Little–Parks effect for two-band superconductors

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The Ginsburg–Landau approach is used to construct a theory of the Little–Parks effect for two-band superconductors. A general relation reflecting the dependence of the relative shift Δt_c of the superconducting transition temperature on the external magnetic flux Φ is obtained. In a particular case, the relation describes the classical Little–Parks effect for single-band superconductors. In spite of the assertion made in the literature, the flux dependence $\Delta t_c = \Delta t_c(\Phi)$ for two-band superconductors is strictly periodic, just as in the classical effect. The main difference from the classical effect, which can be checked experimentally, is a nonparabolic character of the relation $\Delta t_c = \Delta t_c(\Phi)$. In the case where the physical parameters are the same for both bands, additional observable features appear in the plot $\Delta t_c = \Delta t_c(\Phi)$. The investigation of the external properties of the free-energy functional established an important limitiation, previously unmentioned in the literature, on one of the phenomenological parameters. *© 2008 American Institute of Physics*.
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I. INTRODUCTION

The present article is devoted to constructing a theory of the Little–Parks effect for two-band superconductors (such as, for example, the recently discovered[1] superconductor MgB_2). It is pertinent to recall that the classical Little–Parks effect for single-band superconductors is well-known in the literature as one of the most striking demonstrations of macroscopic phase coherence of the superconducting order parameter.[2–4] It is observed in open thin-wall superconducting cylinders in the presence of a constant external magnetic field oriented along the axis of the cylinder. Under conditions where the field is essentially unscreened the superconducting transition temperature $T_{c\Phi}$ (Φ is the magnetic flux through the cylinder) undergoes strictly periodic oscillations (Little–Parks oscillations)

$$\frac{T_c - T_{c\Phi}}{T_c} \propto \min_n \left(\frac{\Phi}{\Phi_0} - n \right)^2 \; (n = 0, \pm 1, \pm 2, \dots), \qquad (1)$$

where $T_c \equiv T_{c\Phi}|_{\Phi=0}$ and $\Phi = \pi\hbar c/e$ is the quantum of magnetic flux.

The Little–Parks effect for single-band superconductors has been discussed repeatedly in the last 10 years in an unconventional formulation. For example, oscillations of the critical temperature in a nonuniform superconducting cylinder[5] and superconducting cylinders with magnetic inclusions[6,7] have been investigated, and the possibility of the effect for a superconducting film in the form of a Möbius strip has been analyzed in Ref. 8.

Unfortunately, thus far the Little–Parks effect in two-band superconducting structures has not been studied either experimentally or theoretically. Moreover, the brief remarks made concerning this subject in the theoretical literature raise serious questions. For example, it is asserted in a recently published review,[9] which is entirely devoted to the application of the Ginsburg–Landau theory to two-band superconductivity, that "Little–Parks oscillations of T_c in two-band superconductors are not periodic." In the light of this, one of our objectives in the present work is to elucidate the situation within the framework of the simple (and, at the same time, quite general) approach of the Ginsburg–Landau theory.

The Gibbs free-energy functional in the Ginsburg–Landau approximation is the starting point of Sec. II. An important limitation on one of the phenomenological parameters of the functional, making possible the existence of an absolute minimum, is established. The complete system of equations for the mean field is obtained by minimizing the functional. In Sec. III the stability of the trivial solution of the mean-field equations, which corresponds to the normal state, is analyzed and a general relation for the superconducting transition temperature $T_{c\Phi}$, which exhibits strictly periodic Little–Parks oscillations, is obtained. Important particular cases are investigated and a graphical illustration is given. Finally, in Sec. IV the main results are discussed and some conclusions are proposed.

II. BASIC EQUATIONS

Let us consider a superconducting film in the form of a hollow circular cylinder (2) with inner and outer radii R_1 and R_2, respectively (see Fig. 1). An external constant magnetic field $\mathbf{H}$ is applied along the symmetry axis of the cylinder: $\mathbf{H} = (0, 0, H)$, where the sign of H is arbitrary. The length L of the generatrix of the cylinder satisfies the condition

$$L \gg R_2. \qquad (2)$$

The limitations on the inner radius of the cylinder R_1 and the thickness of the film $R_2 - R_1$ are discussed below (the conditions (5) and (6), respectively).

We proceed from the Gibbs free-energy functional in the Ginsburg–Landau approximation:

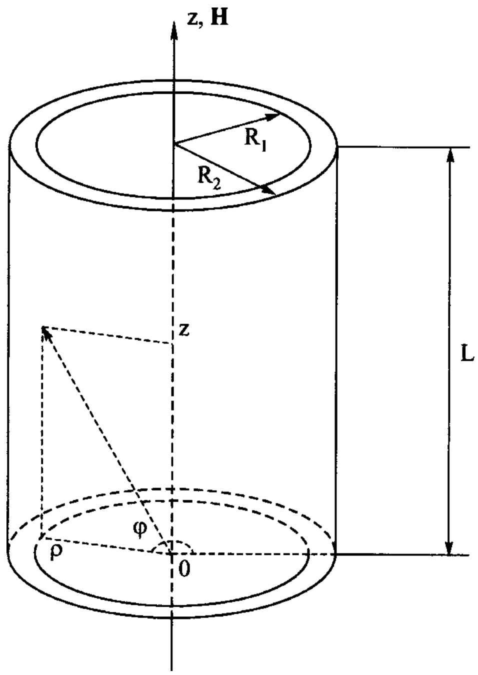

FIG. 1. Geometry of the problem (schematically). The following conditions hold: $L \gg R_2$, $R_1 \gg \max\{\xi_1(0), \xi_2(0)\}$ and $R_2 - R_1 \ll \min\{\xi_{1,2}, \lambda_{\min}, R_1\}$ (see discussion in the text).

$$\begin{aligned}
\mathcal{F}[|\psi_1|,|\psi_2|,\mathbf{A};\mathbf{H}] &= \mathcal{F}_1[|\psi_1|,\mathbf{A}] + \mathcal{F}_2[|\psi_2|,\mathbf{A}] + \mathcal{F}_{12}[|\psi_1|,|\psi_2|,\mathbf{A}] + \int_{\Omega_s+\Omega_a} \frac{(\mathrm{rot}\,\mathbf{A} - \mathbf{H})^2}{8\pi}\, dv, \\
\mathcal{F}_1[|\psi_1|,\mathbf{A}] &= \int_{\Omega_s} \left[\alpha_1|\psi_1|^2 + \frac{1}{2}\beta_1|\psi_1|^4 + \frac{1}{2m_1}\left| \left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\psi_1 \right|^2 \right] dv, \\
\mathcal{F}_2[|\psi_2|,\mathbf{A}] &= \int_{\Omega_s} \left[\alpha_2|\psi_2|^2 + \frac{1}{2}\beta_2|\psi_2|^4 + \frac{1}{2m_2}\left| \left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\psi_2 \right|^2 \right] dv, \\
\mathcal{F}_{12}[|\psi_1|,|\psi_2|,\mathbf{A}] &= -\int_{\Omega_s} \Bigg[\gamma(\psi_1^*\psi_2 + \psi_1\psi_2^*) \\
&\quad - \eta\Bigg[\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\psi_1 \left(i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\psi_2^* \\
&\quad + \left(i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\psi_1^* \left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\psi_2 \Bigg]\Bigg] dv.
\end{aligned} \tag{3}$$

Here $dv \equiv \rho d\rho d\varphi dz$ is an element of volume in a cylindrical coordinate system, $\psi_{1,2} = |\psi_{1,2}|\exp(i\chi_{1,2})$ are complex-valued components of the superconducting order parameter in the bands 1 and 2, and $\mathbf{A}$ is the vector potential of the local magnetic field $\mathbf{h} = h(\mathbf{r})$: $\mathbf{h} = \mathrm{curl}\mathbf{A}$. The three-dimensional integration in the first three terms in the expression (3) extends over the film region (Ω_s), and the integration in the magnetic-energy term extends over the region of the film together with the opening ($\Omega_s + \Omega_a$). The coefficients $\beta_{1,2}$ and γ are temperature independent, and $\beta_{1,2} > 0$. The temperature dependence of the coefficients $\alpha_{1,2} = \alpha_{1,2}(T)$ is given by the relations

$$\alpha_1(T) = -a_1\left(1 - \frac{T}{T_{c_1}}\right), \quad \alpha_2(T) = a_{20} - a_2\left(1 - \frac{T}{T_{c_1}}\right)$$

$$(a_{1,2} > 0,\ a_{20} \geqslant 0), \tag{4}$$

where the parameter T_{c_1} is the superconducting transition temperature for the band 1 in the absence of interband interaction ($\gamma = \eta = 0$) and $\mathbf{H} = 0$. (We note in passing that the band 2 also becomes superconducting in the absence of interband interaction, if $a_{20}/a_2 < 1$, at the temperature $T_{c2} = T_{c1}(1 - a_{20}/a_2)$ for $\mathbf{H} = 0$). To remain within the bounds of the Ginsburg-Landau approximation we required the inner radius of the cylinder to satisfy the condition

$$R_1 \gg \max\{\xi_1(0), \xi_2(0)\}, \tag{5}$$

and we assume the thickness of the superconducting film to be small in the sense that

$$d \equiv R_2 - R_1 \ll \min\{\xi_{1,2}(T), \lambda_{\min}(T), R_1\}, \tag{6}$$

where $\xi_{1,2}(T) \equiv \hbar/\sqrt{2m_{1,2}|\alpha_{1,2}(T)|}$ are the Ginsburg–Landau coherence lengths for the bands 1 and 2, respectively, and

$$\lambda_{\min}(T) \equiv \frac{c}{\sqrt{4\pi}e} \frac{1}{\sqrt{\dfrac{|\bar{\psi}_1(T)|^2}{m_1} + \dfrac{|\bar{\psi}_2(T)|^2}{m_2} + 4|\eta||\bar{\psi}_1(T)||\bar{\psi}_2(T)|}}$$

is the lower limit of the penetration depth of the magnetic field[9] ($|\bar{\psi}_{1,2}|$ denote the coordinate- and sample-size-independent equilibrium values of the variables $|\bar{\psi}_{1,2}|$ for $\mathbf{H} = 0$).

The use of functionals of the type (3) to describe two-band superconductivity has been discussed, for example, in Refs. 9–16. A microscopic approach was used in Refs. 14 and 15. Specifically, it has been shown[14] that the coefficient γ (generally speaking, different from zero irrespective of the presence of interband scattering) can have an arbitrary sign. The coefficient η is nonzero only in the presence of interband scattering.[15] For greater generality we shall assume that the sign of this coefficient can be arbitrary. However, as will be shown below, in order for an absolute minimum of (3) to exist it is necessary that

$$|\eta| < \frac{1}{2\sqrt{m_1 m_2}}. \tag{7}$$

The conditions for $\mathcal{F}$ to be stationary

$$\frac{\delta\mathcal{F}}{\delta\mathbf{A}} = 0, \quad \frac{\delta\mathcal{F}}{\delta\psi_{1,2}} = 0, \tag{8}$$

and the subsequent choice of gauge, in principle, give a complete system of equations for the mean field to determine the equilibrium values of $\mathbf{A}$ and $\psi_{1,2}$. However, it is more convenient to simplify Eq. (3) first, using the symmetry of the problem and the conditions (2) and (6).

On the strength of the condition (2) $\mathbf{h} = \mathbf{H}$, practically everywhere in the region Ω_a (opening). We choose the edge of the vector potential so that inside the region Ω_a

$$\mathbf{A}=(0,A_{\varphi}(\rho),0), \quad A_{\varphi}(\rho)=\frac{H\rho}{2}. \tag{9}$$

In view of the condition (6), the screening $\mathbf{h}=\mathbf{h}(\rho)$ and the variations $A_{\varphi}=A_{\varphi}(\rho)$ in Ω_c can be neglected. Thus, the expression (9) is actually valid in the entire region $\Omega_s+\Omega_a$, and the contribution of the magnetic energy in Eq. (3) is negligibly small. We shall also take account of the fact that as a result of the symmetry of the problem and the condition (6) $|\psi_{1,2}|$ are coordinate independent, and coordinate dependence of the phases of the components of the order parameter reduces to, on account of Eq. (9), a dependence on the angle φ ($\chi_{1,2}=\chi_{1,2}(\varphi)$) with the continuity conditions

$$\chi_{1,2}(2\pi)-\chi_{1,2}(0)=2\pi n_{1,2}, \quad \frac{d\chi_{1,2}}{d\varphi}(2\pi)=\frac{d\chi_{1,2}}{d\varphi}(0),$$

$$n_{1,2}=0,\pm 1,\pm 2,\ldots . \tag{10}$$

We note also that it is convenient to introduce new functional variables φ and θ instead of the phases $\chi_{1,2}$:[16]

$$\varphi=\chi_1-\chi_2,$$

$$\theta=c_1(|\psi_1|,|\psi_2|,\chi_1-\chi_2)\chi_1+c_2(|\psi_1|,|\psi_2|,\chi_1-\chi_2)\chi_2, \tag{11}$$

where

$$c_1(|\psi_1|,|\psi_2|,\chi_1-\chi_2) \equiv \frac{\dfrac{|\psi_1|^2}{m_1}+2\eta|\psi_1||\psi_2|\cos(\chi_1-\chi_2)}{\dfrac{|\psi_1|^2}{m_1}+\dfrac{|\psi_2|^2}{m_2}+4\eta|\psi_1||\psi_2|\cos(\chi_1-\chi_2)},$$

$$c_2(|\psi_1|,|\psi_2|,\chi_1-\chi_2) \equiv \frac{\dfrac{|\psi_2|^2}{m_2}+2\eta|\psi_1||\psi_2|\cos(\chi_1-\chi_2)}{\dfrac{|\psi_1|^2}{m_1}+\dfrac{|\psi_2|^2}{m_2}+4\eta|\psi_1||\psi_2|\cos(\chi_1-\chi_2)}. \tag{12}$$

Since $D(\phi,\theta)/D(\chi_1,\chi_2)\neq 0$ for all χ_1 and χ_2, the new functional variables are independent and can be regarded as arbitrary smooth functions of φ (i.e. $\phi=\phi(\varphi)$ and $\theta=\theta(\varphi)$) satisfying the boundary conditions

$$\phi(2\pi)-\phi(0)=2\pi(n_1-n_2), \quad \frac{d\phi}{d\varphi}(2\pi)=\frac{d\phi}{d\varphi}(0) \tag{13}$$

and

$$\theta(2\pi)-\theta(0)=2\pi[c_1(|\psi_1|,|\psi_2|,\phi(0))n_1+c_2(|\psi_1|,|\psi_2|,\phi(0))n_2],$$

$$\frac{d\theta}{d\varphi}(2\pi)=\frac{d\theta}{d\varphi}(0), \quad n_{1,2}=0,\pm 1,\pm 2,\ldots \tag{14}$$

respectively (see Eq. (11)).

As a result we arrive at the desired simplified relation (3):

$$\frac{\mathcal{F}[|\psi_1|,|\psi_2|,\theta,\phi;\Phi]}{V_s}=\alpha_1|\psi_1|^2+\frac{1}{2}\beta_1|\psi_1|^4+\alpha_2|\psi_2|^2+\frac{1}{2}\beta_2|\psi_2|^4-2\gamma\cos\phi|\psi_1||\psi_2|+\frac{\hbar^2 Q[\theta;\Phi]}{2}\left(\frac{|\psi_1|^2}{m_1}+\frac{|\psi_2|^2}{m_2}+4\eta\cos\phi|\psi_1||\psi_2|\right)+\frac{1}{\pi R^2}\int_0^{2\pi}C(|\psi_1|,|\psi_2|,\phi)\times\left(\frac{d\phi}{d\varphi}\right)^2 d\varphi, \tag{15}$$

where

$$Q[\theta;\Phi]=\frac{1}{2\pi R^2}\int_0^{2\pi}\left(\frac{d\theta}{d\varphi}-\frac{\Phi}{\Phi_0}\right)^2 d\varphi.$$

$$C(|\psi_1|,|\psi_2|,\phi)\equiv\left(\frac{c_2^2}{m_1}|\psi_1|^2+\frac{c_1^2}{m_2}|\psi_2|^2-4c_1c_2\eta\cos\phi|\psi_1|\times|\psi_2|\right)\frac{\hbar^2}{2},$$

$$c_{1,2}=c_{1,2}(|\psi_1|,|\psi_2|,\phi),$$

$R=(R_1+R_2)/2$, $V_s=2\pi RLd$ is the volume of the superconducting cylinder, $\Phi_0=\pi\hbar c/e$ is the quantum of magnetic flux, and $\Phi=\pi HR^2$ is the magnetic flux through the opening of the cylinder.

We shall now prove that the condition (7) ensures the existence of an absolute minimum of the functional (15) [and therefore a minimum of the initial functional (3)]. Indeed, on the basis of this condition $C(|\psi_1|,|\psi_2|,\phi)>0$ (if $|\psi_1|+|\psi_2|\neq 0$), and in addition the quadratic form of the variables $|\psi_1|$ and $|\psi_2|$, which, being a factor multiplying the function $Q[\theta;\Phi]$, is positive definite. Consequently, the right-hand side of Eq. (15) is bounded from below by the density of the equilibrium free energy of the cylinder in the absence of an external field, F_0/V_s:

$$
\begin{aligned}
\frac{\mathcal{F}}{V_s} &\geq \alpha_1|\psi_1|^2+\frac{1}{2}\beta_1|\psi_1|^4+\alpha_2|\psi_2|^2+\frac{1}{2}\beta_2|\psi_2|^4+\frac{\hbar^2\min\limits_\theta Q[\theta;\Phi]}{2}\left(\frac{|\psi_1|^2}{m_1}+\frac{|\psi_2|^2}{m_2}\right)-2|\psi_1||\psi_2|[\gamma-\eta\hbar^2\min_\theta Q[\theta;\Phi]]\cos\phi\\
&\geq \alpha_1|\psi_1|^2+\frac{1}{2}\beta_1|\psi_1|^4+\alpha_2|\psi_2|^2+\frac{1}{2}\beta_2|\psi_2|^4+\frac{\hbar^2\min\limits_\theta Q[\theta;\Phi]}{2}\left(\frac{|\psi_1|^2}{m_1}+\frac{|\psi_2|^2}{m_2}\right)-2|\psi_1||\psi_2||\gamma-\eta\hbar^2\min_\theta Q[\theta;\Phi]|\\
&\geq \alpha_1|\psi_1|^2+\frac{1}{2}\beta_1|\psi_1|^4+\alpha_2|\psi_2|^2+\frac{1}{2}\beta_2|\psi_2|^4-2|\psi_1||\psi_2||\gamma|+\frac{\hbar^2\min\limits_\theta Q[\theta;\Phi]}{2}\left(\frac{|\psi_1|^2}{m_1}+\frac{|\psi_2|^2}{m_2}-4|\eta||\psi_1||\psi_2|\right)\\
&\geq \alpha_1|\psi_1|^2+\frac{1}{2}\beta_1|\psi_1|^4+\alpha_2|\psi_2|^2+\frac{1}{2}\beta_2|\psi_2|^4-2|\psi_1||\psi_2||\gamma|\\
&\geq \alpha_1|\bar\psi_1|^2+\frac{1}{2}\beta_1|\bar\psi_1|^4+\alpha_2|\bar\psi_2|^2+\frac{1}{2}\beta_2|\bar\psi_2|^4-2|\bar\psi_1||\bar\psi_2||\gamma|\equiv\frac{F_0}{V_s},
\end{aligned}
\tag{16}
$$

where $|\bar\psi_1|$ and $|\bar\psi_2|$ comprise the solution of the system of equations

$$\alpha_1|\psi_1|+\beta_1|\psi_1|^3-|\gamma||\psi_2|=0,$$

$$\alpha_2|\psi_2|+\beta_2|\psi_2|^3-|\gamma||\psi_1|=0,$$

that minimizes the next to last row in Eq. (16). We underscore that if the condition (7) is not satisfied, i.e. if

$$|\eta|\geqslant\frac{1}{2\sqrt{m_1m_2}}.$$

the functional (3) and its simplified forms [for example, (15)] cannot be minimized (they have no lower limit) and for this reason cannot be used in physical applications.

Calculation of the functional derivatives in the conditions of stationarity

$$\frac{\delta\mathcal{F}}{\partial\theta}=0,\quad\frac{\delta\mathcal{F}}{\partial\phi}=0 \tag{17}$$

taking account of Eqs. (13) and (14) gives

$$\left(\frac{|\psi_1|^2}{2m_1}+\frac{|\psi_2|^2}{2m_2}+2\eta|\psi_1||\psi_2|\cos\phi\right)\frac{d^2\theta}{d\varphi^2} -2\eta|\psi_1||\psi_2|\frac{d\phi}{d\varphi}\left(\frac{d\theta}{d\varphi}-\frac{\Phi}{\Phi_0}\right)\sin\phi=0, \tag{18}$$

$$\frac{\partial}{\partial\varphi}\left(C\left(\left|\psi_1,||\psi_2|,\phi\right)\frac{d\phi}{d\varphi}\right)+\eta|\psi_1||\psi_2|\sin\phi\left(\frac{d\theta}{d\varphi}-\frac{\Phi}{\Phi_0}\right)^2 -\frac{1}{2}\frac{\partial C(|\psi_1|,|\psi_2|,\phi)}{\partial\phi}\left(\frac{d\phi}{d\varphi}\right)^2-\gamma|\psi_1||\psi_2|\sin\phi=0. \tag{19}$$

Since we are interested in the absolute minimum of the expression (15), Eqs. (18) the condition for continuity of the superconducting current circulating over the surface of the cylinder[16,19]) can be substantially simplified. Specifically, it is obvious from the chain of inequalities (16) that the absolute minimum of the expression (15) will obtain at the point where $d\phi/d\varphi=0(\phi=\text{const})$. Hence, we obtain instead of Eqs. (18) and (19) the elementary system

$$\sin\phi=0 \tag{20}$$

$$\frac{d^2\theta}{d\varphi^2}=0 \tag{21}$$

with the boundary conditions [see Eqs. (13) and (14)]

$$\theta(2\pi)-\theta(0)=2\pi n,\quad\frac{d\theta}{d\varphi}(2\pi)=\frac{d\theta}{d\varphi}(0), \tag{22}$$

$$n=0,\pm1,\pm2,\ldots$$

The choice of physically nonequivalent solutions (20) ($\phi=0\bmod 2\pi$ or $\phi=\pi\bmod 2\pi$) is also determined by the requirement that the expression (15) have a minimum [see the equalities (16)] and reduces to the condition[16]

$$\cos\phi=\mathrm{sgn}[\gamma-\eta\hbar^2\min_\theta Q[\theta;\Phi]]. \tag{23}$$

Evidently, the quantity Φ can influence the choice of the values of ϕ only in the cases $\eta\neq0$ and $\gamma\eta>0$. However, if $\gamma\eta\leqslant0$ ($\gamma\neq0$), the correct solution will be $\phi=\pi\bmod 2\pi$ for $\gamma<0$ and $\phi=0\bmod 2\pi$ for $\gamma>0$, irrespective of the values of Φ. The solution of Eq. (21) under the conditions (22) is obvious:

$$\theta(\varphi)=n\varphi+\theta(0),\quad n=0,\pm1,\pm2,\ldots,\quad\varphi\in[0,2\pi].$$

Substituting into the right-hand side of Eq. (15) the minimizing values of the variables ϕ and θ we find

$$\frac{\mathcal{F}[|\psi_1|,|\psi_2|;\Phi]}{V_s}=\left[\alpha_1+\frac{\hbar^2Q(\Phi)}{2m_1}\right]|\psi_1|^2+\frac{\beta_1}{2}|\psi_1|^4 +\left[\alpha_2+\frac{\hbar^2Q(\Phi)}{2m_2}\right]|\psi_2|^2+\frac{\beta_2}{2}|\psi_2|^4-2|\gamma-\eta\hbar^2Q(\Phi)||\psi_1||\psi_2|, \tag{24}$$

where the function

$$Q(\Phi)\equiv\frac{1}{R^2}\min_n\left(n-\frac{\Phi}{\Phi_0}\right)^2$$

is bounded and periodic:

$$0\leqslant Q(\Phi)\leqslant\frac{1}{4R^2},\quad Q(\Phi+\Phi_0)=Q(\Phi).$$

The conditions of stationarity

$$\frac{\partial\mathcal{F}}{\partial|\psi_1|}=0,\quad \frac{\partial\mathcal{F}}{\partial|\psi_2|}=0$$

give a system of equations for determining the equilibrium values of $|\psi_1|$ and $|\psi_2|$:

$$\left[\alpha_1+\frac{\hbar^2Q(\Phi)}{2m_1}\right]|\psi_1|+\beta_1|\psi_1|^3-|\gamma-\eta\hbar^2Q(\Phi)||\psi_2|=0, \tag{25}$$

$$\left[\alpha_2+\frac{\hbar^2Q(\Phi)}{2m_2}\right]|\psi_2|+\beta_2|\psi_2|^3-|\gamma-\eta\hbar^2Q(\Phi)||\psi_1|=0. \tag{26}$$

III. OSCILLATIONS OF THE SUPERCONDUCTING TRANSITION TEMPERATURE

The superconducting transition temperature for a prescribed value of the magnetic flux, $T_{c\Phi}$, is defined as the temperature at which the trivial solution $|\psi_1|=|\psi_2|=0$ of Eqs. (25) and (26) which corresponds to the normal phase becomes unstable. To determine the limits of stability of the trivial solution we shall examine the second differential of the functional (24) at the point $|\psi_1|=|\psi_2|=0$:

$$\frac{1}{V_s}\delta^2\mathcal{F}[\delta|\psi_1|,\delta|\psi_2|;T,\Phi]$$
$$=\left[\alpha_1(T)+\frac{\hbar^2Q(\Phi)}{2m_1}\right](\delta|\psi_1|)^2+\left[\alpha_2(T)+\frac{\hbar^2Q(\Phi)}{2m_2}\right](\delta|\psi_2|)^2$$
$$-2|\gamma-\eta\hbar^2Q(\Phi)|\delta|\psi_1|\delta|\psi_2|, \tag{27}$$

where $\delta|\psi_{1,2}|$ are arbitrary nonnegative infinitesimals. Obviously, the condition $\delta^2\mathcal{F}>0$ (positive definiteness of $\delta^2\mathcal{F}$) must hold at temperatures $T>T_{c\Phi}$. At the point $T=T_{c\Phi}$ the quadratic form (27) loses the property of positive-definiteness:

$$\delta^2\mathcal{F}\geqslant 0. \tag{28}$$

The condition (28) means that the smallest of the eigenvalues μ_1, μ_2 ($\mu_1<\mu_2$) of the matrix of the quadratic form

$$\hat{M}=\begin{pmatrix}\alpha_1+\dfrac{\hbar^2Q}{2m_1} & -|\gamma-\eta\hbar^2Q| \\ -|\gamma-\eta\hbar^2Q| & \alpha_2+\dfrac{\hbar^2Q}{2m_2}\end{pmatrix} \tag{29}$$

vanishes (see, for example, Ref. 17). Since

$$\mu_{1,2}=\frac{1}{2}\left(\alpha_1+\alpha_2+\frac{\hbar^2Q}{2m_1}+\frac{\hbar^2Q}{2m_2}\right)\left[1\mp\sqrt{1-4\frac{\left(\alpha_1+\dfrac{\hbar^2Q}{2m_1}\right)\left(\alpha_2+\dfrac{\hbar^2Q}{2m_2}\right)-(\gamma-\eta\hbar^2Q)^2}{\left(\alpha_1+\alpha_2+\dfrac{\hbar^2Q}{2m_1}+\dfrac{\hbar^2Q}{2m_2}\right)^2}}\right], \tag{30}$$

where the $-$ and $+$ signs refer to μ_1 and μ_2, respectively, we find from the condition $\mu_1(T_{c\Phi})=0$

$$T_{c\Phi}=T_c-\frac{T_c}{1+\sqrt{\left(\dfrac{a_{20}}{2a_2}\right)^2+\dfrac{\gamma^2}{a_1a_2}}-\dfrac{a_{20}}{2a_2}}$$
$$\times\left\{\sqrt{\left(\frac{a_{20}}{2a_2}\right)^2+\frac{\gamma^2}{a_1a_2}}+\frac{a_1\dfrac{\hbar^2Q(\Phi)}{2m_2}+a_2\dfrac{\hbar^2Q(\Phi)}{2m_1}}{2a_1a_2}-\frac{\sqrt{\left[a_1a_{20}+a_1\dfrac{\hbar^2Q(\Phi)}{2m_2}-a_2\dfrac{\hbar^2Q(\Phi)}{2m_1}\right]^2+4a_1a_2[\gamma-\eta\hbar^2Q(\Phi)]^2}}{2a_1a_2}\right\}, \tag{31}$$

where we have taken into account the definition (4) as well as the expression from Ref. 16 for the critical temperature of a two-band superconductor for $\Phi=0$ ($T_{c\Phi}|_{\Phi=0}\equiv T_c$):

$$T_c=T_{c1}\left(1+\sqrt{\left(\frac{a_{20}}{2a_2}\right)^2+\frac{\gamma^2}{a_1a_2}}-\frac{a_{20}}{2a_2}\right). \tag{32}$$

The relative shift of the critical temperature $\Delta t_c\equiv(T_c-T_{c\Phi})/T_c$ is of interest. For convenience in analyzing the function $\Delta t_c=\Delta t_c(\Phi)$ we introduce the dimensionless parameters

$$p\equiv\frac{a_{20}}{a_2},\quad l\equiv\frac{a_1}{a_2},\quad \bar{\gamma}\equiv\frac{\gamma}{a_1},\quad k\equiv\frac{m_1}{m_2},$$

$$\bar{\xi}\equiv\frac{\hbar}{R\sqrt{2m_1a_1}}\ll 1,\quad \bar{\eta}\equiv 2\eta m_1(|\bar{\eta}|<\sqrt{k}) \tag{33}$$

and the dimensionless function

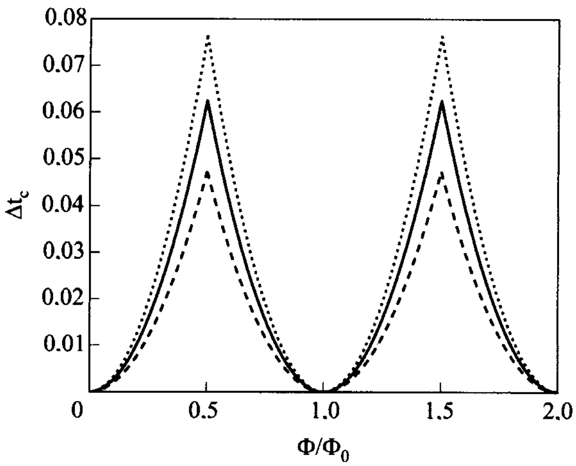

FIG. 2. $\Delta t_c = \Delta t_c(\Phi)$ for single-band (solid line) and two-band superconductors ($p = 1.1765$, $l = 0.8333$, $\bar{\gamma} = 0.7059$) with $\bar{\eta} = 0$ (dashed line) and $\bar{\eta}$ 0.9 (dotted line). The remaining parameters are: $\bar{\xi} = 0.5$, $k = 1$.

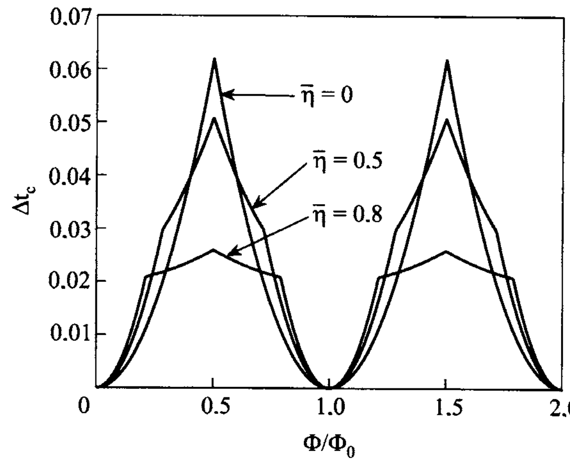

FIG. 3. $\Delta t_c = \Delta t_c(\Phi)$ for the case where the bands 1 and 2 have the identical parameters: $p = 0$; $l = k = 1$ and the values of $\bar{\eta}$ are indicated on the curves. The remaining parameters are: $\bar{\gamma} = 0.01$ and $\bar{\xi} = 0.5$.

$$\bar{Q}(\Phi) \equiv \min_n \left(n - \frac{\Phi}{\Phi_0} \right)^2 \times \left(0 \leq \bar{Q}(\Phi) \leq \frac{1}{4}, \bar{Q}(\Phi + \Phi_0) = \bar{Q}(\Phi) \right). \tag{34}$$

In terms of the parameters (33) and the function (34) the desired flux dependence has the form

$$\Delta t_c = \frac{1}{2 + \sqrt{p^2 + 4l\bar{\gamma}^2} - p} \{ \sqrt{p^2 + 4l\bar{\gamma}^2} + \bar{\xi}^2 (kl + 1) \bar{Q}(\Phi) - \sqrt{[p + \bar{\xi}^2 (kl - 1) \bar{Q}(\Phi)]^2 + 4l[\bar{\gamma} - \bar{\eta}\bar{\xi}^2 \bar{Q}(\Phi)]^2} \}. \tag{35}$$

The relations (31) and (35) give a complete mathematical description of the Little–Parks effect in two-band superconductors. As a particular case, they contain the classical Little–Parks effect for a single-band superconductor.[2–4] For definiteness, we shall examine the relation (35). Let $\bar{\gamma} = \bar{\eta} = 0$ (no interband interaction) and $p > 1$ (the band 2 is superconducting at all temperatures). As expected (see Refs. 2–4 and Introduction), we obtain:

$$\Delta t_c = \bar{\xi}^2 \bar{Q}(\Phi), \tag{36}$$

Contrary to the assertions made in Ref. 9, a strict periodic (with the period Φ_0) flux dependence $\Delta t_c = \Delta t_c(\Phi)$ for two-band superconductors follows immediately from the general relation (35), in complete analogy to the classical case (36). Just as in the classical case, the shift of the superconducting transition temperature vanishes at the points $\Phi/\Phi_0 = n$ ($n = 0, \pm 1, \pm 2, \ldots$) and it reaches its maximum value at the points $\Phi/\Phi_0 = n \pm 1/2$ ($n = 0, \pm 1, \pm 2, \ldots$), where the optimal values of the discrete time n change by 1. (At these latter points the function $\Delta t_c = \Delta t_c(\Phi)$ has singularities (non-differentiability).

The main qualitative difference from the classical case (36) is the nonparabolic character of the flux dependence $\Delta t_c = \Delta t_c(\Phi)$ in regions with the fixed optimal value of n, where $|\Phi/\Phi_0 - n| < 1/2$ (see Fig. 2). This characteristic effect of two-band superconductivity can be checked experimentally.

In Sec. II we indicated the possibility that the states $\phi = 0 \bmod 2\pi$ and $\phi = \pi \bmod 2\pi$ alternate when $\overline{\gamma\eta\xi} > 0$ (see the discussion of the condition (23)). Such alternation occurs if $0 < |\bar{\gamma}| < |\bar{\eta}|\bar{\xi}^2/4$. In particular, if $0 < |\bar{\gamma}| < |\bar{\eta}|\bar{\xi}^2/4$, the superconducting cylinder is in the state $\phi = 0 \bmod 2\pi$ in the regions $|\Phi/\Phi_0 - n| < \sqrt{\bar{\gamma}/(|\bar{\eta}|\bar{\xi}^2)}$ and $\phi = \pi \bmod 2\pi$ in the regions $\sqrt{\bar{\gamma}/(|\bar{\eta}|\bar{\xi}^2)} < |\Phi/\Phi_0 - n| < 1/2$. However, if $\bar{\eta}\bar{\xi}^2/4 < \bar{\gamma} < 0$, the states $\phi = 0 \bmod 2\pi$ and $\phi = \pi \bmod 2\pi$ alternate in the opposite order. Even though in the general case, the transitions between the states $\phi = 0 \bmod 2\pi$ and $\phi = \pi \bmod 2\pi$ are not accompanied by any new observable singularities of the function $\Delta t_c = \Delta t_c(\Phi)$, we shall give one special case where such singularities nonetheless arise.

Let the bands 1 and 2 have identical physical parameters:

$$\alpha_1 = \alpha_2 \equiv \alpha = -a \left(1 - \frac{T}{T_{c_1}} \right) (a_{20} = 0), \quad m_1 = m_2,$$

$$\beta_1 = \beta_2 \equiv \beta; \quad p = 0, l = k = 1. \tag{37}$$

Substituting the expressions (37) into the general relation (35) gives

$$\Delta t_c = \frac{1}{1 + |\bar{\gamma}|} [|\bar{\gamma}| + \bar{\xi}^2 \bar{Q}(\Phi) - |\bar{\gamma} - \bar{\eta}\bar{\xi}^2 \bar{Q}(\Phi)|]. \tag{38}$$

If

$$\overline{\gamma\eta} > 0, \quad 0 < |\bar{\gamma}| < \frac{|\bar{\eta}|\bar{\xi}^2}{4}, \tag{39}$$

the nondifferentiability of the right-hand side of Eq. (38) at the points

$$\frac{\Phi}{\Phi_0} = n \pm \sqrt{\frac{\bar{\gamma}}{|\bar{\eta}|\bar{\xi}^2}}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{40}$$

(the transition points between the states $\phi = 0 \bmod 2\pi$ and $\phi = \pi \bmod 2\pi$) engenders new observable singularities of the function $\Delta t_c = \Delta t_c(\Phi)$ which are completely absent in the classical case[2–4] (see Fig. 3).

IV. DISCUSSION AND CONCLUSIONS

The Ginzburg–Landau approach has been used to construct a theory of the Little–Parks effect for two-band superconductors. A quite simple relation (35) describing the dependence of the relative shift Δt_c of the superconducting transition temperature on the external magnetic flux Φ was obtained. The relation (35) contains as a particular case the well-known Little-Parks effect for single-band superconductors (36).[2–4]

The main difference from the classical effect (36), as can be checked experimentally, is the nonparabolic dependence $\Delta t_c = \Delta t_c(\Phi)$ in regions where $|\Phi/\Phi_0 - n| < 1/2$ ($n = 0, \pm 1, \pm 2, \ldots$). The assertion made in Ref. 9 that the Little–Parks oscillations are not periodic because the physical parameters of the band 1 are different from those of the band 2 is not confirmed: strict periodicity (with the period Φ_0) of the flux dependence $\Delta t_c = \Delta t_c(\Phi)$, just as in the classical case (36), follows uniquely from the general relation (35). Conversely, the new observed features (nondifferentiability) of the function $\Delta t_c = \Delta t_c(\Phi)$, which are completely absent in the classical case, appear precisely when the bands 1 and 2 have the same parameters: see Eq. (38) with the conditions (39) and Fig. 3.

The investigation of the extremal properties of the initial free-energy functional (3) led to the conclusion that the phenomenological parameter η is subject to an important limitation (7). If the condition (7) does not hold, the functional (3) has no lower bound and no minimum and cannot be used in physical applications. As far as we know, the condition (7) has not been noted previously in the literature.

In addition, it has been shown that the absolute minimum of the free-energy functional (3) corresponds to states with constant difference of the phases φ of the complex-valued components of the order parameter ψ_1 and ψ_2 ($\varphi = 0 \bmod 2\pi$ or $\varphi = \pi \bmod 2\pi$). For this reason the soliton states associated with the gradients of ϕ and discussed in the literature[18,19] do not contribute to the Little–Parks oscillations.

Summarizing, it is our hope that the results obtained in the present work will stimulate experimental studies of the Little–Parks effect in two-band superconductors and will lead to further elaboration of the theory, both in the Ginzburg–Landau approach as well as on a consistently microscopic basis.

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