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# Theoretical study of the resonant excitations in coupled Josephson-junction qubits

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**Abstract.** The superconducting qubits are nowadays considered as good candidates both for building a quantum computer and for the study of the quantum effects in macroscopic systems. The system of two coupled qubits can be described by the Liouville equation. We convert the Liouville equation in the system of real equations by making use of the convenient parametrization of the reduced density matrix. Phenomenological inclusion of the relaxation is presented. The time evolution of the levels population in coupled flux qubits is calculated by solving the Bloch-Redfield equations.

There is renewed interest in the superconducting circuits containing Josephson junctions due to their non-linearity and ultra-low dissipation, which is required for practical implementations of the quantum integrated circuits [1]. One of the most interesting possible implementation is a superconducting qubit [2]. The natural and challenging method to probe the Josephson inductance, which is related with the qubit state, is the so-called impedance measurement technique [3, 4]. This technique consists in that the qubit (or another system) is weakly coupled to the tank circuit; changes of the resonant circuit impedance reflect the state of the system. The impedance measurement technique may be useful for probing the state of different systems (see e.g. [5]). The theory of the technique in application to superconducting qubits was recently developed (see [6] and references therein). We have applied this theoretical description with the Bloch equations to describe the resonant excitation of single superconducting qubit in Refs. [7] and [8]. Recently experimental results for the resonant excitation of two coupled flux qubits, probed by the tank circuit, were published in Ref. [9]. In order to describe results of Ref. [9], we convert the Liouville equation for two coupled qubits into the system of real equations in the convenient parametrization in this paper. This parametrization, in particular, is shown to be reasonable for taking relaxation processes into account. Another approach is to solve the Bloch-Redfield equations, which describe the two-qubit system coupled to the bath of harmonic oscillators [10]; our purpose is to compare both approaches on the example of coupled flux qubits.

Consider first the ordinary way of taking into account the hermeticity of the density matrix, by separating the real and imaginary parts of its elements: $\rho_{ij} = x_{ij} + iy_{ij}$. Then the condition $\rho_{ij} = \rho_{ji}^*$ results in the following: $x_{ji} = x_{ij}$, $y_{ji} = -y_{ij}$, $y_{ii} = 0$. Then the Liouville equation, $i\dot{\rho} = [H, \rho]$, can be written in the form of the system of 16 equations:

$$-\dot{y}_{ij} = [H, x]_{ij}\,, \quad \dot{x}_{ij} = [H, y]_{ij}\,. \tag{1}$$

Furthermore, one equation can be reduced from the system taking into account the normalization condition: $\sum \rho_{ii} = \sum x_{ii} = 1$. However in what follows we will consider another parametrization of the density matrix.

Recall that for single qubit (two-level system) the natural parametrization is the decomposition with the Pauli matrices [11]: $\rho = \frac{R_\beta}{2}\sigma_\beta$. Here and below it is assumed that Greek letters run over values from 0 to 3 and Latin letters - from 1 to 3, $\sigma_a$ stand for the Pauli matrices [with $a = 1, 2, 3$] and $\sigma_0$ denotes the unity matrix, the summation over twice repeating indices is assumed. The normalization condition ($Sp\rho = 1$) results in $R_0 = 1$. And the Liouville equation with the Hamiltonian $H = \frac{h_\alpha}{2}\sigma_\alpha$ can be rewritten as following:

$$i\dot{\rho} = i\frac{\dot{R}_\beta}{2}\sigma_\beta = \frac{R_\alpha}{2}\frac{h_\beta}{2}[\sigma_\beta, \sigma_\alpha] = \frac{1}{4}R_j h_i [\sigma_i, \sigma_j] = \frac{i}{2}\varepsilon_{ijk} R_j h_i \sigma_k,$$

where we have taken into account that $[\sigma_0, \sigma_\alpha] = [\sigma_\beta, \sigma_0] = 0$. Thus the Liouville equation is reduced to the system of three real equations: $\dot{R}_i = \varepsilon_{mni} h_m R_n$. Then relaxation can be introduced phenomenologically, so that $R_{1,2} \overset{t\to\infty}{\longrightarrow} R_{1,2}^{(0)} = 0$, $R_3 \to R_3^{(0)}(T)$. In this way we obtain the so-called Bloch equations:

$$\dot{R}_i = \varepsilon_{mni} h_m R_n - \delta_{in}\Gamma_n (R_n - R_n^{(0)}), \tag{2}$$

where $\Gamma_{1,2} = \Gamma_\phi$ is the dephasing rate and $\Gamma_3 = \Gamma_{relax}$ is the energy relaxation rate.

In what follows we consider analogous procedure for two coupled two-level systems (qubits). The four-level general Hamiltonian can be written with Pauli matrices:

$$H = \frac{h_{\alpha\beta}}{2}\sigma_\alpha \otimes \sigma_\beta. \tag{3}$$

In particular case of two interacting qubits it is convenient to rewrite Eq. (3) in the following form

$$H = \frac{1}{2}\mathbf{B}^{(i)}\sigma^{(i)} + \frac{J_{ab}}{2}\sigma_a^{(1)}\sigma_b^{(2)}, \tag{4}$$

where $\sigma_a^{(1)} = \sigma_a \otimes \sigma_0$, $\sigma_a^{(2)} = \sigma_0 \otimes \sigma_a$, $\sigma^{(i)} = (\sigma_1^{(i)}, \sigma_2^{(i)}, \sigma_3^{(i)})$; the vectors $\mathbf{B}^{(i)}$ describe the "local magnetic fields", acting on $i$-th qubit; for the flux qubit: $\mathbf{B}^{(i)} = (-\Delta_i, 0, -\epsilon_i)$. The interaction term describes different types of the interaction: Ising ($J_{ab} = J\delta_{a1}\delta_{b1}$), Heisenberg ($J_{ab} = J\delta_{a3}\delta_{b3}$), isotropic ($J_{ab} = J\delta_{ab}$); note: $\sigma_3^{(1)}\sigma_3^{(2)} = \sigma_3 \otimes \sigma_3$.

We use the following parameterization of the density matrix

$$\rho = \frac{R_{\gamma\delta}}{4}\sigma_\gamma \otimes \sigma_\delta, \tag{5}$$

which allows to benefit from the properties of the density matrix, namely from that it is Hermitian (then $R_{\gamma\delta}$ are the real numbers) and that it is normalized, $Sp\rho = 1$ (then $R_{00} = 1$). Then it follows that the density matrix is parameterized by 15 independent real values.

It is useful to note that $R_{a0} = Sp\left(\rho\sigma_a^{(1)}\right)$ and $R_{0a} = Sp\left(\rho\sigma_a^{(2)}\right)$; indeed

$$Sp\left(\rho\sigma_a^{(1)}\right) = \frac{R_{\gamma\delta}}{4}Sp\left(\sigma_\gamma\sigma_a \otimes \sigma_\delta\right) = \frac{R_{\gamma\delta}}{4}Sp\left(\sigma_\gamma\sigma_a\right)Sp\left(\sigma_\delta\right) = \frac{1}{4}R_{a0}Sp\left(\sigma_0\right)Sp\left(\sigma_0\right) = R_{a0}.$$

Then Liouville equation can be written as following:

$$\frac{i}{4}\dot{R}_{\gamma\delta}\sigma_\gamma \otimes \sigma_\delta = \frac{1}{8}h_{\alpha\beta}R_{\gamma\delta}\Xi_{\alpha\beta\gamma\delta}, \tag{6}$$

$$\begin{aligned}\Xi_{\alpha\beta\gamma\delta} &\equiv \sigma_\alpha \otimes \sigma_\beta \cdot \sigma_\gamma \otimes \sigma_\delta - \sigma_\gamma \otimes \sigma_\delta \cdot \sigma_\alpha \otimes \sigma_\beta = \qquad (7)\\ &= \sigma_\alpha\sigma_\gamma \otimes \sigma_\beta\sigma_\delta - \sigma_\gamma\sigma_\alpha \otimes \sigma_\delta\sigma_\beta. \qquad (8)\end{aligned}$$

When the indices are non-zero:

$$\Xi_{abgd} = \delta_{ag}\sigma_0 \otimes [\sigma_b, \sigma_d]_- + i\varepsilon_{agk}\sigma_k \otimes \{\sigma_b, \sigma_d\}_+ = \delta_{ag}\sigma_0 \otimes 2i\varepsilon_{bdk}\sigma_k + i\varepsilon_{agk}\sigma_k \otimes 2\delta_{bd}\sigma_0.$$

When there are zeros amongst the indices, the tensor $\Xi_{\alpha\beta\gamma\delta}$ is non-zero only when in Eq. (8) there is not only one Pauli matrix to the left or to the right from $\otimes$ (e.g., $\Xi_{00\gamma\delta} = 0$):

$$\begin{aligned}\Xi_{0b0d} &= \sigma_0 \otimes [\sigma_b, \sigma_d] = 2i\varepsilon_{bdk}\sigma_0 \otimes \sigma_k, \quad & \Xi_{a0g0} = [\sigma_a, \sigma_g] \otimes \sigma_0 = 2i\varepsilon_{agk}\sigma_k \otimes \sigma_0,\\ \Xi_{0bgd} &= \sigma_g \otimes [\sigma_b, \sigma_d] = 2i\varepsilon_{bdk}\sigma_g \otimes \sigma_k, \quad & \Xi_{a0gd} = [\sigma_a, \sigma_g] \otimes \sigma_d = 2i\varepsilon_{agk}\sigma_k \otimes \sigma_d,\\ \Xi_{ab0d} &= \sigma_a \otimes [\sigma_b, \sigma_d] = 2i\varepsilon_{bdk}\sigma_a \otimes \sigma_k, \quad & \Xi_{abg0} = [\sigma_a, \sigma_g] \otimes \sigma_b = 2i\varepsilon_{agk}\sigma_k \otimes \sigma_b.\end{aligned}$$

Then equating the coefficients before $\sigma_\alpha \otimes \sigma_\beta$ to the left and to the right, from Eq. (6) we obtain:

$$\left\{\begin{array}{r}\dot{R}_{i0} = \varepsilon_{mni}\left(h_{m0}R_{n0} + h_{mk}R_{nk}\right),\\ \dot{R}_{0j} = \varepsilon_{mnj}\left(h_{0m}R_{0n} + h_{km}R_{kn}\right),\\ \dot{R}_{ij} = \varepsilon_{mni}\left(h_{m0}R_{nj} + h_{mj}R_{n0}\right) + \varepsilon_{mnj}\left(h_{0m}R_{in} + h_{im}R_{0n}\right).\end{array}\right. \qquad (9)$$

Note that for a single qubit from the first equation we obtain $\dot{R}_{i0} = \varepsilon_{mni}h_{m0}R_{n0}$: cf. Eq. (2).

If equation (9) for $R_{\alpha\beta}$ is considered in the eigenstate representation, the relaxation processes can be taken into account phenomenologically by introducing the relaxation term in the r.h.s. analogously to Bloch equation (2). Such a consideration was developed in Ref. [13]. It was shown [13] that such method is convenient for the qualitative study of the stationary states since the result weakly depend on the specific form of the relaxation matrix. For example, the most simple case is when the relaxation constants are identical for all values $R_{\alpha\beta}$: $\Gamma_{\alpha\beta} = \Gamma$; then we have to add the term $-\Gamma\left(R_{\alpha\beta} - R^{(0)}_{\alpha\beta}\right)$ in the r.h.s. of Eq. (9).

For the two-qubit Hamiltonian (4) the system (9) can be rewritten (note: $h_{m0} = B^{(1)}_m$, $h_{0m} = B^{(2)}_m$, $h_{km} = J_{km}$):

$$\left\{\begin{array}{r}\dot{R}_{i0} = \varepsilon_{mni}\left(B^{(1)}_m R_{n0} + J_{mk}R_{nk}\right),\\ \dot{R}_{0j} = \varepsilon_{mnj}\left(B^{(2)}_m R_{0n} + J_{km}R_{kn}\right),\\ \dot{R}_{ij} = \varepsilon_{mni}\left(B^{(1)}_m R_{nj} + J_{mj}R_{n0}\right) + \varepsilon_{mnj}\left(B^{(2)}_m R_{in} + J_{im}R_{0n}\right).\end{array}\right. \qquad (10)$$

Consider particular cases:

(i) Isotropic interaction ($J_{km} = J\delta_{km}$) [12]:

$$\left\{\begin{array}{r}\dot{R}_{i0} = \varepsilon_{mni}\left(B^{(1)}_m R_{n0} + JR_{nm}\right),\\ \dot{R}_{0j} = \varepsilon_{mnj}\left(B^{(2)}_m R_{0n} + JR_{mn}\right),\\ \dot{R}_{ij} = \varepsilon_{mni}B^{(1)}_m R_{nj} + \varepsilon_{mnj}B^{(2)}_m R_{in} + \varepsilon_{ijn}J(R_{n0} - R_{0n}).\end{array}\right. \qquad (11)$$

(ii) Heisenberg interaction ($J_{ab} = J\delta_{a3}\delta_{b3}$):

$$\left\{\begin{array}{r}\dot{R}_{i0} = \varepsilon_{mni}B^{(1)}_m R_{n0} + \varepsilon_{3ni}JR_{n3},\\ \dot{R}_{0j} = \varepsilon_{mnj}B^{(2)}_m R_{0n} + \varepsilon_{3nj}JR_{3n},\\ \dot{R}_{ij} = \varepsilon_{mni}B^{(1)}_m R_{nj} + \varepsilon_{mnj}B^{(2)}_m R_{in} + \delta_{j3}\varepsilon_{3ni}JR_{n0} + \delta_{i3}\varepsilon_{3nj}JR_{0n}.\end{array}\right. \qquad (12)$$

For calculations it is constructive to rewrite this for the flux qubits, for $\mathbf{B}^{(i)} = (-\Delta_i, 0, -\epsilon_i)$:

$$\begin{cases} \dot{R}_{i0} = -\varepsilon_{1ni}\Delta_1 R_{n0} - \varepsilon_{3ni}\epsilon_1 R_{n0} + \varepsilon_{3ni} J R_{n3}, \\ \dot{R}_{0j} = -\varepsilon_{1nj}\Delta_2 R_{0n} - \varepsilon_{3nj}\epsilon_2 R_{0n} + \varepsilon_{3nj} J R_{3n}, \\ \dot{R}_{ij} = -\varepsilon_{1ni}\Delta_1 R_{nj} - \varepsilon_{3ni}\epsilon_1 R_{nj} - \varepsilon_{1nj}\Delta_2 R_{in} - \varepsilon_{3nj}\epsilon_2 R_{in} + \\ \quad + \delta_{j3}\varepsilon_{3ni} J R_{n0} + \delta_{i3}\varepsilon_{3nj} J R_{0n}. \end{cases} \tag{13}$$

To illustrate the time evolution of the system of coupled qubits, in Fig. 1 we plot the diagonal elements of the density matrix, which characterize the populations of the respective energy levels. At $t = 0$ the system was assumed to be in the ground state. After the transient dynamics, the energy levels populations oscillate with the frequency of the ac driving flux, which characterizes the stationary solution.

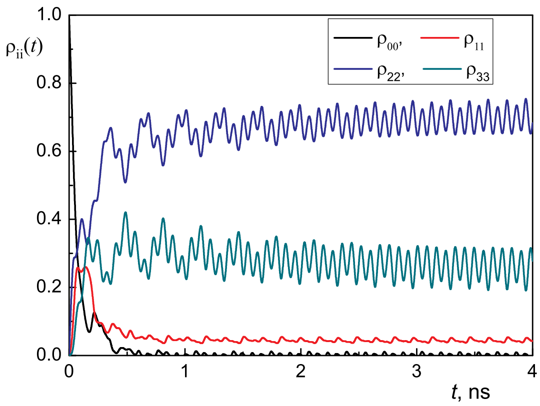


**Figure 1.** Time-dependence of the diagonal elements of the density matrix in the energy representation for the following set of parameters. Frequency and amplitude of ac flux: $\omega/2\pi = 6$ GHz and $\Phi_{ac}/\Phi_0 = 10^{-2}$, dc flux biases: $\Phi_{dc}^{(1,2)}/\Phi_0 = 10^{-3}$, persistent currents: $I_p^{(1)}\Phi_0/h = 375$ GHz, $I_p^{(2)}\Phi_0/h = 700$ GHz, temperature: $T/h = 1$ GHz, strength of dissipation $\alpha = 10^{-2}$.

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