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DYNAMICS OF SUPERCONDUCTING QUBITS WITH EXACTLY SOLVABLE BIAS PULSES

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The dynamics of a quantum two-state system (qubit) with external control bias pulses of special shapes is considered. The bias pulses represent the potentials for which the Schrödinger equation can be solved exactly. The probability to register a definite direction of the current in a loop and its time-averaged values are calculated for the flux qubit; calculations are performed both analytically and numerically in the presence of relaxation and decoherence within the framework of the density matrix formalism. It is demonstrated that there exist external bias pulses for which the definite current direction probability is a monotonically increasing function of time that approaches a limiting value exceeding 1/2. The probability to find the system in the excited state is calculated, and the possibility of inverse population in a properly driven two-state system is demonstrated given that the relaxation and decoherence rates are small enough.

INTRODUCTION

In the last few years, superconducting Josephson-junction–based devices have been studied intensively as candidates for the implementation of a quantum computer (for example, see [1–9]). Indeed, under certain conditions (low temperatures and small dimensions) the Josephson-junction devices are macroscopic quantum two-state systems, that is, they can be used as a model for quantum bits (qubits). The quantum properties of the macroscopic Josephson-junction devices make them more attractive in comparison with microscopic qubits (spins and ions [10, 11]) for the development of scalable quantum systems. Among them are high-sensitive superconducting quantum interference devices (SQUID) of magnetic fields and single-electron transistors (SET) for electric charges [12] that admit preparation in a preset initial state or a superposition of states.

One common approach to control the qubit dynamics is to drive a two-state system, as well as a particle in a double-well potential, with a sinusoidal field. In some cases, an interesting physical phenomenon can be established. Instead of oscillating between the wells, the particle can be localized in one of them by a driving external field. This unusual behavior of the probability is referred to as coherent destruction of tunneling in double-well potential studies [13–18], dynamic localization in transport theory [19, 20], and population trapping in atomic physics [21]. It is important to note that up to the present time, most researches have considered systems in a periodic driving field with possible amplitude and frequency modulation.

In the present work, we demonstrate that the oscillating character of the field is not obligatory for the appearance of this phenomenon. We present a class of non-periodic time-dependent bias pulses that lead to the similar behavior of the qubit.

The quantum evolution of the qubit state in the two-state approximation is described by the Schrödinger equation (for example, see [12, 22])

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$$i\frac{\partial\Psi}{\partial t} = H\Psi ,$$

where the Hamiltonian

$$H = \Delta\sigma_x + \varepsilon(t)\sigma_z,\ \hbar = 1, \tag{1}$$

has been written in the basis of the eigenvectors ("physical" states) $\{|0\rangle,|1\rangle\}$ of the Pauli matrix ξ $(\sigma_z|0\rangle = |0\rangle$ and $\sigma_z|1\rangle = -|1\rangle)$ and $\Psi = (\psi_1,\psi_2)^T$. For the charge qubit [1], these states correspond to a definite number of the Cooper pairs on the superconducting island (the Cooper-pair box). For the flux qubit [23], the vectors $|0\rangle$ and $|1\rangle$ correspond to coherent superconducting currents circulating in the loop in opposite directions. We assume that the tunneling amplitude Δ is constant and the bias ε is a function of time $\varepsilon = \varepsilon(t)$. The bias $\varepsilon(t)$ is governed by the gate voltage $V_g(t)$ for the charge qubit or depends on the magnetic flux $\Phi_x(t)$ passing through the loop for the flux qubit.

For definiteness, we further consider the flux qubit. The results obtained can be easily generalized to the charge qubit. Then the states $\{|0\rangle,|1\rangle\}$ of the flux qubit have the definite (clockwise or counterclockwise) direction of superconducting current circulating in the loop. The above-mentioned non-periodic time-dependent bias pulses are the potentials for which Schrödinger's equation (1) can be solved exactly [24–27]. Thus, the probability calculated using these exact solutions is, for example, the probability $P^{\uparrow}$ of the clockwise current direction. We demonstrate that for some special non-periodic potentials, the probability of the clockwise current direction at time t under condition that at the initial moment of time $t = 0$ it was counterclockwise is a monotonically increasing function of time approaching 3/4. This behavior of the probability is observed only at critical values of the system parameters. For small derivations of biases from their critical values, the probability oscillates; however, its minimal value still exceeds 1/2. It is demonstrated that such behavior of the probability remains unchanged even in the presence of dissipation. The evolution of the time-averaged probability values is also considered. An analysis of solutions of Eq. (1) with special non-periodic potentials in the energy representation indicates the possibility of inverse population in the two-state system even in the presence of dissipation. Our main result is that using a properly chosen non-periodic time-dependent potential, the qubit state can be "frozen" in one of the two possible qubit states for a long time. We note that the calculated occupation probability of the qubit states is directly related to the experimentally measurable quantities such as the phase shift of the resonant circuit weakly coupled with the qubit (for example, see [28]). We hope that our results offer new opportunities for controlling the qubit dynamics. Some other aspects of the qubit level population control can be found in [29–32].

EXACTLY SOLVABLE BIAS PULSES

The intertwining operator technique provides a universal approach to the construction of new exactly solvable equations. The idea of this method goes back to Darboux [33] and finds wide application in soliton theory [34]. Its quantum mechanical application (see, for example, [35]) is closely related to the fact that the one-dimensional Schrödinger equation is an ordinary second-order differential equation defined by the potential energy operator. The method is based on the possibility of finding an operator (an intertwining operator) that relates solutions of the Schrödinger equation with different potentials. Thus, if solutions of the Schrödinger equation with a given potential have been known and the intertwining operator has been found, solutions of the same equations with another potential can be constructed by applying the intertwining operator to the solution of the initial equation. Matrix generalization of the given method [34] is widely used for solving quantum-mechanical problems [36].

In [24] it was demonstrated how to construct the differential-matrix intertwining operators for the system of two differential equations of type (1). To this end, nonstationary Schrödinger equation (1) was reduced to the one-dimensional stationary Dirac equation with an effective non-Hermitian Hamiltonian, and the mathematical apparatus developed in [36] was used. Starting from the simplest case $\varepsilon = \varepsilon_0 = \text{const}$, a family of new nontrivial potentials (biases), for which Schrödinger's equation (1) could be solved exactly, was found. In the present work, we take advantage of the results obtained in [24–27] to describe the time evolution of the flux qubit and the time dependence of the qubit localization probability and to calculate its time-averaged values.

Let us consider first the behavior of the flux qubit with the bias

$$\varepsilon_1(t) = \varepsilon_0 - \frac{4\varepsilon_0}{1+4\varepsilon_0^2 t^2}. \tag{2}$$

Solutions of Eq. (1) with this potential were analyzed in detail in [24]. Imposing the initial conditions $|\psi_1(0)|^2 = 1$ and $|\psi_2(0)|^2 = 1$, we obtain the probability $P^{\uparrow}(t) = |\psi_2(t)|^2$ of the clockwise current direction at time t under condition that it was counterclockwise at the initial moment of time $t = 0$. For $\tau = \Delta t$ and $\xi = \frac{\varepsilon_0}{\Delta}$, we obtain the following expression for the probability:

$$P_1^{\uparrow}(\tau) = \left[\Theta^6\left(1+4\xi^2\tau^2\right)\right]^{-1}$$

$$\times\left[16\xi^4\Theta^2\tau^2\cos^2\Theta\tau + 4\xi^2\Theta\tau\left(1-3\xi^2\right)\sin 2\Theta\tau + \left(4\xi^2\Theta^4\tau^2 + \left(1-3\xi^2\right)^2\right)\sin^2\Theta\tau\right],$$

$$\Theta = \sqrt{1+\xi^2}.$$

It can be seen that $P^{\uparrow}$ oscillates provided $\xi^2 \neq 1/3$. For $\xi^2 = 1/3$, the probability is described by the function

$$P_1^{\uparrow}(\tau) = \frac{\tau^2}{1+\frac{4}{3}\tau^2},$$

monotonically increasing from zero at the initial moment of time to $3/4$ for $\tau >> 1$ or $t >> \Delta^{-1}$ (see the heavy curve in Fig. 1*a*). It is important to note that for ξ^2 close to 1/3, the minimal value of the probability $P_1^{\uparrow}$ exceeds $1/2$ very quickly after the bias application (see the thin and dotted curves in Fig. 1*a*).

The expression for the time-averaged probability has the form

$$\overline{P_1^{\uparrow}} = \frac{1+5\xi^2}{2\left(1+\xi^2\right)^2}.$$

For $\xi^2 = 3/5$, the average probability behaves similar to the resonance one: at this point it reaches its maximal value $\overline{P_1^{\uparrow}} \approx 0.78$ (see the heavy curve in Fig. 2*a*), whereas $P_1^{\uparrow}(\tau)$ oscillates with frequency $4\sqrt{2/5}$ (see Fig. 1).

Using chains of the above simple transformations, new families of more exactly solvable potentials can be obtained. The properties of such chains were analyzed in detail in [26]. The results obtained in the present work allow a large family of new exactly solvable biases to be constructed for equation (1). For example, for a two-fold transformation leading to the bias of the shape

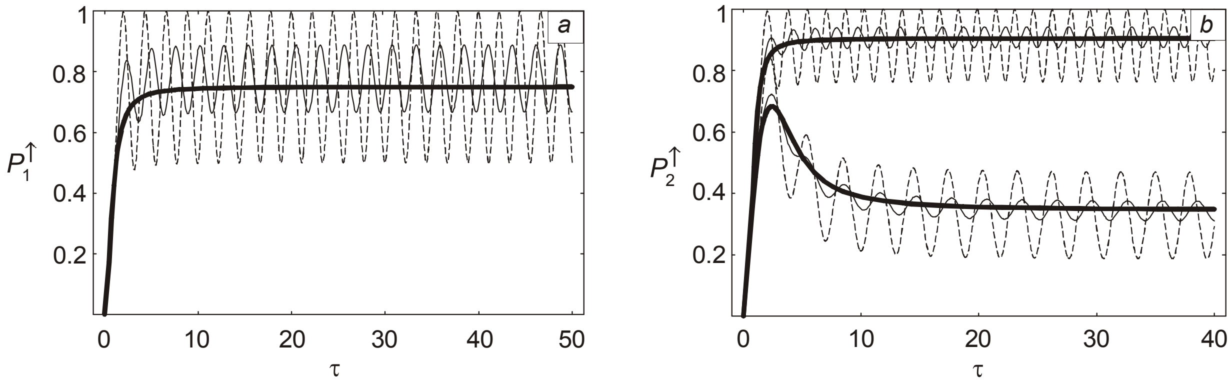

Fig. 1. Time dependence of the clockwise current direction probability: *a*) probability $P_1^{\uparrow}$ for $\xi = 1/\sqrt{3}$ (the thin curve), 1 (the dashed curve), and $1/\sqrt{3}$ (the heavy curve) and *b*) probability $P_2^{\uparrow}$ for $\xi = \sqrt{2+\frac{2}{\sqrt{5}}}$, $\xi = \sqrt{1.2+\frac{2}{\sqrt{5}}}$, and $\xi = \sqrt{1+\frac{2}{\sqrt{5}}}$ in the upper plot and for $\xi = \sqrt{1.05-\frac{2}{\sqrt{5}}}$, $\sqrt{1.01-\frac{2}{\sqrt{5}}}$, and $\sqrt{1-\frac{2}{\sqrt{5}}}$ in the lower plot (the dashed, thin, and heavy curves, respectively).

$$\varepsilon_2(t) = \frac{\varepsilon_0\left(45 - 180\varepsilon_0^2 t^2 - 144\varepsilon_0^4 t^4 + 64\varepsilon_0^6 t^6\right)}{9 + 108\varepsilon_0^2 t^2 + 48\varepsilon_0^4 t^4 + 64\varepsilon_0^6 t^6}, \tag{3}$$

the clockwise current direction probability assumes the form

$$\begin{gathered} P_2^{\uparrow}(\tau) = \left[\Theta^{10}\left(9 + 108\varepsilon_0^2 t^2 + 48\varepsilon_0^4 t^4 + 64\varepsilon_0^6 t^6\right)\right]^{-1} \\ \times\Big(16\xi^4\Theta^2\tau^2\left[16\xi^4\Theta^4\tau^4 + 24\xi^2\left(3 - 14\xi^2 + 7\xi^4\right)\tau^2 + 9\left(9 - 6\xi^2 + \xi^4\right)\right] \\ + Q_1\left[Q_2 \sin^2\Theta\tau + Q_3 \sin 2\Theta\tau\right]\Big), \end{gathered} \tag{4}$$

where

$$Q_1 = 1 - 10\xi^2 + 5\xi^4,$$

$$Q_2 = 64\xi^6\Theta^4\tau^6 + 48\xi^4\left(1 - 18\xi^2 - 19\xi^4\right)\tau^4 + 36\xi^2\left(3 - 2\xi^2 + 11\xi^4\right)\tau^2 + 9Q_1,$$

$$Q_3 = 12\xi^2\Theta\tau\left(\Theta^2\left(16\xi^4\tau^4 + 7\right) + 2\left(4\xi^2\tau^2 + 1\right)\left(1 - 5\xi^2\right)\right).$$

The last term in Eq. (4) describes oscillations with frequency 2Θ, consequently, the probability oscillations vanish at $Q_1 = 0$, and the probability acquires a monotonic character. Thus, with the bias ε described by Eq. (3), in contrast with Eq. (2), two possibilities can be indicated with $Q_1 = 0$. This means that the probability of the clockwise current direction ceases to oscillate and acquires a monotonic character at $\xi = \sqrt{1 - 2/\sqrt{5}}$ or $\xi = \sqrt{1 + 2/\sqrt{5}}$. This

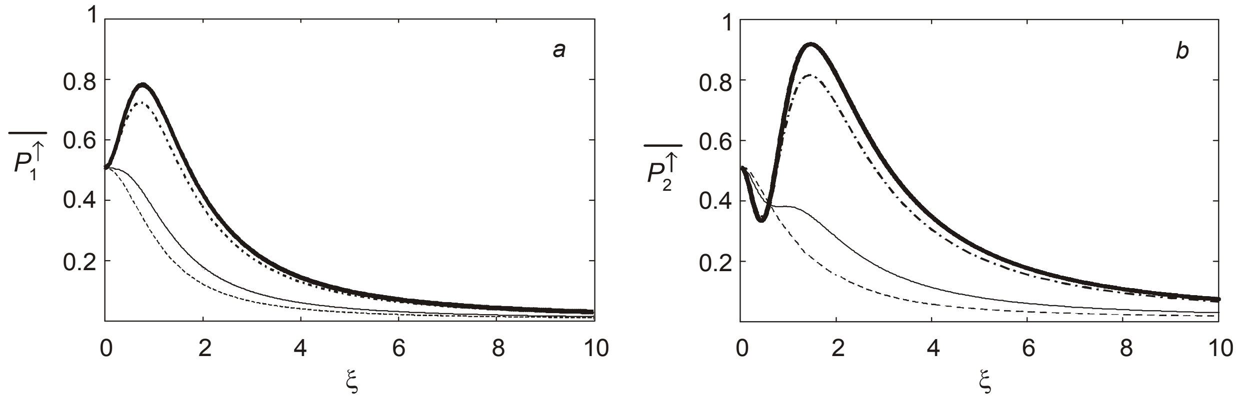

Fig. 2. Dependence of the time-averaged clockwise current direction probabilities on ξ.

situation is illustrated in Fig. 1 (the heavy curves), where the oscillating behavior of the probability for ε values close to the critical ones is also demonstrated (see the thin and dashed curves).

For $\varepsilon(t)$ given by Eq. (3), time averaging yields the following expression for the probability:

$$\overline{P_2^{\uparrow}} = \frac{1-2\xi^2+13\xi^4}{2\left(1+\xi^2\right)^3}.$$

The dependence of $\overline{P_2^{\uparrow}}$ on the parameter ξ is shown in Fig. 2 by the heavy curve. It has a maximum $\overline{P_2^{\uparrow}} \approx 0.91$ at $\xi \approx 1.46$.

We now consider a more complicated case in which the bias is a function of three parameters [24, 27]:

$$\varepsilon_3(t) = \varepsilon_0 + \frac{2\omega^2}{b\cos(2\omega t+\varphi)-\varepsilon_0}, \quad b^2 = \varepsilon_0^2 - \omega^2 > 0. \tag{5}$$

Here φ is arbitrary and ε_0 and ω satisfy inequality (5). We note that in this case, $\varepsilon = \varepsilon_3(t)$ is a periodic function with amplitude related to the frequency. Expression (5) is a generalization of formula (2). Indeed, setting $\varphi = \arctan\frac{\omega}{2\varepsilon_0} - \frac{1}{2}\arctan\frac{\omega}{b}$ in Eq. (5), in the limit $\omega \to 0$ we obtain the ε dependence on time considered above.

The analytical expression for the probability $P_3^{\uparrow}(\tau)$ is cumbersome, and we restrict ourselves only to graphic illustrations of the clockwise current direction probability for $\Theta \approx \frac{\omega}{\Delta}$ (see Fig. 3). The probability $P_3^{\uparrow}(\tau)$ was examined in more detail in [24].

INFLUENCE OF THE DISSIPATION ON THE PROBABILITY

We have already considered the evolution of the system isolated from the environment and driven by the external field described by the function $\varepsilon(t)$. The system is described by the wave function which is a solution of the Schrödinger equation with Hamiltonian (1). However, in actual experiments the quantum system also interacts with the external reservoir, which leads to the irreversible processes of energy dissipation and destruction of the quantum coherence or to the so-called dephasing of the quantum states. In this case, the system states should be described within the density matrix formalism [37]. Since the main obstacle to the experimental implementation of quantum information

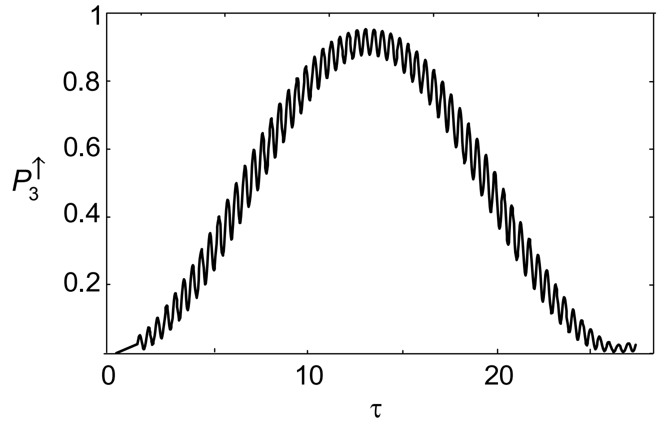

Fig. 3. Evolution of the clockwise current direction probability $P_3^{\uparrow}$ calculated for $\theta = 6.88$, $\Theta = 7$, $\xi = \sqrt{48}$, and $\varphi = 0$.

processing is environment-induced decoherence (for example, see [38]), a study of the qubits as open systems seems more urgent.

In this Section we study the behavior of the qubit with the bias $\varepsilon(t)$ given by Eqs. (2), (3), and (5) with allowance for the dephasing and relaxation processes. We analyze the possibility of population inversion in the two-level system with dissipation. We note that unlike [29] where a three-level system was analyzed, we study the possibility of population inversion in the two-level system itself, thereby demonstrating the possibility of exciting laser generation in the system of two-level atoms at low temperature.

The quantum dynamics of the qubit is described by the Liouville equation for the density matrix. For the density operator of the form

$$\rho = \frac{1}{2}\begin{pmatrix} 1+Z & X-iY \\ X+iY & 1-Z \end{pmatrix},$$

we solve the equation of motion

$$i\frac{\partial \rho}{\partial t} = [H\rho]$$

to obtain the probability $P^{\uparrow} = [1 - Z(t)]/2$. The effect of the relaxation processes on the system weakly coupled with the environment can be phenomenologically described by two parameters, namely, dephasing (Γ_{φ}) and relaxation rates (Γ_{relax}) (for example, see [39]), thereby, one can derive the following system of equations for $X(t)$, $Y(t)$, and $Z(t)$:

$$\frac{dX}{dt} = -2\varepsilon(t)Y - \Gamma_{\varphi}X,$$

$$\frac{dY}{dt} = -2\Delta Z + 2\varepsilon(t)X - \Gamma_{\varphi}Y,$$

$$\frac{dZ}{dt} = 2\Delta Y - \Gamma_{\text{relax}}\left(Z - Z(0)\right).$$

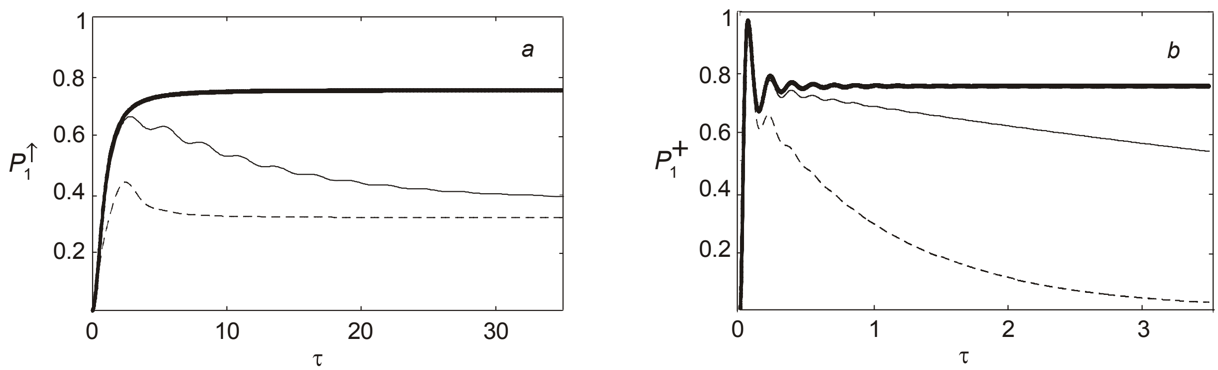

Fig. 4. Influence of relaxation on the probabilities: (*a*) clockwise current direction probability $P_1^{\uparrow}$ at $\xi = 1/\sqrt{3}$ and (*b*) upper level occupation probability P_1^{+} at $\xi = 1/\sqrt{3}$.

To investigate energy level populations, we calculate the probability P^{+} to find the system in the excited state [39]. The effect of the relaxation processes on the behavior of the probabilities $P_1^{\uparrow}$ and P_1^{+}, exhibiting a monotonic time dependence for $\varepsilon(t)$ given by Eq. (2) for $\xi = 1/\sqrt{3}$, is illustrated in Fig. 4. We do not show plots of $P_2^{\uparrow}$ and P_2^{+} for $\varepsilon(t)$ given by Eq. (5) here, because they are analogous to the plots shown in Fig. 4. The dependence of the probabilities $P_3^{\uparrow}(t)$ and P_3^{+} for $\varepsilon(t)$ given by Eq. (5) at $\xi = 1/\sqrt{3}$, $\beta = \sqrt{15}/2$, and $\varphi = 0$ is illustrated in Fig. 5. The solid curves are drawn for $\Gamma_{\varphi} = \Gamma_{\text{relax}} = 0$, that is, without relaxation processes in the system. The thin and dashed curves are drawn for $\Gamma_{\varphi} = \Gamma_{\text{relax}} = 0.01$ and $\Gamma_{\varphi} = \Gamma_{\text{relax}} = 0.1$, respectively. All values are in units of Δ. From the figures it can be seen that the inverse population is possible even in the system with dissipation, through for a short time interval τ.

The time-averaged probabilities $\overline{P_1^{\uparrow}}$ and $\overline{P_2^{\uparrow}}$ as functions of the dimensionless parameter ξ are shown in Fig. 2. Here the solid curves are drawn without relaxation in the system, and the thin and dashed curves are drawn for $\Gamma_{\varphi} = \Gamma_{\text{relax}}$, as in Figs. 4 and 5. The dashed curves are drawn for $\Gamma_{\varphi}/\Delta = \Gamma_{\text{relax}}/\Delta = 0.001$. According to these figures, the inverse population is possible only for small enough dissipation (see the dash-dotted curves) and for the parameter ξ close to the critical value at which the time-averaged probabilities reach their maximal values.

CONCLUSIONS

The quantum dynamics of the two-state system subjected to biases of special shapes for which the Schrödinger equation can be solved exactly has been considered. By way of example, the time evolution of the superconducting flux qubit was investigated. The nontrivial behavior of the qubit with biases of special shapes was demonstrated. Varying the shape of the time-dependent bias, the dynamic behavior of the occupation probability can be changed qualitatively. In particular, the amplitude of oscillating probability describing the definite current direction in the loop can be tuned to zero, thereby tuning the probability to a monotonic function of time. The possibility of inverse population in the two-level system with dissipation was demonstrated. Such behavior of the occupation probability is directly related to the experimentally measurable quantities, that makes experimental verification of our theoretical predictions possible.

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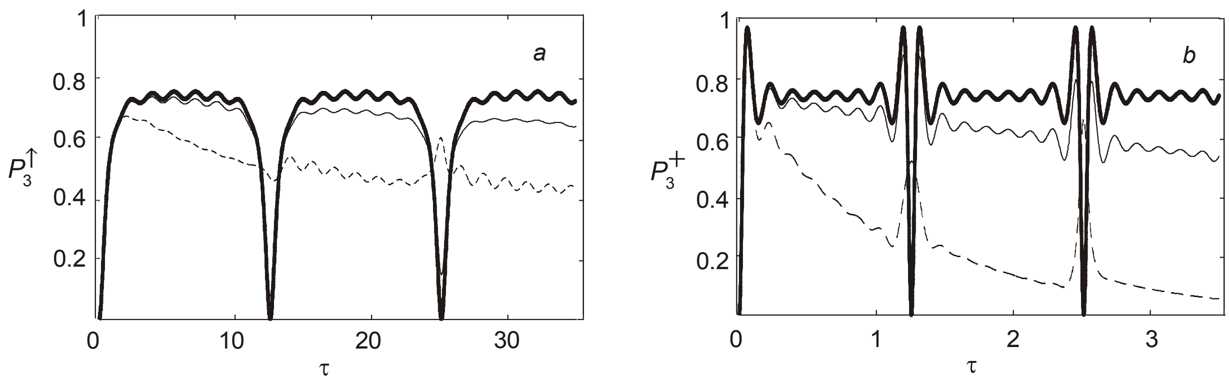

Fig. 5. Influence of relaxation on the probabilities: (*a*) clockwise current direction probability $P_3^{\uparrow}$ at $\xi = 1/\sqrt{3}$, $\beta = \sqrt{15}/2$, and $\varphi = 0$; (*b*) upper level occupation probability P_3^{+} at $\xi = 1/\sqrt{3}$, $\beta = \sqrt{15}/2$, and $\varphi = 0$.

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