SUPERCONDUCTIVITY, INCLUDING HIGH-TEMPERATURE SUPERCONDUCTIVITY

Stochastic-parametric amplification of narrow-band signals in a single-junction SQUID interferometer

O. G. Turutanov,[a)] V. I. Shnyrkov, and A. M. Glukhov

B. Verkin Institute for Low Temperature Physics and Engineering, National Academy of Sciences of Ukraine, pr. Lenina 47, Kharkov 61103, Ukraine

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The features of the response of a single-junction superconducting quantum interferometer to a low-frequency harmonic signal in the presence of noise and a high-frequency electromagnetic field are investigated through numerical solution of the equations of motion. It is shown that in this situation a system described by a double-well potential will display stochastic-parametric amplification of weak harmonic signals owing to the cooperative effects of noise and the high-frequency field. The gain is a nonmonotonic function of the amplitude of the high-frequency field and the variance of the noise flux and passes through a maximum. A detailed numerical analysis of the dependence of the gain on the noise intensity and on the frequency and amplitude of the high-frequency field is carried out in the stochastic, parametric, and stochastic-parametric amplification regimes. It is shown that at optimal amplitude of the high-frequency field the gain for a weak harmonic signal reaches rather high values (10–30). The specific properties of stochastic-parametric amplification are discussed, and the possible applications of this effect for constructing input circuits of detectors based on SQUIDs with registration of weak narrow-band information signals are considered. © *2008 American Institute of Physics*.
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INTRODUCTION

Because of their high sensitivity, SQUIDS are widely used in carrying out unique physical experiments, and they have therefore been stimulating considerable interest in the study of these devices and in the refinement of methods of using them to detect weak low-frequency signals. Among the topics of study in the physics of SQUIDs is the noise-induced increase of the degree of order in a superconducting ring closed by a Josephson junction (an rf SQUID interferometer) under the joint influence of a weak information signal and noise.[1–3] In this case an unusual situation can arise in which thermodynamic fluctuations with energy k_BT (k_B is Boltzmann's constant, and T is the temperature), exciting a noise flux with a variance $\langle\Phi_N^2\rangle \approx k_BTL$ in an interferometer with inductance L, play a constructive role, amplifying the response of the system to a weak external signal. This phenomenon, called the stochastic resonance (SR), is of a rather general character and is manifested in a large class of open nonlinear dynamical systems.[4–6] The underlying basis of this effect is that an external periodic signal causes stochastic synchronization between individual random events of transition between metastable states under the influence of the noise.

In the analysis of the SR it is usually assumed that the optimal stochastic amplification can be obtained by varying the noise intensity,[4–6] but in the construction of detectors based on superconducting quantum interferometers the value of the inductance is chosen from the condition of weak suppression of the interference[7] by thermal fluctuations. Moreover, for complex multichannel systems based on SQUIDs it is desirable for the bath temperature to be rather low and remain constant in the course of the measurements. At a constant temperature one can control the SR gain by varying the parameters of the interferometer, but that, as a rule, leads to marked complication of the whole measurement scheme, and it is therefore necessary to analyze alternative mechanisms for controlling the parameters of a nonlinear dynamical system for the purpose of achieving the maximum gain at a nonoptimal (weak) influence of fluctuations on the interferometer.

The potential energy of an interferometer containing a single Josephson junction is described by the expression

$$U(x,x_e) = \frac{1}{2}(x - x_e)^2 - \frac{\beta}{4\pi^2}\cos 2\pi x, \qquad (1)$$

where $x_e=\Phi_e/\Phi_0$ and $x=\Phi/\Phi_0$ are the external and internal magnetic fluxes, normalized to the flux quantum $\Phi_0=h/2e$ (we use the SI system; h is Planck's constant, and e is the charge of the electron), $\beta=2\pi LI_c/\Phi_0$ is a basic parameter of the interferometer, L is the inductance of the ring, and I_c is the critical current of the Josephson junction. If the external bias x_{dc} is equal to half the flux quantum, i.e., $x_e=x_{\mathrm{dc}}=0.5$, then for moderate values of β the potential energy has a symmetric form with two minima separated by a potential barrier of a height ΔU (see Fig. 1) that depends on β. With increasing value of β the number of local minima increases (inset in Fig. 1), but for low-frequency, weak signals and moderate values of the thermal fluctuations, one may consider only the two lowest metastable states, i.e., reduce the problem to motion in a double-well potential.

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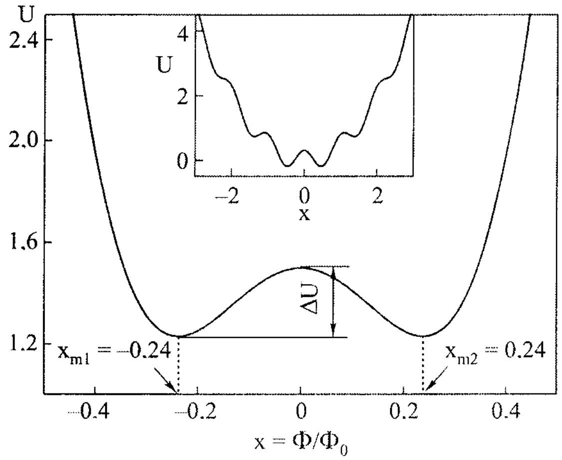

FIG. 1. Potential energy U of a single-junction interferometer as a function of the normalized magnetic flux x for $x_e = x_{dc} = 0.5$; the parameter $\beta = 1.5$. The inset shows the same for $\beta = 12$.

For sufficiently high barriers $\Delta U/D \gg 1$ the mean frequency r_k (the probability per unit time) of thermally activated transitions between two metastable current states is determined by the Kramers law:[8]

$$r_k = \frac{\omega_0 \omega_b}{2\pi\gamma} \exp\left(-\frac{\Delta U}{D}\right), \tag{2}$$

where ω_0 and ω_b are the characteristic frequencies of oscillations at the extremal points of the potential, γ is the dissipation coefficient, and D is the noise intensity (variance). The relaxation dynamics of the motion in a critical-current-dependent potential is determined by the capacitance C and normal resistance R of the junction and is characterized by the dimensionless parameter $\beta_C = 2\pi R^2 I_C C/\Phi_0$. In superconducting interferometers, to reduce the effects due to inertial oscillations, the values of the normal resistance, capacitance, and critical current are chosen from the condition $\beta_C < 1$.

In the most interesting case, that of an interferometer with a low-resistance π junction, the strong-dissipation limit $\beta_C \ll 1$ is realized, and the inertial oscillations do not arise. In this limit the probability of decay (2) of metastable current states of the interferometer owing to thermal fluctuations can be written in the form

$$r_k = (\omega^2/2\pi\gamma)\exp(-\Delta U/k_B T), \tag{3}$$

where $\omega \simeq (LC)^{-1}$ is the frequency of the natural electromagnetic oscillations of a ring with a junction in the absence of dissipation, and $\gamma = 1/(RC)$. The pre-exponential factor thus determines the frequency of attempts of the ring to undergo a transition from one metastable state to another, with allowance for dissipation in the contact. A weak harmonic signal periodically varies the external magnetic flux by Δx_e, modulating the barrier height as $\Delta U \sim 1/L(\Delta x_e)^{3/2}$ (Ref. 9). It follows from expression (3) that under the influence of a signal, the mean probability of transitions per unit time, r_k, will vary periodically with a frequency equal to twice the signal frequency f_s, mainly because of the dependence of the barrier height ΔU on the signal amplitude. If at a certain optimal noise power the condition $r_k \cong 2f_s$ is satisfied, a sort of synchronization of the information signal and the random process of decay of metastable states arises, so that transitions with the frequency of the weak information signal become the most probable. The "synchronization" of the random and periodic processes leads to the circumstance that the spectral density of current circulating in the interferometer (and of the magnetic moment) acquires an amplified component at the frequency of the information signal and its harmonics.[1–3]

Control of the SR by varying the critical current in interferometers with a special 4-terminal Josephson junction has been discussed previously.[10] Recently it has been proposed[11] to use the SR in a system consisting of an ensemble of coupled interferometers on the basis of π junctions[12] for the creation of special stochastic antennas intended for amplification of weak harmonic signals. There it was assumed that the optimal noise intensity for SR is specified in the antenna independently of the temperature of the measuring SQUID itself. The latter condition is difficult to meet experimentally, especially in an integrated implementation of the measuring channels used in modern multichannel biomagnetic systems, SQUID microscopes, defectoscopes, etc. If it is taken into account that these devices are rather complex, optimization of the values of all the parameters and the choice of measurement strategy with optimization of the SR through an increase in temperature becomes an extremely difficult problem.

In Ref. 13, in a discussion of the interaction of an rf SQUID with a microwave field, it was shown that the influence of the field largely reduces to a renormalization of the parameter β, i.e., the effective height of the potential barrier. Subsequently[14] this effect was used as a basis for calculating the parametric gain of a weak information signal in a single-junction interferometer with microwave pumping without taking thermodynamic fluctuations into account. Recently[15] a general analysis of an analogous process was carried out and particularized to the case of a superconducting quantum interferometer with three Josephson junctions. The authors of Ref. 16 investigated the dynamic symmetry-breaking of a model potential due to the mixing of two harmonics with variable amplitude and initial phase shift for control of a stochastic resonance. In a recent paper[17] the case of stochastic amplification of a weak monochromatic signal in the presence of a high-frequency signal in a system with a biquadratic model potential, which is often used for analytical investigations of SR. In that paper it is shown asymptotically and numerically that one can obtain a gain greater than unity for a suitable choice of the the ratio of the amplitude of the high-frequency signal to its frequency.

Our goal in this paper is to analyze the dynamic processes in a double-well potential of a single-junction interferometer in the limit of strong dissipation ($\beta_C \ll 1$) under the simultaneous influence of a weak information signal, a random force (noise), and a high-frequency electromagnetic field. Attention is devoted mainly to an investigation of the gain of a weak information signal in relation to the noise intensity and the frequency and amplitude of the high-frequency field.

MODEL AND NUMERICAL SIMULATION TECHNIQUE

In this paper we carry out a numerical analysis of the amplification of a weak harmonic signal by a single-junction interferometer under two main assumptions. One is that the thermodynamic fluctuations are not large, $r_k \ll 2f_s$, and the other is that the frequency of the high-frequency electromagnetic field $f_p \gg f_s$. The high-frequency field $x_p = A \sin 2\pi f_p t$ is the source of energy for amplification of the information signal $x_s = a \sin 2\pi f_s t$, and it can therefore be treated as a "high-frequency pump." Furthermore, at a temperature T there is a noise flux in the interferometer, with an rms deviation

$$\langle x_N^2 \rangle^{1/2} = (k_B T L)^{1/2} \Phi_0, \tag{4}$$

and the inductance L of the interferometer satisfies the condition[18]

$$L \ll L_F = \frac{1}{k_B T}\left(\frac{\Phi_0}{2\pi}\right)^2. \tag{5}$$

We shall assume that the weak link in the interferometer is a Josephson π junction. Such an interferometer has a symmetric double-well potential in the ground state even without biasing by a dc flux bias $\Phi_{dc} = n\Phi_0$, $n = 0, 1, 2, \ldots$. Experimental and theoretical studies[12] show that the majority of the properties of π junctions are described to good accuracy by a modified resistive model. In this model the total conduction current through the junction is written in the form of a sum of the supercurrent $I_c \sin \varphi$, the "normal" current V/R, and the bias current. Substitution of the total current into the basic equation for a single-junction interferometer brings that equation to the form[19]

$$LC\frac{d^2}{dt^2}\frac{\Phi(t)}{\Phi_0} + \frac{L}{R}\frac{d}{dt}\frac{\Phi(t)}{\Phi_0} + \frac{\beta}{2\pi}\sin 2\pi\frac{\Phi(t)}{\Phi_0} = \frac{\Phi_e(t)}{\Phi_0}, \tag{6}$$

where the external flux $\Phi_e = \Phi_{dc} + \Phi_{ac}(t) + \Phi_N(t)$ consists of a sum of the magnetic fluxes of the signal, high-frequency pump $\Phi_{ac} = a\Phi_0 \sin 2\pi f_s t + A\Phi_0 \sin 2\pi f_p t$ and the noise flux $\Phi_N(t) = \Phi_0 \xi(t)$, $\langle \xi(t)\xi(t')\rangle = 2D_x \delta(t-t')$, D_x is the variance of the normalized noise flux, and Φ_{dc} is set equal to zero.

Choosing the inductance of the interferometer from the condition $L \ll L_F$, for the case of a π junction with a characteristic capacitance $C \sim 10^{-15}$ F and normal resistance $R \sim 1\Omega$, we obtain $\beta_C \ll 1$. In this case the second-derivative term is negligible in comparison with the rest, and we shall drop it. Since the Josephson junction based on a high-T_c superconductor usually have small values of RC, the analogous conditions are easily satisfied, even for interferometers with cooling at the nitrogen level, through a suitable decrease of L_F and increase of the parameter β. Going over to normalized fluxes $x_i = \Phi_i/\Phi_0$, we finally obtain an equation of motion of the system which is convenient for numerical analysis:

$$\frac{dx}{dt} = \frac{1}{\tau_L}\left(x_e - x + \frac{\beta}{2\pi}\sin 2\pi x\right). \tag{7}$$

In the analysis of the influence of weak fluctuations on an interferometer it is usually assumed that thermodynamic fluctuations of the flux are in the form of delta-correlated noise $\xi(t)$ with a normal distribution with variance $D_x = k_B T L/\Phi_0^2$ and an unbounded frequency band. In our calculations the noise band is bounded above by a value f_c determined by the discreteness of the representation of the time of the process and which is chosen in the range 2–32 kHz for the harmonic signal with frequency $f_s = 2$ Hz. In other words, we specify an exponentially correlated noise with a correlation time $1/f_c$: $\langle \xi(t)\xi(t')\rangle \sim \exp(-f_c(t-t'))$. The justification for choosing the value of this parameter will be discussed in the next Section.

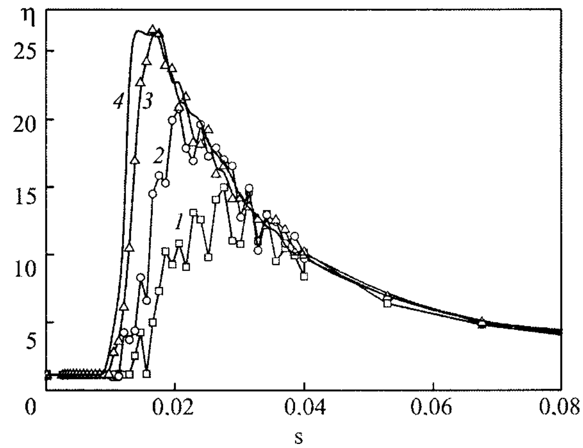

FIG. 2. Stochastic resonance: gain η of the magnetic field in the interferometer as a function of the noise level s with a frequency band bounded above by the cutoff frequency f_c [Hz]: 128 (*1*), 512 (*2*), 2048 (*3*), 4096 (*4*) for $f_s = 2$ Hz, $a = 0.01$.

Equation (7) belongs to the class of stiff problems, since it simultaneously contains at least two highly different time scales (the flux damping time in the ring, $\tau_L = L/R \sim 10^{-10}$ s, and the period of the weak information signal $1/f_s = 0.5$ s) for changes of the variable φ. For solving it we have chosen the standard method based on a modified second-order Rosenbrock formula[20] with rapid convergence. The initial condition is specified in the form $x(0) = x_{min}$, where x_{min} is the value of the flux corresponding to one of the two minima of the potential $U(x)$ at the specified value of β. The total input and output signals are subjected to fast Fourier transformation to obtain the corresponding amplitude spectrum. The gain was determined as the ratio of the fundamental spectral harmonic of the output signal (i.e., at the frequency of the information signal) to the corresponding value for the input signal. All of the case, unless specifically noted, were done for an interferometer with parameter $\beta = 1.5$.

STOCHASTIC RESONANCE INTERFEROMETER IN A HIGH-FREQUENCY FIELD. RESULTS AND DISCUSSION

The solution of equation (7) with an external flux of the form $x_e = a \sin 2\pi f_s t + s\xi(t)$ for different values of the noise intensity, specified by the standard deviation $s \equiv D_x^{1/2}$, gives the well-known (see, e.g., Refs. 1–4) "resonance" form of the gain η as a function of s (Fig. 2). As we have said above, in the numerical analysis the noise spectrum is bounded above by a cutoff frequency f_c. Additional calculations show that the gain η is practically independent of f_c provided that it is at least 1000 times larger than the signal frequency f_s (curve *4* in Fig. 2). In the majority of calculations the value $f_c = 2048$ Hz was chosen at a signal frequency $f_s = 2$ Hz (see curve *3* in Fig. 2). At lower values of f_c the gain fell off, and the scatter of its values increased sharply (curves *2* and *3* in Fig. 2). This result is easily explained if the noise process is treated as a kind of mechanism that brings about "sampling" of the weak, low-frequency information signal. The cutoff frequency for the noise determines the maximum frequency of attempts of the system to jump from one potential well to

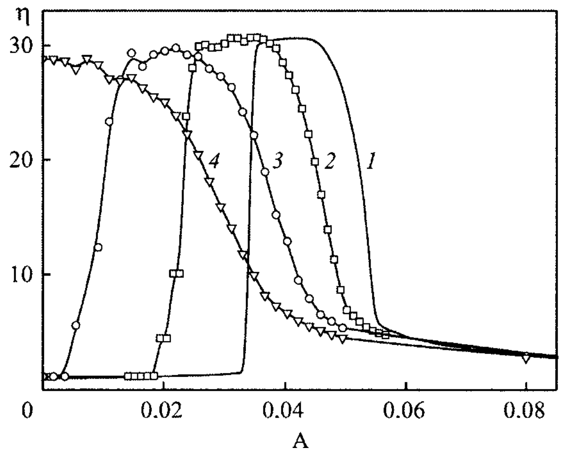

FIG. 3. Gain η of the magnetic flux in the interferometer as a function of the amplitude of the high-frequency field A for different values of the thermal noise s: 0 (*1*), 0.025 (*2*), 0.05 (*3*), 0.075 (*4*) for $\beta=1.5$, $f_s=2$ Hz, $a=0.01$, $f_p=256$ Hz, $f_c=32768$ Hz.

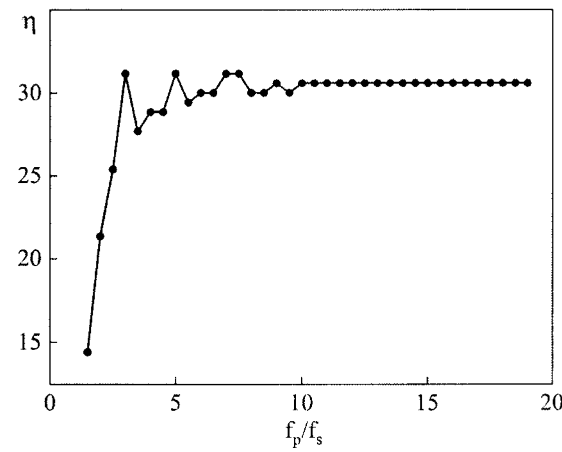

FIG. 4. Maximum parametric gain η of the interferometer as a function of the ratio of the pump frequency f_p to the information signal frequency f_s at $f_s=2$ Hz, $a=0.01$.

another. Owing to a certain (e.g., Gaussian) distribution of instantaneous values of the noise, some of these attempts turn out to be successful and can indeed effect "sampling" and promote amplification of a weak signal. The probability of transition between wells (and the residence time of the system in each well) at every point in time depends on the instantaneous value of the weak periodic information signal. In order for amplification to occur upon realization of SR, the frequency of successful attempts must be appreciably (theoretically, according to the Kotel'nikov theorem, at least a factor of 2) greater than the frequency of the information signal. Further increase of the frequency of successful attempts leads to an increase of the frequency of asymmetric transitions from well to well but does not alter the average residence time of the system in each of the wells and therefore does not affect the amplitude of the signal-frequency Fourier component of the total flux of the ring.

Figure 3 shows the curves of the gain η versus amplitude A of the high-frequency pump at various levels of thermodynamic fluctuations of the system. It is seen that if the noise power in the system is less than the value necessary for obtaining maximum stochastic amplification, then the maximum of the gain can be reached through additional high-frequency pumping. A practical criterion for choosing the frequency f_p of the high-frequency pump is the value at which the weak-signal gain η stops varying upon further increase of the frequency f_p.

Figure 4 shows the curve of the maximum parametric gain $\eta_{\max}$ (achieved at some frequency f_p which is individual for each point) versus the frequency ratio f_p/f_s at a given frequency f_p. The nonmonotonic nature of the dependence $\eta_{\max}(f_p/f_s)$ reflects the influence of the commensurability or inncommensurability of close frequencies f_p and f_s and the initial phase difference between the information and pump signals. As the ratio f_p/f_s is increased, the commensurability of the frequencies and the value of the initial phase difference of the two signals play an ever smaller role, and the gain η ceases to depend on the frequency of the pump signal. One can assume η to be constant for $f_p/f_s>10$. With the the noise and high-frequency pump corresponding to maximum gain, the Fourier amplitude spectrum of the output signal (Fig. 5) displays a large number of higher harmonics of the useful signal. In view of this circumstance, the pump frequency f_p should be chosen 2–3 orders of magnitude higher than the signal frequency in order to avoid the appearance of components with combination frequencies near the fundamental frequency of the signal. The spectrum presented in Fig. 5 was calculated at a level of internal noise in the ring somewhat lower than the value corresponding to the maximum on the stochastic resonance curve. It is seen in Figs. 4 and 5 that the gain η goes to a constant level in that region where the level of the harmonics of the useful signal with frequency close to f_p is sufficiently low. The ideal case would be when the frequencies of the signals are incommensurate and very different. Then the points in time at which the pump signal brings about "sampling" of the information signal in its different periods correspond to different, nonrepeating phases of the information signal. Thus the "sampling" of the information signal is done quasi-randomly, and the effect of the regular pump signal in this process become similar to the effect of noise in the SR effect. These considerations underscore certain general features of the behavior

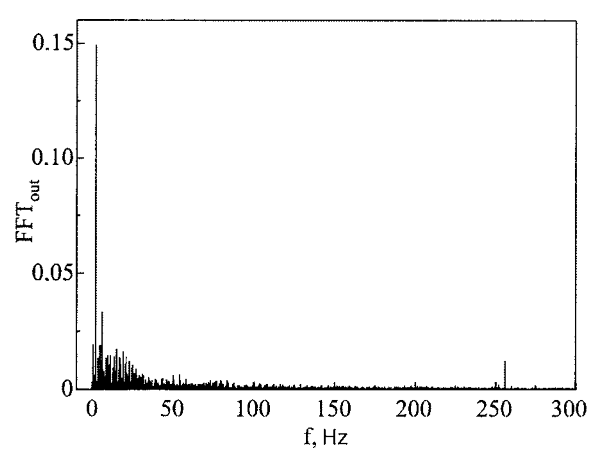

FIG. 5. Fourier amplitude spectrum of the output signal under the joint influence of noise and a high-frequency pump signal. The spectrum is calculated at the point of maximum stochastic-parametric amplification at a fixed noise level, for $a=0.01$, $f_s=2$ Hz, $A=0.02$, $f_p=256$ Hz, $s=0.08$, $f_c=2048$ Hz.

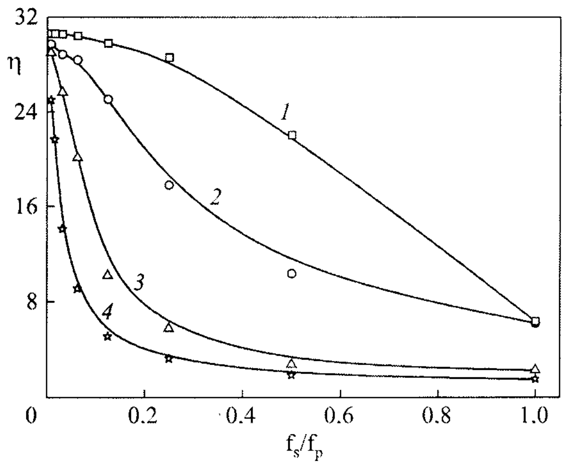

FIG. 6. Frequency dependence of the stochastic-parametric gain of the magnetic flux in an interferometer for different values of the ratio of the noise and high-frequency pump: $s=0$ (*1*), $s/A=4$ (*2*), $s/A=10$ (*3*), $A=0$ (*4*), for $a=0.01$, $f_p=256$ Hz. Each curve is calculated at a fixed noise level.

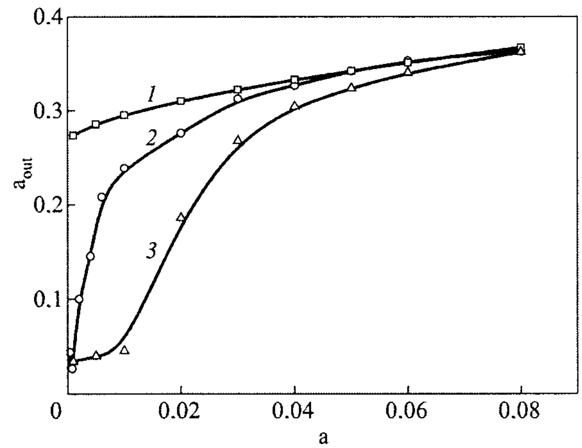

FIG. 7. Dependence of the amplitude of the fundamental harmonic of the high-frequency amplified signal a_{out} on the amplitude of the input low-frequency information signal a (transfer characteristics of the interferometer) for different shapes of the high-frequency pump signal: meander (*1*), thermal noise (*2*), sine (*3*) for $f_s=2$ Hz, $f_p=256$ Hz, $f_c=2048$ Hz.

of a system in the presence of noise and a periodic high-frequency field. However, despite the similarity of the weak-signal amplification in the presence of additional noise or a high-frequency field, the nature of these effects is nevertheless different. The high-frequency field causes a periodic variation of the Josephson inductance, which determines the shape of the potential, including the value of the barrier. Thus the mechanism of amplification in this case is a kind of parametric amplification. The differences of the mechanisms of amplification are manifested, e.g., in the dependence of the maximum gain on the frequency of the useful signal. It is known (see, e.g., Ref. 4) that in the case of SR the maximum amplification of a weak signal increases as the frequency of that signal decreases. In the case of a high-frequency pump the gain is practically independent of the frequency of the useful signal at sufficiently high pump frequencies, as is seen in Fig. 4. If the frequency of the high-frequency pump is not high enough in relation to the signal frequency, then one observes intermediate curves of the stochastic-parametric gain as a function of the frequency of the information signal (Fig. 6). The shape of these curves is determined by the ratio of the intensity of the noise to the amplitude of the high-frequency pump. The two extreme curves correspond to the cases of pure parametric and pure stochastic amplification, and the intermediate curves to the case of mixed stochastic-parametric amplification at different ring inductances, making for different values of the noise flux. Thus the behavior of the superconducting interferometers in a high-frequency field when thermal fluctuations are taken into account presents an opportunity for regulating the amplitude and frequency characteristics of stochastic input circuits of SQUID-based detectors.

An additional analysis shows that $\eta(A)$ is not appreciably influenced by the spectral composition of the high-frequency pump, which depends on the shape of the periodic process (sine, meander, triangle, saw-tooth, etc.). However, the nonlinearity of the amplification is strongly dependent on the spectral composition of the pump (Fig. 7). The curvature of that function has a different sign for different shapes of the pump signal, including that of noise. This means that the nonlinearity of the gain within certain limits can be regulated by choosing a certain spectral composition of the high-frequency pump.

As is seen in Fig. 6, when the condition $L \ll L_F$ is satisfied the gain reaches rather high values ($\sim 10-30$) over a wide range of ratios of the noise and high-frequency pump. In the case when the noise and high-frequency components of the external magnetic flux are comparable in magnitude and together modulate the dynamics of the interferometer, one can speak of a cooperative amplification effect, which is logically called stochastic-parametric amplification. Apparently it is possible to treat this effect as a stochastic resonance effect but with a dynamically renormalized barrier height due to the influence of the high-frequency pump. Taking into consideration that the noise flux due to thermal fluctuations usually does not vary in the course of an experiment, one can treat this effect as a way of controlling the stochastic amplification with the aid of an additional high-frequency field. Such an effect should be expected in other bistable systems in which there is an energy parameter that depends on the input signal.

CONCLUSION

The behavior of a superconducting interferes with a π junction under the simultaneous influence of a weak harmonic low-frequency signal, a high-frequency electromagnetic field, and thermodynamic fluctuations creating a noise flux has been studied by numerical simulation. It was shown that the joint influence of noise and a high-frequency field leads to a cooperative phenomenon, which we have called stochastic-parametric amplification. This effect permits one to obtain the maximum amplification of a weak signal by varying the amplitude of the high-frequency field against a background of nonoptimal SR amplification determined by the internal noise flux of the interferometer. From the results obtained with high-frequency excitation, it follows that the dynamic renormalization of the probability of decay of metastable current states permits amplification of signals using interferometers with rather high potential barriers, i.e., with large β.

Since a symmetric double-well potential and the strong-dissipation limit $\beta_C \ll 1$ are easily realized in interferometers closed by other types of Josephson junctions, i.e., shunted SIS, ScS, and SNS, the results obtained above are of a general character. The most important distinction of interferometers with a π junction is that one can implement stochastic-parametric amplification without a static external magnetic field. Furthermore, π junctions prepared in a single technological cycle have a small spread in values of the critical current,[12] making it possible to build large arrays of almost identical interferometers. A superconducting antenna based on an array of interferometers[11] in a practically uniform external electromagnetic field in the megahertz range can be used for cooperative amplification of weak information signals. As to the antennas of small area used to build SQUID microscopes,[21] the foregoing analysis shows that the stochastic-parametric gain for a weak information signal can amount to $\sim 10-30$ at noise spectral densities arbitrarily lower than the amplitude of the high-frequency pump.

The frequency band for amplification of a signal through a high-frequency field is bounded by the pump frequency. By decreasing (increasing) this frequency in relation to the frequency of the useful signal, one can in principle narrow (widen) the gain band. However, this question, like that of the dependence of the gain on the spectral properties of the pump signal and degree of correlation of random processes in the system, is in need of further detailed investigation.

[a)]Email: turutano@ilt.kharkov.ua

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