e} = \frac{1}{2e}\nabla\langle\mathbf{j}\rangle_\eta = 0. \tag{45}$$

Equation (45), which is a conservation law for a current in the presence of sources, is a consequence of two facts: the equality $\theta=\phi$, and the entry of ϕ in the functional argument $\Omega_{|\eta|}$ in the form $\nabla\phi$ (the absence of a functional dependence on the zero Fourier component $\phi_0 = \frac{1}{V}\int_V d^3\mathbf{r}\phi(\mathbf{r})$). The existence of the conservation law (45) is an indicator of the internal self-consistency of our theory.

Based on the definition (38), we have

$$\frac{\delta\Omega_{|\eta|}}{\delta|\eta(\mathbf{r})|} = |\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle_\eta|. \tag{46}$$

Taking the functional Legendre transform, we proceed from the potential $\Omega_{|\eta|} = \Omega_{|\eta|}\left[|\eta|,\mathbf{A}_e - \frac{1}{2e}\nabla\phi\right]$ to the "effective potential" $\Omega = \Omega\left[|\langle\psi_\downarrow\psi_\uparrow\rangle_\eta|,\mathbf{A}_e - \frac{1}{2e}\nabla\phi\right]$ (compare this with the definition of "effective action" in quantum field theory:[4,6,26]

$$\Omega\left[|\langle\psi_\downarrow\psi_\uparrow\rangle_\eta|,\mathbf{A}_e - \frac{1}{2e}\nabla\phi\right] = \Omega_{|\eta|}\left[|\eta|,\mathbf{A}_e - \frac{1}{2e}\nabla\phi\right] - \int_V d^3\mathbf{r}|\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle_\eta||(\eta(\mathbf{r}))|. \tag{47}$$

In light of the general properties of Legendre transforms,[30] variation of Eq. (47) gives

$$\frac{\delta\Omega}{\delta|\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle_\eta|} = -|(\eta(\mathbf{r}))|. \tag{48}$$

Given the definition (38), in the absence of sources ($\|\eta\| \to 0$), we obtain

$$\left.\frac{\delta\Omega}{\delta|\langle\psi_\downarrow\psi_\uparrow\rangle_\eta|}\right|_{|\langle\psi_\downarrow\psi_\uparrow\rangle_\eta| = |\langle\psi_\downarrow\psi_\uparrow\rangle|} = 0. \tag{49}$$

In other words, the quasi-average $|\langle\psi_\downarrow\psi_\uparrow\rangle|$ satisfies the condition of time independence for the effective potential Ω.

If the functional dependence $\Omega = \Omega\left[|\langle\psi_\downarrow\psi_\uparrow\rangle_\eta|, \ \mathbf{A}_e - \frac{1}{2e}\nabla\phi\right]$ were known exactly, we could find $|\langle\psi_\downarrow\psi_\uparrow\rangle|$ from the condition of time independence (49). Substituting $|\langle\psi_\downarrow\psi_\uparrow\rangle|$ in the effective potential would give the directly measured thermodynamic potential of the superconducting structure. Although a determination of the exact functional dependence $\Omega = \Omega\left[|\langle\psi_\downarrow\psi_\uparrow\rangle|, \mathbf{A}_e - \frac{1}{2e}\nabla\phi\right]$ is impossible in principle, the problem does allow a simple solution in the mean field approximation (see Section 5).

4. The "disappearance" of the unique part of the phase of the quasi-average (the Higgs mechanism)

As should be clear from the previous remarks, the zero component of the phase ϕ_0 of the average $\langle\psi_\downarrow\psi_\uparrow\rangle_\eta$ can be set equal to zero without loss of generality. (In the model of Eq. (1) this evidently corresponds to a certain choice of "vacuum" (3).) We now represent the phase $\phi = \phi(\mathbf{r})$ of the average $\langle\psi_\downarrow\psi_\uparrow\rangle_\eta$ in the form $\phi = \tilde{\phi} + \phi_s$, where $\tilde{\phi} = \tilde{\phi}(\mathbf{r})$ is a unique function satisfying

$$\frac{1}{2\pi}\oint_\Gamma (\nabla\tilde{\phi}.d\mathbf{l}) = 0, \tag{50}$$

for an arbitrary closed contour $\Gamma \in V$, while $\phi_s = \phi_s(\mathbf{r})$ is a nonunique function which, on some closed contours $\Gamma_n \in V$, satisfies the condition

$$\frac{1}{2\pi}\oint_{\Gamma_n} (\nabla\phi_s.d\mathbf{l}) = n, \quad n = \pm 1, \pm 2, ... \tag{51}$$

(Recall[12,13] that contours of type Γ_n exist in superconducting structures with a multiply connected geometry and even in single-connected structures in the presence of Abrikosov vortices.) We emphasize that the choice of function $\tilde{\phi} = \tilde{\phi}(\mathbf{r})$ is still arbitrary: this arbitrariness reflects the invariance of the original Hamiltonian $\mathcal{H}$ with respect to a local gauge transformation U_1. On the other hand, the function $\phi_s = \phi_s(\mathbf{r})$ should be considered given.

Consider the chain of equalities

$$\begin{aligned}|\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle_\eta| &= \exp(-i\phi(\mathbf{r}))\mathrm{Tr}\left[\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rho_\eta[\mathbf{A}_t]\right] \\ &= \exp(-i\phi(\mathbf{r}))\mathrm{Tr}\left[U_\psi\left[\frac{\phi}{2e}\right]\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})U_\psi^{-1}\right. \\ &\quad\times\left.\left[\frac{\phi}{2e}\right]U_\psi\left[\frac{\phi}{2e}\right]\rho_\eta[\mathbf{A}_t]U_\psi^{-1}\left[\frac{\phi}{2e}\right]\right] \\ &= \exp(-i\phi(\mathbf{r}))\mathrm{Tr}\left[U_\psi\left[\frac{\phi}{2e}\right]\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\right. \\ &\quad\times\left.U_\psi^{-1}\left[\frac{\phi}{2e}\right]\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi\right]\right] \\ &= \mathrm{Tr}\left[\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi\right]\right].\end{aligned} \tag{52}$$

Let us define a function $\alpha_\parallel \equiv \alpha_\parallel(\mathbf{r})$ specified over the entire space which falls off as $|\mathbf{r}| \to \infty$ and coincides with ϕ for $\mathbf{r} \in V$. Continuing the transformation (52)

$$\begin{aligned}&|\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle_\eta| \\ &\quad= \mathrm{Tr}\left[\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi\right]\right] \\ &\quad= \mathrm{Tr}\left[\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})U_\mathbf{A}\left[\frac{\alpha_\parallel}{2e}\right]\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi\right]U_\mathbf{A}^{-1}\left[\frac{\alpha_\parallel}{2e}\right]\right] \\ &\quad= \mathrm{Tr}\left[\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi_s\right]\right].\end{aligned} \tag{53}$$

We now explain these calculations. In the first step, using the unitary transformation U_ψ under the Tr sign (Eq. (52)) made it possible for us to avoid the phase ϕ of the source η in the average $\langle\psi_\downarrow\psi_\uparrow\rangle_\eta$. However, in the Hamiltonian $\mathcal{H}_e$, the operator $\mathbf{A}_t$ is replaced by the combination $\mathbf{A}_t - \frac{1}{2e}\nabla\phi$ (see the definitions (14) and (38)). In the second stage, we have used the unitary transformation $U_\mathbf{A}$ as a result, in the Hamiltonian $\mathcal{H}_0$ instead of the combination $\mathbf{A}_t - \frac{1}{2e}\nabla\phi$ only the combination $\mathbf{A}_t - \frac{1}{2e}\nabla\phi_s$ remains with a nonunique (physical) part of the phase.

In order to understand where the unique part of the phase ϕ from the last row of Eq. (53) has "gone," we examine the quantity $\widetilde{\langle\mathbf{A}(\mathbf{r})\rangle}_{|\eta|}$, defined as the average over the quantum statistical ensemble $\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi\right]$ (see the last row of Eq. (52)). We have

$$\begin{aligned}\widetilde{\langle\mathbf{A}(\mathbf{r})\rangle}_{|\eta|} &\equiv \mathrm{Tr}\left[\mathbf{A}(\mathbf{r})\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi\right]\right] \\ &= \mathrm{Tr}\left[U_\mathbf{A}\left[\frac{\alpha_\parallel}{2e}\right]\mathbf{A}(\mathbf{r})U_\mathbf{A}^{-1}\left[\frac{\alpha_\parallel}{2e}\right]U_\mathbf{A}\left[\frac{\alpha_\parallel}{2e}\right]\rho_{|\eta|}\right. \\ &\quad\times\left.\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi\right]U_\mathbf{A}^{-1}\left[\frac{\alpha_\parallel}{2e}\right]\right] \\ &= \mathrm{Tr}\left[U_\mathbf{A}\left[\frac{\alpha_\parallel}{2e}\right]\mathbf{A}(\mathbf{r})U_\mathbf{A}^{-1}\left[\frac{\alpha_\parallel}{2e}\right]\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi_s\right]\right] \\ &= \mathrm{Tr}\left[\mathbf{A}(\mathbf{r})\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi_s\right]\right] + \frac{1}{2e}\nabla\alpha_\parallel \\ &\equiv \langle\mathbf{A}(\mathbf{r})\rangle_{|\eta|} + \frac{1}{2e}\nabla\alpha_\parallel.\end{aligned} \tag{54}$$

On the other hand, we have still not used the gauge freedom remaining in the choice of $\tilde{\phi} = \tilde{\phi}(\mathbf{r})$. Given the representation (19), we write

$$\begin{aligned}\widetilde{\langle\mathbf{A}(\mathbf{r})\rangle}_{|\eta|} &= \widetilde{\langle\mathbf{A}(\mathbf{r})\rangle}_{|\eta|} + \widetilde{\langle\mathbf{A}(\mathbf{r})\rangle}_{|\eta|} \\ &\equiv \mathrm{Tr}\left[\mathbf{A}_\perp(\mathbf{r})\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi\right]\right] \\ &\quad+ \mathrm{Tr}\left[\mathbf{A}_\parallel(\mathbf{r})\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi\right]\right] \\ &= \mathrm{Tr}\left[U_\mathbf{A}\left[\frac{\alpha_\parallel}{2e}\right]\mathbf{A}_\perp(\mathbf{r})U_\mathbf{A}^{-1}\left[\frac{\alpha_\parallel}{2e}\right]\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi_s\right]\right] \\ &\quad+ \mathrm{Tr}\left[U_\mathbf{A}\left[\frac{\tilde{\tilde{\chi}}}{2e}\right]\mathbf{A}_\parallel(\mathbf{r})U_\mathbf{A}^{-1}\left[\frac{\tilde{\tilde{\chi}}}{2e}\right]U_\mathbf{A}\left[\frac{\tilde{\tilde{\chi}}}{2e}\right]\rho_{|\eta|}\right. \\ &\quad\times\left.\left[\mathbf{A}_t - \frac{1}{2e}\nabla\phi\right]U_\mathbf{A}^{-1}\left[\frac{\tilde{\tilde{\chi}}}{2e}\right]\right] \\ &= \langle\mathbf{A}_\perp(\mathbf{r})\rangle_{|\eta|} + \mathrm{Tr}\left[\mathbf{A}_\parallel(\mathbf{r})\rho_{|\eta|}\left[\mathbf{A}_t - \frac{1}{2e}\nabla\left(\phi - \tilde{\tilde{\chi}}\right)\right]\right] \\ &\quad+ \frac{1}{2e}\nabla\tilde{\tilde{\chi}} \equiv \widetilde{\langle\mathbf{A}(\mathbf{r})\rangle}_{|\eta|} + \widetilde{\widetilde{\langle\mathbf{A}_\parallel(\mathbf{r})\rangle}}_{|\eta|} + \frac{1}{2e}\nabla\tilde{\tilde{\chi}},\end{aligned} \tag{55}$$

where the invariance $U_{\mathbf{A}}\mathbf{A}_\perp U_{\mathbf{A}}^{-1}=\mathbf{A}_\perp$ is used. Since $\widetilde{\widetilde{\langle \mathbf{A}_\parallel(\mathbf{r})\rangle}}_{|\eta|}$ can be written in as the gradient of a scalar, it is always possible to choose $\tilde{\tilde{\chi}}=\tilde{\tilde{\chi}}(\mathbf{r})$ so that the equality $\widetilde{\widetilde{\langle \mathbf{A}_\parallel(\mathbf{r})\rangle}}_{|\eta|}+\frac{1}{2e}\nabla\tilde{\tilde{\chi}}=0.$ is satisfied in the last row of Eq. (55). (This choice of $\tilde{\tilde{\chi}}$ means complete elimination of the remaining gauge freedom.)

This result should be compared with the result of the last row of Eq. (54), which is valid for an arbitrary choice of $\tilde{\phi}$. From this, we find

$$\langle \mathbf{A}(\mathbf{r})\rangle_{|\eta|}=\langle \mathbf{A}_\perp(\mathbf{r})\rangle_{|\eta|}-\frac{1}{2e}\nabla\alpha_\parallel,$$

or, on taking the limit $\|\eta\|\to 0$,

$$\langle \mathbf{A}(\mathbf{r})\rangle=\langle \mathbf{A}_\perp(\mathbf{r})\rangle-\frac{1}{2e}\nabla\alpha_\parallel\equiv\langle \mathbf{A}_\perp(\mathbf{r})\rangle+\langle \mathbf{A}_\parallel(\mathbf{r})\rangle. \quad (56)$$

Equation (56) is the major result of our analysis. It means that the unique part of the phase of the superconducting order parameter is "exchanged" for the longitudinal component of the vector potential induced over the entire space, in complete analogy with what happens in the Higgs' Abelian model (1).

We now elaborate on the significance of the average in Eq. (56). According to the definitions given above

$$\langle \mathbf{A}(\mathbf{r})\rangle=\lim_{\|\eta\|\to 0}\mathrm{Tr}\left[\mathbf{A}(\mathbf{r})\rho_{|\eta|}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]\right]$$

$$\rho_{|\eta|}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]=\frac{1}{Z_{|\eta|}}\exp\left(\frac{\mathcal{H}_{\|\eta\|}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]-\mu\mathcal{N}}{T}\right),$$

$$Z_{|\eta|}=\exp\left(\frac{\mathcal{H}_{\|\eta\|}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]-\mu\mathcal{N}}{T}\right),$$

$$\mathcal{H}_{|\eta|}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]=\mathcal{H}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]+\frac{1}{2}\int_V d^3\mathbf{r}|\eta(\mathbf{r})|\times\left[\psi_\uparrow^+(\mathbf{r})\psi_\downarrow^+(\mathbf{r})+\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\right],$$

$$\mathcal{H}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]=\mathcal{H}_e\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]+\mathcal{H}_b+\mathcal{H}_{\rm imp}+\mathcal{H}_{BCS}+\mathcal{H}_{\rm em}. \quad (57)$$

Given the invariance of $\mathbf{A}$ with respect to the global gauge transformation $(U_0\mathbf{A}U_0^{-1}=\mathbf{A})$ in Eq. (57), we can switch the operations lim and Tr. As a result, we obtain

$$\langle \mathbf{A}(\mathbf{r})\rangle=\mathrm{Tr}\lim_{\|\eta\|\to 0}\left[\mathbf{A}(\mathbf{r})\rho_{|\eta|}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]\right]=\mathrm{Tr}\left[\mathbf{A}(\mathbf{r})\rho\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]\right]$$

$$\rho\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]=\frac{1}{Z}\exp\left(-\frac{\mathcal{H}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]-\mu\mathcal{N}}{T}\right),$$

$$Z=\mathrm{Tr}\,\exp\left(-\frac{\mathcal{H}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]-\mu\mathcal{N}}{T}\right), \quad (58)$$

where the Hamiltonian $\mathcal{H}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]$ is defined by the last row of Eq. (57).

It is perfectly obvious that averaging of the remaining electrodynamic operators (n_e, $\mathbf{j}$, $\mathbf{H}$, and $\mathbf{E}$) for $T<T_c$ also reduces to averaging over the grand canonical ensemble $\rho\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]$. Thus, Eqs. (33)–(35) remain valid. In particular, despite a contrary claim in the literature,[31] the strict equality $\langle\mathbf{E}\rangle=0$ uniquely implies the absence of any kind of static electric potentials in equilibrium superconducting systems.

5. The mean-field approximation

As usual,[10–14] the closed system of equations of the theory of superconductivity can be obtained only in the mean-field approximation. The exact results of Sections 3 and 4 can be used to derive these equations in a strict and consistent manner. We begin with the mean-field approximation for the quantities $\mathcal{H}_{|\eta|}=\mathcal{H}_{|\eta|}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]$ and $\Omega_{|\eta|}=\Omega_{|\eta|}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]$.

Given that the quantum fluctuations in the induced vector potential $\mathbf{A}$ are small on the scale of a small ratio $\frac{v_F}{c}$ (v_F is the Fermi velocity and c is the speed of light), in the term $\mathcal{H}_e\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]$ of the Hamiltonian $\mathcal{H}_{|\eta|}\left[\mathbf{A}_t-\frac{1}{2e}\nabla\phi_s\right]$ we make the replacement

$$\mathcal{H}_e\to\mathcal{H}_e^{MF}\ :\ \mathbf{A}+\mathbf{A}_e\to\langle\mathbf{A}\rangle\equiv\langle\mathbf{A}\rangle_\perp+\langle\mathbf{A}\rangle_\parallel, \quad (59)$$

where the averages $\langle\ldots\rangle$ are defined by Eq. (56). Since the vector potential of the external field $\mathbf{A}_e$ satisfies the continuity condition $\nabla\mathbf{A}_e=0$, here and in the following we consider it to be included in the definition of $\langle\mathbf{A}\rangle_\perp$. We omit the energy $\mathcal{H}_{\rm em}$ of the electromagnetic field. (Here we assume, of course, that Maxwell's equation (33) are satisfied.)

In the term $\mathcal{H}_{BCS}$ we first replace the real electron-electron interaction parameter $g=g(\mathbf{r})$ by the auxiliary function $g_\varepsilon=g_\varepsilon(\mathbf{r})$ (see Fig. 2) and the limiting transition $\varepsilon\to+0$ will be taken in the equations for the mean field. Then we consider the elementary identity

$$\psi_\uparrow^+\psi_\downarrow^++\psi_\downarrow\psi_\uparrow=(\psi_\uparrow^+\psi_\downarrow^+-|\langle\psi_\downarrow\psi_\uparrow\rangle_\eta|)(\psi_\downarrow\psi_\uparrow-|\langle\psi_\downarrow\psi_\uparrow\rangle_\eta|)+(\psi_\uparrow^+\psi_\downarrow^++\psi_\downarrow\psi_\uparrow)|\langle\psi_\downarrow\psi_\uparrow\rangle_\eta|-|\langle\psi_\downarrow\psi_\uparrow\rangle_\eta|^2. \quad (60)$$

Given that the first term on the right of Eq. (60) is "small" in a certain sense, we arrive at a quadratic (in terms of the electron creation and annihilation operators) approximation for $\mathcal{H}_{BCS}$

$$\mathcal{H}_{BCS}\to\mathcal{H}_{BCS}^{MF}=\int_V d^3\mathbf{r}|g_\varepsilon(\mathbf{r})|\left[|\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle_\eta|^{MF}\right]^2-\int_V d^3\mathbf{r}|g_\varepsilon(\mathbf{r})||\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle_\eta|^{MF}\times\left[\psi_\uparrow^+(\mathbf{r}t)\psi_\downarrow^+(\mathbf{r}t)+\psi_\downarrow(\mathbf{r}t)\psi_\uparrow(\mathbf{r}t)\right]. \quad (61)$$

In order to determine the average $|\langle\psi_\downarrow\psi_\uparrow\rangle_\eta|^{MF}$, we require Eqs. (46)–(49) in the mean-field approximation. We introduce the definitions

$$\Omega^{MF}_{|\eta|} = -T\ln Z^{MF}_{|\eta|},\ \ Z^{MF}_{|\eta|} = \mathrm{Tr}\exp\left(\frac{-\mathcal{H}^{MF}_{|\eta|} - \mu\mathcal{N}}{T}\right). \quad (62)$$

Equations (46)–(49) will be retained if the potential $\Omega^{MF}_{|\eta|}$ is independent of $|\langle\psi_\downarrow\psi_\uparrow\rangle_\eta|^{MF}$, i.e.,

$$\frac{\delta\Omega^{MF}_{|\eta|}}{\delta|\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle_\eta|^{MF}} = 0. \quad (63)$$

From here, we arrive at the self-consistency condition

$$|\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle_\eta|^{MF} = \frac{1}{2Z^{MF}_{|\eta|}|_{\varepsilon=0}}\mathrm{Tr}\left[\left[\psi^+_\uparrow(\mathbf{r})\psi^+_\downarrow(\mathbf{r}) + \psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\right]\times\exp\left(-\frac{\mathcal{H}^{MF}_{|\eta|}|_{\varepsilon=0} - \mu\mathcal{N}}{T}\right)\right], \quad (64)$$

where we have taken the limit $\varepsilon \to +0$. on the right hand side of Eq. (64). (The role of the auxiliary function $g_\in$ is clear from the above derivation of the self-consistency condition: this function is necessary in order to be able to take the variational derivative (63) over the entire region V, including the nonsuperconducting barriers.)

The above discussion leads to the equality

$$\Omega^{MF}\left[|\langle\psi_\downarrow\psi_\uparrow\rangle|^{MF}, \langle\mathbf{A}\rangle - \frac{1}{2e}\nabla\phi_s\right] = \Omega^{MF}_{|\eta|=0}\left[|\langle\psi_\downarrow\psi_\uparrow\rangle|^{MF}, \langle\mathbf{A}\rangle - \frac{1}{2e}\nabla\phi_s\right] = \Omega^{MF}_{|\eta|=0}\left[\langle\mathbf{A}\rangle - \frac{1}{2e}\nabla\phi_s\right],$$

$$\Omega^{MF}_{|\eta|=0} = -T\ln Z^{MF},$$

$$Z^{MF} \equiv Z^{MF}_{|\eta|=0} = \mathrm{Tr}\exp\left(-\frac{\mathcal{H}^{MF}_{|\eta|}\Big|_{\varepsilon=0} - \mu\mathcal{N}}{T}\right), \quad (65)$$

where $|\langle\psi_\downarrow\psi_\uparrow\rangle_\eta|^{MF} \equiv |\langle\psi_\downarrow\psi_\uparrow\rangle|^{MF}$ satisfies the self-consistency condition in the absence of sources ($\|\eta\| \to 0$), with

$$|\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle|^{MF} = \frac{1}{2Z^{MF}|_{\varepsilon=0}}\mathrm{Tr}\left[\left[\psi^+_\uparrow(\mathbf{r})\psi^+_\downarrow(\mathbf{r}) + \psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\right]\times\exp\left(-\frac{\mathcal{H}^{MF}_{|\eta|}|_{\varepsilon=0} - \mu\mathcal{N}}{T}\right)\right]. \quad (66)$$

Before taking the trace with respect to the electron fields in Eqs. (65) and (66), a slight generalization of these results is appropriate. To do this, we introduce a real, continuous, and non-negative function $|\tilde{F}| = |\tilde{F}|(\mathbf{r})$, where $\mathbf{r} \in V$. Without taking the limit $\varepsilon \to +0$ and rejecting the self-consistency condition (66), we now determine the nonequilibrium effective potential $\Omega = \Omega\left[|\tilde{F}|, \langle\mathbf{A}\rangle - \frac{1}{2e}\nabla\phi_s\right]$ using the formula

$$\Omega\left[|\tilde{F}|, \langle\mathbf{A}\rangle - \frac{1}{2e}\nabla\phi_s\right] = \Omega^{MF}_{|\eta|=0}\left[|\tilde{F}|, \langle\mathbf{A}\rangle - \frac{1}{2e}\nabla\phi_s\right]. \quad (67)$$

In a state of thermodynamic equilibrium, we shall have

$$\left.\frac{\delta\Omega}{\delta|\widetilde{F(\mathbf{r})}|}\right|_{|\tilde{F}|=|F|} = 0, \quad |F(\mathbf{r})| = |\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle^{MF}|. \quad (68)$$

The trace with respect to the electron fields, which figures in the right hand side of Eq. (67), is easily calculated by functional integration.[4–6,26,32,33] As a result, we obtain

$$\Omega\left[|\tilde{F}|, \langle\mathbf{A}\rangle - \frac{1}{2e}\nabla\phi_s\right] = \int_V d^3\mathbf{r}|g_\varepsilon(\mathbf{r})||\widetilde{F(\mathbf{r})}|^2 - \frac{T}{2}\mathrm{Tr}\,\ln\hat{\hat{G}}^{-1} + \frac{1}{4}\mathrm{Tr}\left[(\hat{1}+\hat{\tau}_3)\hat{1}\hat{\hat{H}}_{|g|\equiv 0}\right], \quad (69)$$

$$\hat{\hat{G}}^{-1}(\mathbf{r},\mathbf{r}';\tau-\tau') = \left[-\hat{\hat{1}}\frac{\partial}{\partial\tau} - \hat{\hat{H}}_{|g_\varepsilon|}(\mathbf{r})\right]\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'),$$

$$\hat{\hat{H}}_{|g_\varepsilon|}(\mathbf{r}) = -\mu\hat{\tau}_3\hat{1} - \frac{1}{2m}\left[\hat{\hat{1}}\nabla + \frac{i}{2}\left[\nabla\phi_s(\mathbf{r}) - 2e\langle\mathbf{A}(\mathbf{r})\rangle\right]\hat{\tau}_3\hat{1}\right]^2\hat{\tau}_3\hat{1} + \frac{1}{2}(\hat{1}+\hat{\tau}_3)\hat{U}(\mathbf{r}) - \frac{1}{2}(\hat{1}-\hat{\tau}_3)\hat{U}^t(\mathbf{r}) - \hat{\tau}_2\hat{\sigma}_2|g_\varepsilon(\mathbf{r})||F\tilde{(}\mathbf{r})|,$$

$$\hat{U}(\mathbf{r}) \equiv \hat{U}^b(\mathbf{r}) + \hat{U}^{\mathrm{imp}}(\mathbf{r}),\ \ 0 < \tau < \frac{1}{T}. \quad (70)$$

Here $\hat{\tau}_i, \hat{\sigma}_i$ $(i = 1,2,3)$ are the Pauli matrices in Gorkov-Nambu space and spin space, respectively; $\hat{1}$ is the 2×2 unit matrix; $\hat{\hat{1}}$ is the 4×4 unit matrix; and $\hat{\tau}_i\hat{\sigma}_k$ and $\hat{\tau}_i\hat{1}$, $\hat{1}\hat{\sigma}_i$ must be understood as the right product of the corresponding matrices

$$\hat{\tau}_1\hat{\sigma}_2 \equiv \begin{pmatrix}\hat{0} & \hat{\sigma}_2\\ \hat{\sigma}_2 & \hat{0}\end{pmatrix} \equiv \begin{pmatrix}0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & -i & 0 & 0\\ i & 0 & 0 & 0\end{pmatrix},$$

$$\hat{\tau}_3\hat{1} \equiv \begin{pmatrix}\hat{1} & \hat{0}\\ \hat{0} & -\hat{1}\end{pmatrix} \equiv \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \quad \text{etc.}$$

The last term on the right of Eq. (69) arises from the need to symmetrize the electron creation and annihilation operators before taking the functional integral.[33]

The spectrum of the integral (in the sense of the theory of generalized functions) operator $\hat{\hat{G}}^{-1}$, defined by Eq. (70), does not contain a zero, so that this operator has a uniquely determined inverse operator $\hat{G}$, the kernel $\hat{\hat{G}}(\mathbf{r},\mathbf{r}';\tau-\tau')$ of which satisfies the conditions

$$\int_0^\beta d\tau_1\int_V d^3\mathbf{r}_1\hat{\hat{G}}^{-1}(\mathbf{r},\mathbf{r}_1;\tau-\tau_1)\hat{\hat{G}}(\mathbf{r}_1,\mathbf{r}';\tau_1-\tau') = \int_0^\beta d\tau_1\int_V d^3\mathbf{r}_1\hat{\hat{G}}(\mathbf{r},\mathbf{r}_1;\tau-\tau_1)\hat{\hat{G}}^{-1}(\mathbf{r}_1\tau_1,\mathbf{r}';-\tau_1-\tau') = \hat{\hat{1}}\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'),$$

$$\hat{\hat{G}}\left(\mathbf{r};\mathbf{r}',\tau+\frac{1}{T}\right) = -\hat{\hat{G}}(\mathbf{r},\mathbf{r}'\tau),\ \hat{\hat{G}}(\mathbf{r},\mathbf{r}';\tau-\tau') = \begin{bmatrix}\hat{G}(\mathbf{r},\mathbf{r}';\tau-\tau') & \hat{F}(\mathbf{r},\mathbf{r}';\tau-\tau')\\ \hat{F}^+(\mathbf{r},\mathbf{r}';\tau-\tau') & -\left[\hat{G}(\mathbf{r}',\mathbf{r};\tau'-\tau)\right]^t\end{bmatrix}. \quad (71)$$

The conditions (71) mean that the kernel $\hat{\hat{G}}(\mathbf{r};\mathbf{r}',\tau-\tau')$ is a matrix thermodynamic Green function that satisfies equations of the Gorkov type (the first row of Eq. (71)).[11]

The time-independence condition (68) now yields

$$|F(\mathbf{r})| = \frac{T}{2}\left|\mathrm{Sp}\left[i\hat{\sigma}_2\hat{F}(\mathbf{r},\mathbf{r};0)\right]\right|, \tag{72}$$

where Sp denotes taking the trace over the spin indices. In addition, varying Eq. (69) with respect to $\langle\mathbf{A}\rangle$ gives the observed curren

$$\begin{aligned}\langle\mathbf{j}(\mathbf{r})\rangle &= -\frac{\delta\Omega}{\delta\langle\mathbf{A}(\mathbf{r})\rangle}\\ &= \frac{ie}{2m}T\lim_{\mathbf{r}'\to\mathbf{r}}(\nabla_{\mathbf{r}'}-\nabla_{\mathbf{r}})\mathrm{Sp}\,\hat{G}\left(\mathbf{r},\mathbf{r}';-0^+\right)\\ &\quad -\frac{e^2\langle\mathbf{A}(\mathbf{r})\rangle}{m}T\mathrm{Sp}\,\hat{G}\left(\mathbf{r},\mathbf{r}';-0^+\right).\end{aligned} \tag{73}$$

The functions $\hat{F}$ and $\hat{G}$ are found by solving Eq. (71), in which it is first necessary to go to the limit $\varepsilon\to+0$. Thus, Eqs. (71)–(73), supplemented by the Maxwell equations $\nabla\times\langle\mathbf{H}\rangle = 4\pi\langle\mathbf{j}\rangle$, and $\langle\mathbf{H}\rangle = \nabla\times\langle\mathbf{A}\rangle$ and the appropriate boundary conditions, form a complete and closed system and can be used to solve any problem in the theory of equilibrium superconductivity. In particular, in the simplest case of a structurally uniform superconductor without impurities and electromagnetic interactions, Eq. (69), together with the self-consistency conditions (72), yields the well-known expression[13] for the thermodynamic potential obtained by the Bogolyubov grand canonical transformation method.[34]

6. The Josephson effect without a "phase difference"

The theory developed in the previous sections of this paper applied in varying degrees to structurally uniform superconductors and to arbitrary types of superconducting structures containing Josephson junctions (provided, of course, that the correlation between the electrons with antiparallel spins is not disrupted inside the nonsuperconducting barriers; see Section 7). Here, however, the fundamental interest is in the question of how the interpretation of the Josephson effects changes when the customary "phase difference" of the literature[14,21,35] is absent.

Without significant loss of generality, it is enough to study a singly connected Josephson structure in the presence of fields and currents. More specifically, let us examine (Fig. 3) a plane-parallel Josephson structure that is uniform along the z axis and has a low-transparency tunnel barrier of thickness d ($x\in[-d/2,\,d/2]$) and width $L(y\in[-L/2,\,L/2])$. An external magnetic field H_e is applied along the z axis. A transport current I flows along the x axis. It is required to determine the distribution of the magnetic field inside the barrier, $\langle\mathbf{H}(x,y)\rangle = \langle\mathbf{H}(0,y)\rangle \equiv \langle\mathbf{H}(y)\rangle(x\in[-d/2,\,d/2])$, and the maximum transport current, $I_c = I_c(H_e)$.

If the magnetic field H_e and current I depend weakly on $|F|$ (the superconducting order parameter), then the following approximation is sufficiently accurate:[32]

$$\hat{G}(\mathbf{r},\mathbf{r};\omega_n)_{\langle\mathbf{A}\rangle} \approx \hat{G}(\mathbf{r},\mathbf{r};\omega_n)_{\langle\mathbf{A}\rangle=0}\exp\left[ie\int_{\mathbf{r}'}^{\mathbf{r}}(d\mathbf{l}\cdot\langle\mathbf{A}(\mathbf{l})\rangle)\right], \tag{74}$$

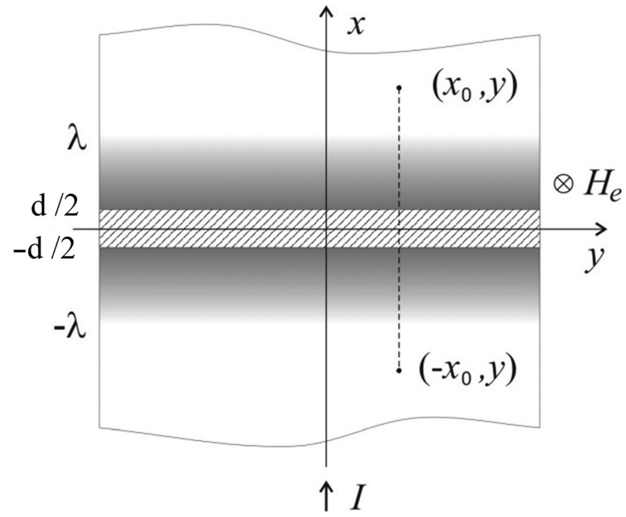

Fig. 3. The geometry of a plane-parallel tunnel Josephson junction (schematic). Here $\mathbf{H}_e = (0,0,H_e)$ is the external magnetic field, λ is the penetration depth of the field into the depth of the superconducting shores, d is the thickness of the tunnel barrier ($d \ll 2\lambda$), L is the width of the barrier ($0<L<\infty$), and $\mathbf{I} = (I,0,0)$ is the specified transport current.

where the integral is taken along the straight line joining the points $\mathbf{r}$ and $\mathbf{r}'$. Given the geometry of the problem, $\langle\mathbf{H}\rangle = (0,0,\langle H\rangle)$. Let the magnetic field $\langle H\rangle$ penetrate into the interior of the superconducting shores (along the x axis) to a depth λ given by

$$\lambda = \frac{1}{\langle H_0\rangle}\int_0^{+\infty}dx\langle H(x)\rangle. \tag{75}$$

Since $I = \int_{-L/2}^{L/2}dy j_J(y)$, where $j_J(y) \equiv \langle j_x(0,y)\rangle$—is the Josephson current through the junction, we begin by calculating j_J. Using Eq. (74) and the method of Refs. 36 and 37, we find

$$\begin{aligned} j_J &= j_c\sin\Phi(y),\\ \Phi(y) &= -2e\int_{(-x_0,y)}^{(x_0,y)}d\xi\langle A_x(\xi,y)\rangle\\ &= \int_{(-x_0,y)}^{(x_0,y)}d\xi\left[\frac{\partial\alpha_{\parallel}(\xi,y)}{\partial\xi} - 2e\langle A_{\perp x}(t,y)\rangle\right],\\ &= \alpha_{\parallel}(x_0,y) - \alpha_{\parallel}(-x_0,y) - 2e\int_{(-x_0,y)}^{(x_0,y)}d\xi\langle A_{\perp x}(\xi,y)\rangle, \end{aligned} \tag{76}$$

where $j_c>0$—is the critical current[14,21] and $x_0 \gg \lambda$ (see Fig. 3).

In order to find the dependence $\langle H_0\rangle = \langle H_0(y)\rangle$, we use the general definition

$$\begin{aligned}\langle H(x,y)\rangle &= \frac{\partial}{\partial x}\langle A_y(x,y)\rangle - \frac{\partial}{\partial y}\langle A_x(x,y)\rangle\\ &= \frac{\partial}{\partial x}\langle A_{\perp y}(x,y)\rangle - \frac{\partial}{\partial y}\langle A_{\perp x}(x,y)\rangle.\end{aligned} \tag{77}$$

Integrating Eq. (77) over the interval $x\in(-x_0,x_0)$ and assuming that $2\lambda \gg d$, we find

$$\int_{(-x_0,y)}^{(x_0,y)} d\xi \langle H(\xi,y)\rangle \approx 2\lambda\langle H_0(y)\rangle = \langle A_{\perp y}(x_0,y)\rangle - \langle A_{\perp y}(-x_0,y)\rangle - \int_{(-x_0,y)}^{(x_0,y)} d\xi \frac{\partial\langle A_{\perp x}(\xi,y)\rangle}{\partial y}. \quad (78)$$

Now we note that at the points (x_0, y) and $(-x_0, y)$ there is no component of the superconducting current parallel to the tunnel barrier, so that

$$\langle A_y(x_0,y)\rangle = \langle A_y(-x_0,y)\rangle = 0. \quad (79)$$

Thus,

$$\langle A_{\perp y}(x_0,y)\rangle = \frac{1}{2e}\frac{\partial\alpha_{\parallel}(x_0,y)}{\partial y}, \quad \langle A_{\perp y}(-x_0,y)\rangle = \frac{1}{2e}\frac{\partial\alpha_{\parallel}(-x_0,y)}{\partial y}. \quad (80)$$

Now we calculate $\frac{1}{2e}\frac{\partial\Phi}{\partial y}$ using Eq. (80)

$$\frac{1}{2e}\frac{\partial\Phi(y)}{\partial y} = \frac{1}{2e}\frac{\partial\alpha_{\parallel}(x_0,y)}{\partial y} - \frac{1}{2e}\frac{\partial\alpha_{\parallel}(-x_0,y)}{\partial y} - \int_{(-x_0,y)}^{(x_0,y)} d\xi \frac{\partial\langle A_{\perp x}(\xi,y)\rangle}{\partial y} = \langle A_{\perp y}(x_0,y)\rangle - \langle A_{\perp y}(-x_0,y)\rangle - \int_{(-x_0,y)}^{(x_0,y)} d\xi \frac{\partial\langle A_{\perp x}(\xi,y)\rangle}{\partial y}. \quad (81)$$

On comparing this with Eq. (78), we obtain Josephson's result[35]

$$\langle H_0(y)\rangle = \frac{1}{4e\lambda}\frac{\partial\Phi(y)}{\partial y}. \quad (82)$$

(Our derivation of Eq. (82) should be compared with the literature.[21,35])

Substituting Eq. (82) in the Maxwell equation $\nabla \times \langle \mathbf{H}\rangle = 4\pi\langle \mathbf{j}\rangle$, we obtain the well-known Ferrel-Prange equation[21]

$$\frac{d^2\Phi}{dy^2} = \frac{1}{\lambda_J^2}\sin\Phi, \qquad \lambda_J^2 = \frac{1}{16\pi|e|\lambda j_c}, \quad (83)$$

where λ_J- is the Josephson penetration depth for the magnetic field along the tunnel barrier. A complete, rigorous solution of this equation for the external conditions formulated above has been obtained elsewhere.[38–40]

In the case of small junctions ($L \ll 2\lambda_J$) without an eternal field, the intrinsic field of the Josephson current can be neglected[39] and Eq. (76) reduces to Josephson's classical result

$$j_J = j_c \sin\left[\alpha_{\parallel}(x_0,0) - \alpha_{\parallel}(-x_0,0)\right], \quad (84)$$

which is interpreted in terms of a "phase difference."[14,21,35] If a superconducting structure with a tunneling barrier has a doubly coupled geometry (the case of SQUIDs), the general expression for the Josephson current (76) remains valid, but now $\Phi(y)$ has a more complicated form

$$\Phi(y) = \alpha_{\parallel}(x_0,y) - \alpha_{\parallel}(-x_0,y) - 2e\int_{(-x_0,y)}^{(x_0,y)} d\xi\langle A_{\perp x}(\xi,y)\rangle + \phi_s(x_0,y) - \phi_s(-x_0,y), \quad (85)$$

where $\phi_s = \phi_s(x,y)$ is the nonsinusoidal function of position, defined in Section 4, and has the significance of an external condition.

7. Discussion and some concluding comments.

It has been shown here that the unique part of the phase of the superconducting order parameter (quasi-average) $\langle\psi_{\downarrow}\psi_{\uparrow}\rangle$ transforms into the longitudinal component of the vector potential of the induced magnetic field over the entire space. Thus, a complete analogy has been established between the Meissner effect in the physics of superconductivity[12–14] and the Higgs mechanism[1] in high-energy physics.

The starting point for our analysis (Section 2) was the microscopic Hamiltonian of the system in the presence of impurities, nonsuperconducting barriers, and a static, external magnetic field (14). (The justification for the Hamiltonian (14) is given in Appendix A.) The distinctive feature of our approach (compared to those in the literature[10–14,21]) is accounting for the quantum nature of the induced electromagnetic field in the Hamiltonian.[14] The latter is quantized under the condition $A_0 = 0$ (A_0 is the scalar potential) without destroying the invariance with respect to the time-independent gauge transformations. (The gauge $A_0 = 0$ is compared with the "traditional" noncovariant gauge $\nabla\mathbf{A} = 0$ in Appendix B.)

In Section 3, we moved from the Hamiltonian (14) to the corresponding Hamiltonian with external "Cooper pair sources" (Eq. (38)) and gave a strict definition of the superconducting order parameter $\langle\psi_{\downarrow}\psi_{\uparrow}\rangle$ as a quasi-average which generalizes Bogolyubov's definition for the spatially uniform case.[28] In that section, a rigorous proof was obtained for the conservation of the superconducting current (Eq. (45)) and it was shown that $\langle\psi_{\downarrow}\psi_{\uparrow}\rangle$ satisfies the condition of time independence for the thermodynamic potential (Eq. (49)). Using the unitary transform (30) under the trace which appears in the definition of the quantum-statistical averages $\langle\psi_{\downarrow}\psi_{\uparrow}\rangle$ and $\langle\mathbf{A}\rangle$, we obtained an algebraic proof of the occurrence of the Higgs mechanism (Section 4).

The rigorous results of Sections 2–4 served as the basis of a new variational formulation of the mean-field approximation in Section 5. The formulas in that section contain only the quantities with physical significance $|\psi_{\downarrow}\psi_{\uparrow}|$, $\langle\mathbf{A}\rangle$, and $(1/2e)\nabla\phi_s$, while the variational method itself is equally applicable to structurally uniform superconductors and to systems containing nonsuperconducting barriers.

In Section 6, a new, field theoretical interpretation of the Josephson effect is proposed which does not use the traditional[14,21,35] concept of a "phase difference" and several examples are discussed. These examples show that, despite the change in the physical interpretation (the appearance of a longitudinal component of the induced vector potential instead of a "phase difference"), the major physical results concerning Josephson structures with nonferromagnetic barriers (calculations of the critical current,[14,21] the critical field for penetration of Josephson vortices,[38–40] oscillatory effects in SQUIDs,[21] etc.) are unchanged. (The special case of Josephson contacts with ferromagnetic barriers, where correlations between electrons with antiparallel spins may be destroyed,[41] requires a separate discussion.)

Our analysis clearly shows, however, that the argument of the sine on the right hand side of Eq. (76), which is a classical quantity by definition, is purely of classical origin for any Josephson structures, with barriers of arbitrary type (including ferromagnetic[41]). In addition, given the assumed smoothness of the potentials $g = g(\mathbf{r})$ and $\hat{U}^b = \hat{U}^b(\mathbf{r})$ (Section 2) for existence of a linear integral when determining $\Phi(y)$ with Eq. (76), it is necessary and sufficient that the superconducting order parameter $|\langle\psi_\downarrow\psi_\uparrow\rangle|$ not go to zero anywhere in the region of the barrier. (This last statement can be regarded as a rigorous mathematical formulation of the necessary and sufficient condition for the existence of a Josephson junction between the shores of a contact.)

The author thanks all the participants in the experimental seminar of A. N. Amel'yanchuk and the theoretical seminar of L. A. Pastur for constructive discussions.

APPENDIX A: THE "CLASSICAL" LAGRANGIAN AND HAMILTONIANS

We determine the "classical" Lagrangian of a superconducting structure with s-wave pairing in an arbitrary gauge using the following formulas:

$$\mathcal{L} = \mathcal{L}_e + \mathcal{L}_b + \mathcal{L}_{\text{imp}} + \mathcal{L}_{BCS} + \mathcal{L}_{\text{em}} + e\int_V d^3\mathbf{r} A_0(\mathbf{r}t) n_i(\mathbf{r}), \tag{A1}$$

$$\begin{aligned}\mathcal{L}_e = \int_V d^3\mathbf{r}\Bigg[\sum_\alpha i\psi_\alpha^*(\mathbf{r}t)\left[\frac{\partial}{\partial t} + ieA_0(\mathbf{r}t)\right]\psi_\alpha(\mathbf{r}t) \\ -\frac{1}{2m}\left[\nabla + ie\mathbf{A}(\mathbf{r}t) + ie\mathbf{A}_e(\mathbf{r})\right]\psi_\alpha^*(\mathbf{r}t) \\ \times\left[\nabla - ie\mathbf{A}(\mathbf{r}t) - ie\mathbf{A}_e(\mathbf{r})\right]\psi_\alpha(\mathbf{r}t)\Bigg],\end{aligned} \tag{A2}$$

$$\mathcal{L}_b + \mathcal{L}_{\text{imp}} = -\int_V d^3\mathbf{r}\sum_{\alpha,\beta}\psi_\alpha^*(\mathbf{r}t)\left[U^b{}_{\alpha\beta}(\mathbf{r}) + U^{\text{imp}}{}_{\alpha\beta}(\mathbf{r})\right]\psi_\beta(\mathbf{r}t), \tag{A3}$$

$$\mathcal{L}_{BCS} = \frac{1}{2}\int_V d^3\mathbf{r}\, g(\mathbf{r})\sum_\alpha \psi_\alpha^*(\mathbf{r}t)\psi_{-\alpha}^*(\mathbf{r}t)\psi_{-\alpha}(\mathbf{r}t)\psi_\alpha(\mathbf{r}t), \tag{A4}$$

$$\mathcal{L}_{\text{em}} = \frac{1}{8\pi}\int_V d^3\mathbf{r}\left[\left[\nabla A_0(\mathbf{r}t) + \frac{\partial\mathbf{A}(\mathbf{r}t)}{\partial t}\right]^2 - \left[\nabla\times\mathbf{A}(\mathbf{r}t)\right]^2\right]. \tag{A5}$$

Here $A_\mu = (A_0, -\mathbf{A})$—is the classical 4-vector potential of the induced electromagnetic field; ψ_α and ψ_α^* are the "classical" Grassman fields which obey the anticommutation relations

$$\begin{aligned}\left[\psi_\alpha(\mathbf{r}t), \psi_\beta(\mathbf{r}'t)\right]_+ &= \left[\psi_\beta^*(\mathbf{r}t), \psi_\beta^*(\mathbf{r}'t)\right]_+ \\ &= \left[\psi_\alpha(\mathbf{r}t), \psi_\beta^*(\mathbf{r}'t)\right]_+ = 0.\end{aligned} \tag{A6}$$

Let us assume that the fields A_μ and ψ_α, ψ_α^* are specified over a finite time interval $t \in [t_1, t_2]$ and the following boundary conditions are satisfied:

$$\begin{aligned}&A_\mu(\mathbf{r}t_2) = A_\mu(\mathbf{r}t_1), \quad \psi_\alpha(\mathbf{r}t_2) = -\psi_\alpha(\mathbf{r}t_1), \\ &\psi_\alpha^*(\mathbf{r}t_2) = -\psi_\alpha^*(\mathbf{r}t_1).\end{aligned} \tag{A7}$$

The last term on the right of Eq. (A1) describes the electromagnetic interaction with the ion "background" in a "jelly" model.[13] All the other notation Eqs. (A1)–(A5) becomes clear upon comparison with the corresponding notation of Section 2.

The density of the Lagrangian (the integrand in Eqs. (A1)–(A5)) is invariant with respect to the global gauge transformation

$$\psi_\alpha \to e^{ie\chi_0}\psi_\alpha, \quad \psi_\alpha^* \to e^{-ie\chi_0}\psi_\alpha^*, \quad \mathbf{A} \to \mathbf{A}, \quad \chi_0 = \text{const.} \tag{A8}$$

This property leads to the charge conservation law

$$\frac{e\partial n_e}{\partial t} + \nabla\mathbf{j} = 0, \tag{A9}$$

where

$$\begin{aligned}n_e(\mathbf{r}t) &= \sum_\alpha \psi_\alpha^*(\mathbf{r}t)\psi_\alpha(\mathbf{r}t), \\ \mathbf{j}(\mathbf{r}t) &= \sum_\alpha\Bigg[\frac{ie}{2m}\left[\psi_\alpha^*(\mathbf{r}t)\nabla\psi_\alpha(\mathbf{r}t) - \left[\nabla\psi_\alpha^*(\mathbf{r}t)\right]\psi_\alpha(\mathbf{r}t)\right] \\ &\quad -\frac{e^2}{m}\left[\mathbf{A}(\mathbf{r}t) + \mathbf{A}_e(\mathbf{r})\right]\psi_\alpha^*(\mathbf{r}t)\psi_\alpha(\mathbf{r}t)\Bigg].\end{aligned} \tag{A10}$$

Without the last term on the right of Eq. (A1), the density of the Lagrangian is also invariant with respect to the local gauge transformation

$$\begin{aligned}&\psi_\alpha \to e^{ie\chi}\psi_\alpha, \quad \psi_\alpha^* \to e^{-ie\chi}\psi_\alpha^*, \\ &\mathbf{A} \to \mathbf{A} + \nabla\chi, \quad A_0 \to A_0 - \frac{\partial\chi}{\partial t}, \quad \chi = \chi(\mathbf{r}t).\end{aligned} \tag{A11}$$

In general, only the action $S = \int_{t_1}^{t_2} dt\mathcal{L}$ will be invariant with respect to the transformation (A11) (because of the boundary conditions (A7)).

We now find the momenta p_ψ, p_{ψ^*} and p_{A_0}, $p_{\mathbf{A}}$ which are canonically conjugate to the fields ψ, ψ^* and A_0, $\mathbf{A}$

$$\begin{aligned}&p_{\psi_\alpha} = \frac{\boldsymbol{\delta}\mathcal{L}}{\delta\dfrac{\partial\psi_\alpha}{\partial t}} = i\psi_\alpha^*, \quad p_{\psi_\alpha^*} = \frac{\boldsymbol{\delta}\mathcal{L}}{\delta\dfrac{\partial\psi_\alpha^*}{\partial t}} \equiv 0; \\ &p_{A_0} = \frac{\delta\mathcal{L}}{\delta\dfrac{\partial A_0}{\partial_t}} \equiv 0, \quad p_{\mathbf{A}} = \frac{\delta\mathcal{L}}{\delta\dfrac{\partial\mathbf{A}}{\partial t}} = \frac{1}{4\pi}\left(\nabla A_0 + \frac{\partial\mathbf{A}}{\partial t}\right) = -\frac{\mathbf{E}}{4\pi},\end{aligned} \tag{A12}$$

where the symbols $\frac{\boldsymbol{\delta}}{\delta\frac{\partial\psi_\alpha}{\partial t}}$ and $\frac{\boldsymbol{\delta}}{\delta\frac{\partial\psi_\alpha^*}{\partial t}}$ denote the left and right variational derivatives, respectively.[5] Using Eq. (A12) and the definition

$$\mathcal{H} \equiv \int_V d^3\mathbf{r}\left[\sum_\alpha p_{\psi_\alpha}(\mathbf{r}t)\frac{\partial\psi_\alpha(\mathbf{r}t)}{\partial t}\right] + \int_V d^3\mathbf{r}p_\mathbf{A}(\mathbf{r}t)\frac{\partial\mathbf{A}(\mathbf{r}t)}{\partial t} - \mathcal{L},$$

we arrive at the "classical" Hamiltonian in an arbitrary gauge:

$$\mathcal{H} = \mathcal{H}_e + \mathcal{H}_b + \mathcal{H}_{\text{imp}} + \mathcal{H}_{BCS} + \mathcal{H}_{\text{em}} + \int_V d^3\mathbf{r}\left[\frac{\mathbf{E}(\mathbf{r}t)\,\nabla A_0(\mathbf{r}t)}{4\pi} + e\left[n_e(\mathbf{r}t) - n_i(\mathbf{r})\right]A_0(\mathbf{r}t)\right], \tag{A13}$$

$$\mathcal{H}_e = \int_V d^3\mathbf{r}\left[\frac{1}{2m}\sum_\alpha\left[\nabla + ie\mathbf{A}(\mathbf{r}t) + ie\mathbf{A}_e(\mathbf{r})\right]\psi^*_\alpha(\mathbf{r}t) \times \left[\nabla - ie\mathbf{A}(\mathbf{r}t) - ie\mathbf{A}_e(\mathbf{r})\right]\psi_\alpha(\mathbf{r}t)\right], \tag{A14}$$

$$\mathcal{H}_b + \mathcal{H}_{\text{imp}} = \int_V d^3\mathbf{r}\sum_{\alpha,\beta}\psi^*_\alpha(\mathbf{r}t)\left[U^b{}_{\alpha\beta}(\mathbf{r}) + U^{\text{imp}}{}_{\alpha\beta}(\mathbf{r})\right]\psi_\alpha(\mathbf{r}t) \tag{A15}$$

$$\mathcal{H}_{BCS} = \frac{1}{2}\int_V d^3\mathbf{r}g(\mathbf{r})\sum_\alpha\psi^*_\alpha(\mathbf{r}t)\psi^*_{-\alpha}(\mathbf{r}t)\psi_{-\alpha}(\mathbf{r}t)\psi_\alpha(\mathbf{r}t), \tag{A16}$$

$$\mathcal{H}_{\text{em}} = \frac{1}{8\pi}\int_V d^3\mathbf{r}\left[\mathbf{E}^2(\mathbf{r}t) + (\nabla\times\mathbf{A})^2(\mathbf{r}t)\right], \quad \mathbf{E} = -\nabla\times A_0 - \frac{\partial\mathbf{A}}{\partial t}. \tag{A17}$$

It is useful to note that, ongoing to the noncovariant quantization scheme for $\mathcal{H}$, the last term on the right of Eq. (A13) can be neglected. In fact, the first term in the square brackets can be integrated by parts, and the integrated surface term vanishes because it has been assumed that the field $\mathbf{E}$ falls off rapidly toward infinity. The remaining expression is identically equal to zero in the $A_0 = 0$ gauge, as in the $\nabla\mathbf{A} = 0$ gauge (because of the operator Gauss law).

APPENDIX B: COMPARISON OF THE GAUGE CONDITIONS ∇A = 0 AND A_0 = 0

When the $\nabla\mathbf{A} = 0$ gauge is used, the quantized Maxwell equations (23)–(26) are supplemented further by the Gauss law in operator form

$$\nabla\mathbf{E} = 4\pi e(n_e - n_i). \tag{B1}$$

Equation (B1) makes it possible to eliminate the scalar potential A_0

$$A_0(\mathbf{r}t) = e\int_V \frac{d^3\mathbf{r}'\left[n_e(\mathbf{r}'t) - n_i(\mathbf{r}')\right]}{|\mathbf{r}-\mathbf{r}'|}. \tag{B2}$$

Thus, the quantum Hamiltonian in this gauge has the form

$$\mathcal{H}_{\nabla\mathbf{A}=0} = \mathcal{H}_e[\mathbf{A}_\perp + \mathbf{A}_e] + \mathcal{H}_b + \mathcal{H}_{\text{imp}} + \mathcal{H}_{BCS} + \mathcal{H}_{\text{em}}, \tag{B3}$$

where

$$\mathcal{H}_e[\mathbf{A}_\perp + \mathbf{A}_e]\int_V d^3\mathbf{r}\left[\frac{1}{2m}\sum_\alpha\left[\nabla + ie\mathbf{A}_\perp(\mathbf{r}t) + ie\mathbf{A}_e(\mathbf{r})\right] \times\psi^+_\alpha(\mathbf{r}t)\left[\nabla - ie\mathbf{A}_\perp(\mathbf{r}t) - ie\mathbf{A}_e(\mathbf{r})\right]\psi_\alpha(\mathbf{r}t)\right], \tag{B4}$$

and the electromagnetic energy breaks up into two components

$$\mathcal{H}_{\text{em}} = \mathcal{H}_{\text{em}\perp} + \mathcal{H}_C = \frac{1}{8\pi}\int_V d^3\mathbf{r}\left[\mathbf{E}^2_\perp(\mathbf{r}t) + \left[\nabla\times\mathbf{A}_\perp(\mathbf{r}t)\right]^2\right] + \frac{e^2}{2}\int_V d^3\mathbf{r}\int_V d^3\mathbf{r}'\frac{\left[n_e(\mathbf{r}t) - n_i(\mathbf{r})\right]\left[n_e(\mathbf{r}'t) - n_i(\mathbf{r}')\right]}{|\mathbf{r}-\mathbf{r}'|},$$

$$\mathbf{E}_\perp = -\frac{\partial\mathbf{A}_\perp}{\partial t}. \tag{B5}$$

The operators $\mathbf{A}_\perp$ and $\partial\mathbf{A}_\perp/\partial t$, in Eqs. (B4) and (B5) obey the commutation relations in the first two lines of Eq. (20). The remaining terms in the Hamiltonian (B3) are the same as in the Hamiltonian of Eq. (14).

We now show that the quantum Hamiltonian $\mathcal{H}$ in an $A_0 = 0$ gauge (Eq. (14)) used in this paper leads to the same value of the grand statistical sum (32) as the quantum Hamiltonian $\mathcal{H}_{\nabla\mathbf{A}=0}$ (Eq. (B3)) introduced above. We begin with the definition of the energy levels $E_{k,N}$

$$E_{k,N} = \langle N, E_{k,N}, n_i|\mathcal{H}|N, E_{k,N}, n_i\rangle,$$

where the operators in $\mathcal{H}$ are assumed to be time independent (i.e., specified in the Schroedinger representation).

We expand the vector potential $\mathbf{A}$ and electric field $\mathbf{E}$ into transverse and parallel components (cf. Eq. (19))

$$\mathbf{A} = \mathbf{A}_\perp + \mathbf{A}_\parallel, \quad \mathbf{E} = \mathbf{E}_\perp + \mathbf{E}_\parallel,$$

where $\mathbf{A} = \mathbf{A}(\mathbf{r})$ and $\mathbf{E} = \mathbf{E}(\mathbf{r})$. Given that

$$\int_V d^3\mathbf{r}\mathbf{E}_\perp(\mathbf{r})\mathbf{E}_\parallel(\mathbf{r}) = 0,$$

and the eigenvalue equation

$$(\nabla\mathbf{E} - 4\pi e n_e)|N, E_{k,N}, n_i\rangle = -4\pi e n_e|N, E_{k,N}, n_i\rangle,$$

we obtain

$$E_{k,N} = \langle N, E_{k,N}, n_i|\tilde{\mathcal{H}}|N, E_{k,N}, n_i\rangle,$$

where

$$\tilde{\mathcal{H}} = \mathcal{H}_e + \mathcal{H}_b + \mathcal{H}_{\text{imp}} + \mathcal{H}_{BCS} + \mathcal{H}_{\text{em}\perp} + \mathcal{H}_C,$$

with

$$\mathcal{H}_e = \int_V d^3\mathbf{r}\frac{1}{2m}\left[\sum_\alpha\left[\nabla + ie\left(\mathbf{A}_\perp(\mathbf{r}) + \mathbf{A}_\parallel(\mathbf{r})\right) + ie\mathbf{A}_e(\mathbf{r})\right]\psi^+_\alpha(\mathbf{r})\left[\nabla - ie\left(\mathbf{A}_\perp(\mathbf{r}) + \mathbf{A}_\parallel(\mathbf{r})\right) - ie\mathbf{A}_e(\mathbf{r})\right]\psi_\alpha(r)\right],$$

$$\mathcal{H}_b + \mathcal{H}_{\text{imp}} = \int_V d^3\mathbf{r}\sum_{\alpha,\beta}\psi^+_\alpha(\mathbf{r})\left[U^b_{\alpha\beta}(\mathbf{r}) + U^{\text{imp}}{}_{\alpha\beta}(\mathbf{r})^{\alpha\beta}\right]\psi_\beta(\mathbf{r}t),$$

$$\mathcal{H}_{BCS} = -\frac{1}{2}\int_V d^3\mathbf{r}|g(\mathbf{r})|\sum_\alpha\psi^+_\alpha(\mathbf{r})\psi^+_{-\alpha}(\mathbf{r})\psi_{-\alpha}(\mathbf{r})\psi_\alpha(\mathbf{r}),$$

$$\mathcal{H}_{\text{em}\perp} + \mathcal{H}_C = \frac{1}{8\pi}\int_V d^3\mathbf{r}\left[E^2_\perp(\mathbf{r}) + \left[\nabla\times\mathbf{A}_\perp(\mathbf{r})^2\right] + \frac{e^2}{2}\int_V d^3\mathbf{r}\int_V d^3\mathbf{r}'\frac{\left[n_e(\mathbf{r}) - n_i(\mathbf{r})\right]\left[n_e(\mathbf{r}') - n_i(\mathbf{r}')\right]}{|\mathbf{r}-\mathbf{r}'|}\right].$$

It can be seen that the energy levels $E_{k,N}$ can be represented as averages of the new Hamiltonian $\tilde{\mathcal{H}}$, in which the longitudinal part of the electric field energy $\int_V d^3\mathbf{r}\mathbf{E}_{\|}^2(\mathbf{r})$ is replaced by the Coulomb energy $\mathcal{H}_C$. The definition of $\mathcal{H}_e$ contains a longitudinal variable $\mathbf{A}_{\|}$ that commutates with all the operators contained in the Hamiltonian $\tilde{\mathcal{H}}$, d can be eliminated with the aid of the unitary transformation

$$U_\psi[\nu] = \exp\left[ie\int d^3\mathbf{r}\nu(\mathbf{r})n_e(\mathbf{r})\right],$$
$$\nu(\mathbf{r}) = -\frac{1}{4\pi}\int_V d^3\mathbf{r}'\frac{\nabla\mathbf{A}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|},\quad [U_\psi,\mathcal{N}] = 0.$$

We finally obtain

$$E_{k,N} = \langle N, E_{k,N}, n_i|U_\psi^{-1}[\nu]U_\psi[\nu]\tilde{H}U_\psi^{-1}[\nu]U_\psi[\nu]|N, E_{k,N}, n_i\rangle$$
$$= \langle N, E_{k,N}|\mathcal{H}_{\nabla\mathbf{A}=\mathbf{0}}|N, E_{k,N}\rangle,$$

where the Hamiltonian $\mathcal{H}_{\nabla\mathbf{A}=0}$ is given by Eq. (B3) (with time-independent operators).

As pointed out in Section 2, the gauge $A_0 = 0$ is preferable for physical reasons. In fact, if we used the condition $\nabla\mathbf{A} = 0$, in all the equations of Section 5 instead of the combination $\langle\mathbf{A}\rangle - \frac{1}{2e}\nabla\phi_s$, this would yield the combination

$$\langle\mathbf{A}_\perp\rangle - \frac{1}{2e}\nabla\phi, \tag{B6}$$

which contains a nonphysical unique part of the phase $\tilde{\phi}$ $(\phi + \tilde{\phi} + \phi_s)$. In addition, Eq. (B6) would contradict the physical interpretation of the Meissner effect as the "acquisition of mass by a photon,"[4,5,9] since the full vector potential would only contain two independent components $(\langle\mathbf{A}\rangle = \langle\mathbf{A}_\perp\rangle)$.

Finally, we note that the electrical neutrality condition (35) in the $\nabla\mathbf{A} = 0$ gauge does not follow automatically from averaging $\mathbf{E} = -\nabla A_0 - \partial\mathbf{A}/\partial t$ over the grand canonical ensemble. This condition must be specified additionally in order to match the result with the situation when $A_0 = 0$.

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Translated by D. H. McNeill