Dynamic behavior of a superconducting flux qubit excited by a series of electromagnetic pulses

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We study theoretically the behavior of a superconducting flux qubit subjected to a series of electromagnetic pulses. The possibility of controlling the system state via changing the parameters of the pulse is studied. We calculated the phase shift in the tank circuit weakly coupled to the qubit which can be measured by the impedance measurement technique. For the flux qubit we consider the possibility of estimating the relaxation rate from the impedance measurements by varying the delay time between the pulses. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2747079]

I. INTRODUCTION

Quantum effects in mesoscopic superconducting circuits based on small Josephson junctions have attracted renewed attention. It has been demonstrated that Josephson devices at low temperature behave like quantum two-level systems. Therefore, ideas developed in atomic and molecular physics can be used for description of artificially fabricated circuits of macroscopic size. These concepts are stimulated further by the prospects for realizing quantum bits (qubits) for quantum information processing. Qubits are effective two-level quantum systems with externally controlled parameters. In the last decade a large number of proposals for implementing qubits based on Josephson elements have been proposed.[1–4] There are three basic types of Josephson-junction circuits that behave quantum mechanically at low temperature. They are charge,[1] phase,[2] and flux[3] qubits. All of them can be fabricated with high precision with the help of modern lithography and can be the basis of the quantum computer. A promising implementation for quantum computations is a 3JJ flux qubit consisting of a superconducting loop with three Josephson junctions.[3] This type of qubit is insensitive to charge noise, and it has been shown that it has a high quality factor.[5] It was predicted that such systems should exhibit various quantum-mechanical effects including macroscopic quantum tunneling of the flux.[6] Indeed, the predicted effects have been observed experimentally.[2,7,8] The quantum dynamics in single qubits was studied in Refs. 3, 4, and 9.

In our work we study the dynamics of the flux qubit subjected to a series of rectangular electromagnetic pulses. We present the model we use for calculations of the phase shift α in the resonant tank circuit based on the density matrix approach. Next we analyze the case which permits analytical solution and obtain small addition to the α in the ground state as function of the relaxation rate Γ_R. For arbitrary parameters we solve equations numerically and compare obtained result with analytical calculations.

II. THE MODEL

Our aim is to study the behavior of a superconducting 3JJ flux qubit excited by a series of rectangular electromagnetic pulses. The flux qubit consists of a superconducting loop with three junctions: two identical and one with the parameters differing by factor β. For all calculations below we take β equal to 0.8.

The Hamiltonian of the flux qubit in the two-level approximation has the form:[10,11]

$$\hat{H} = -\varepsilon\hat{\sigma}_z - \Delta\hat{\sigma}_x, \tag{1}$$

where the diagonal term ε is the bias and the off-diagonal term $\Delta \propto \exp(-E_J/E_C)$ is the tunneling amplitude between the wells. Here $\hat{\sigma}_x$ and $\hat{\sigma}_z$ are Pauli matrices in the basis $\{|\downarrow\rangle, |\uparrow\rangle\}$ of the current operator in the qubit: $\hat{I}=I_0\hat{\sigma}_z$, $I_0 = I_C\lambda(\beta,g)/2\pi$, where I_C is the critical current of the qubit, $g=E_J/E_C$, the explicit formula for $\lambda(\beta,g)$ can be found in Ref. 12. The eigenstates of $\hat{\sigma}_z$ correspond to the clockwise $(\hat{\sigma}_z|\downarrow\rangle=-|\downarrow\rangle$ and counterclockwise $(\hat{\sigma}_z|\uparrow\rangle=|\uparrow\rangle)$ currents in the qubit. The bias

$$\varepsilon = I_0\Phi_0\left(f - \frac{1}{2}\right) \tag{2}$$

is controlled by the dimensionless applied magnetic flux $f = \Phi_x/\Phi_0$ through the qubit; $\Phi_0=h/2e$ is the flux quantum.

The magnetic flux consists of two components:

$$f = f_{DC} + \tilde{f}(t), \tag{3}$$

which describe the adiabatically changing magnetic flux, f_{DC}, and the time-dependent component, $\tilde{f}(t)$. We will study the possibility to control the system state via the series of the rectangular pulses with the amplitude f_A and duration from $t_1^{(n)}=n(T+\tau)$ to $t_2^{(n)}=n(T+\tau)+T$:

$$\tilde{f}(t) = \sum f_A[\theta(t-t_1^{(n)}) - \theta(t-t_2^{(n)})], \tag{4}$$

where $\theta(t)$ stands for the theta function, T is the pulse duration, and τ is the delay between pulses (Fig. 1). The effect of the pulse is to change the level occupation probabilities and make them oscillating functions of time during the pulse. It should be noted that in the basis $\{|\downarrow\rangle, |\uparrow\rangle\}$ of the current operator, which are not eigenstates of the Hamiltonian, the probabilities oscillate both during and after the pulse.

We describe the system's evolution with the Bloch equation for the density matrix $\hat{\rho}$ ($\hbar=1$):

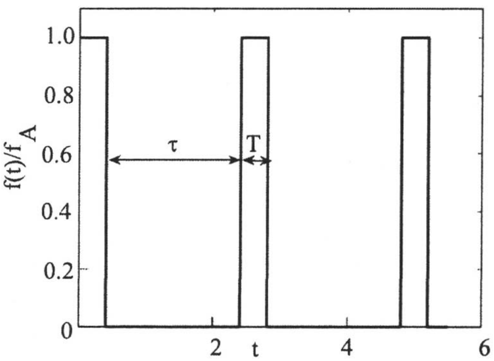

FIG. 1. The series of pulses.

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H},\hat{\rho}] + \hat{\Gamma}\hat{\rho}. \quad (5)$$

The impedance measurement technique[13,14] consists in that the tank circuit probes the effective inductance of the system via measuring the phase shift α between the voltage and current in the tank circuit. The phase shift α is related to the Josephson inductance $\mathcal{L}$ of the qubit as follows:

$$\tan\alpha \approx k^2 Q L \mathcal{L}^{-1}, \quad (6)$$

$$\mathcal{L}^{-1} = \frac{1}{\Phi_0}\frac{\partial\langle\hat{I}\rangle}{\partial f_{DC}} = = \frac{I_0}{\Phi_0}\frac{\partial}{\partial f_{DC}}\mathrm{Tr}(\hat{\rho}\hat{\sigma}_z). \quad (7)$$

Here M is the mutual inductance of the qubit with the tank circuit; $Q=R_T\sqrt{C_T/L_T}$ and $k=M/\sqrt{LL_T}$ are the quality factor and the coupling coefficient for the tank circuit, which consists of the inductor, L_T, capacitor, C_T, and resistor, R_T, connected in parallel (see Ref. 15 for more details).

III. EXCITATION OF THE FLUX QUBIT WITH THE SERIES OF PULSES

In this Section we study the excitation of the flux qubit with the series of rectangular pulses. We start from the general 1-qubit Hamiltonian that has the form of Eq. (1) in the basis of states $\{|\downarrow\rangle,|\uparrow\rangle\}$, assuming $\tilde{f}(t)=0$. For a flux qubit these states correspond to a definite direction of the current circulating in the ring. First the time-independent Hamiltonian is diagonalized in the basis of eigenstates $\{|-\rangle,|+\rangle\}$ with the rotation matrix $\hat{S}$:

$$\hat{S} = \begin{pmatrix} \cos(\eta/2) & \sin(\eta/2) \\ -\sin(\eta/2) & \cos(\eta/2) \end{pmatrix},$$

with $\sin\eta=-\Delta/\sqrt{\Delta^2+\varepsilon^2}$, $\cos\eta=\varepsilon/\sqrt{\Delta^2+\varepsilon^2}$.

For the calculation of the observable value, the phase shift α in the tank circuit, according to Eq. (6), we need the density matrix in the energy representation, where its diagonal components are equal to the probability of the system to be in the ground $|-\rangle$ or excited state $|+\rangle$.

Next we introduce the time-dependent terms into the time-independent Hamiltonian. Making use of the transformation $\hat{H}(t)=\hat{S}^{-1}\hat{H}(t)\hat{S}$, we get the Hamiltonian $\hat{H}(t)$ in the energy representation for the flux qubit:[16]

$$\hat{H}(t) = -\frac{\Delta E}{2}\hat{\tau}_z - 2I_0\Phi_0\tilde{f}(t)(\cos\eta\hat{\tau}_z + \sin\eta\hat{\tau}_x)/\Delta E, \quad (8)$$

$$\Delta E = 2\sqrt{\Delta^2+\varepsilon^2}. \quad (9)$$

The time evolution of the density matrix, which can be taken in the form $\hat{\rho}=(\hat{1}+X\hat{\tau}_x+Y\hat{\tau}_y+Z\hat{\tau}_z)/2$, is described by the equation of motion (5). Initial condition for the density matrix in the $\{|-\rangle,|+\rangle\}$ basis is $X(0)=Y(0)=0$, $Z(0)=1$, which corresponds to the ground state of the system. Solving the system of equations for $X(t)$, $Y(t)$, $Z(t)$ with phenomenologically introduced dephasing and relaxation rates Γ_φ and Γ_R:

$$\frac{dX}{dt} = (\Delta E + h(t)\cos(\eta))Y - \Gamma_\varphi X,$$

$$\frac{dY}{dt} = h(t)\sin(\eta)Z - (\Delta E + h(t)\cos(\eta))X - \Gamma_\varphi Y,$$

$$\frac{dZ}{dt} = -h(t)\sin(\eta)Y - \Gamma_R(Z - Z(0)), \quad (10)$$

with $h(t)=2I_0\Phi_0\tilde{f}(t)$, we obtain the probability of occupation of the upper level $|+\rangle$ (the excited state) $P_+(t)=\rho_{22}(t)=(1-Z(t))/2$. We calculate the density matrix in the flux basis making use of the transformation $\hat{\rho}_{\mathrm{flux}}=\hat{S}\hat{\rho}\hat{S}^{-1}$ and obtain the probability of the current to be circulating in the clockwise direction $P_L(t)$, according to:

$$P_L(t) = \frac{1}{2}(1 - \sin(\eta)X(t) - \cos(\eta)Z(t)). \quad (11)$$

Averaging $P_L(t)$ over t, we calculate phase shift α according to Eq. (6). For arbitrary values of the parameters of applied perturbation the system of equations (10) can be solved numerically. In the limiting case of one pulse (that is $\tau=\infty$) and without relaxation processes taken into account the solution can be found in Ref. 17.

Before presenting the numerical results, consider the limiting case which permits the analytical solution:

$$T \ll T_{R,\varphi} = \Gamma_{R,\varphi}^{-1} \ll \tau. \quad (12)$$

In this case we can neglect the decay rates Γ_R and Γ_φ in (10) during the excitation time T and assume that after the pulse during the delay time τ the system is returned to the ground state. Periodically repeating this process, we will have the input of relaxation processes in the time-averaged characteristics of the system.

With the assumptions (12) and for zero temperature the solution of the equations (10) can be found for the two time intervals: during the pulse ($0<t<T$) and after the pulse ($T<t<T+\tau$).

The solution for $0<t<T$ is the following:

$$X_1(t) = \frac{2AC}{A^2+C^2}\sin^2\left(\frac{1}{2}\sqrt{A^2+C^2}t\right),$$

$$Y_1(t) = -\frac{A}{\sqrt{A^2+C^2}}\sin(\sqrt{A^2+C^2}t),$$

$$Z_1(t)=\frac{1}{A^2+C^2}(C^2+A^2\cos(\sqrt{A^2+C^2}t)), \tag{13}$$

where $A=-h\sin\eta$, $C=-\Delta E-h\cos\eta$, $h=2I_0\Phi_0 f_A$; and, for $T<t<T+\tau$:

$$X_2(t)=\exp[-\Gamma_\varphi(t-T)][X_1(T)\cos(\Delta E(t-T)) + Y_1(T)\sin(\Delta E(t-T))],$$

$$Z_2(t)=1-\exp[-\Gamma_R(t-T)](1-Z_1(T)).$$

Let the duration of the pulse T be equal to $\pi/\sqrt{A^2+C^2}$, which corresponds to one cycle of excitation during the time T.

Then we obtain:

$$X_2(t)=\exp(-\Gamma_\varphi(t-T))\frac{2AC}{A^2+C^2}\cos(\Delta E(t-T)),$$

$$Z_2(t)=1-\frac{2A^2}{A^2+C^2}\exp(-\Gamma_R(t-T)).$$

Taking into account the inequalities (12), we obtain for the time-averaged values $\bar{Z}$ and $\bar{X}$ the following expressions:

$$\bar{X}\approx\frac{2AC}{A^2+C^2}\frac{\Gamma_\varphi}{(\Gamma_\varphi^2+\Delta E^2)\tau}, \tag{14}$$

$$\bar{Z}\approx 1-\frac{2A^2}{A^2+C^2}\frac{1}{\Gamma_R\tau}. \tag{15}$$

Before substituting these values in the (11) and (16) we estimate the contribution of $\bar{X}$ and $\bar{Z}$ to the phase shift α for the parameters we use for calculation. Our evaluation indicates that the contribution of $\bar{X}$ is about three orders lower than that of $\bar{Z}$, so we neglect the term containing $\bar{X}$ in Eq. (11). We calculate the small addition to the phase shift α due to the relaxation process $\alpha^{(1)}$. At the point $f=1/2$ we obtain after some algebra:

$$\alpha^{(1)}=\frac{k^2QL}{\Phi_0}\frac{2f_A^2(I_0\Phi_0)^3}{\Delta((I_0\Phi_0 f_A)^2+\Delta^2)}\frac{1}{\tau\Gamma_R}. \tag{16}$$

Hence from the measurement of the phase shift α at the point $f=1/2$ it is possible to estimate the relaxation rate according to Eq. (16). The behavior of $\alpha^{(1)}$ at the point $f=1/2$ as a function of the product $\tau\Gamma_R$ is presented in Fig. 2. In the calculations we use the same parameters as in the numerical calculations below.

Now we study the excitation of the flux qubit by the series of rectangular pulses numerically. Namely we calculate the phase shift in the tank circuit by making use of the solution of the Eqs. (6) and (10). Fig. 3 is plotted for the following set of the parameters for the qubit: $I_0\Phi_0=200$ GHz, $\Delta=1.4$ GHz, $k^2Q(LI_0)/\Phi_0=2\times10^{-3}$; the excitation $\tilde{f}(t)$ was considered to be a series of pulses with $\tau=T\ll\Gamma^{-1}$, $\tau=0.5\Gamma^{-1}$, $\tau=2\Gamma^{-1}$ (from upper to lower) and the decay rates $\Gamma_R=\Gamma_\varphi=\Gamma=0.1$ GHz. We observe that at $\tau\gtrsim\Gamma^{-1}$ the resonances disappear with increasing the delay time τ. This can be used in practice for the relatively simple estimation of the decay rates by changing the delay time between the pulses.

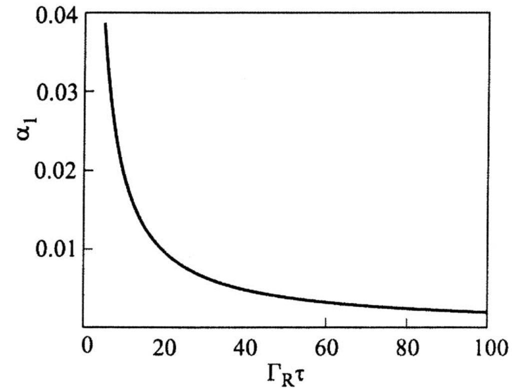

FIG. 2. The addition to the phase shift α_1 for the flux qubit excited by a series of rectangular pulses, due to the relaxation process in the point $f=0.5$.

Next we compare the theoretically (dashed) and numerically (marked by points) calculated curves for the phase shift α calculated under the assumption (12) from the (14) and (15). For the numerical calculations we use the following parameters of the pulse: $T=0.5$, $\tau=100$, $f_A=0.005$ and of the decay rates $\Gamma_R=\Gamma_\varphi=\Gamma=0.1$ GHz. Such values of τ, T, and $\Gamma_{R,\varphi}$ correspond to the limiting case which we considered previously (12), and one can see very good agreement in Fig. 4.

IV. CONCLUSION

We have studied the dynamics of a flux qubit subjected to a series of rectangular electromagnetic pulses. We investigated the changes of the tank circuit phase shift α for the single qubit that appears due to excitation by the pulses. It was demonstrated that the response of the tank circuit essentially depends on the relation between the decay rates $\Gamma_{R,\varphi}$ and the delay time τ, which may be used for estimation of Γ_R by measuring the phase shift α as a function of the delay time τ.

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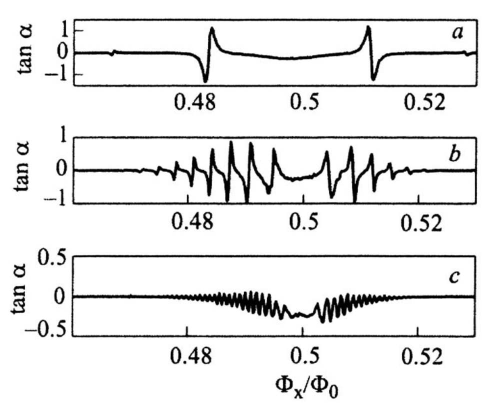

FIG. 3. The phase shift α for the flux qubit excited by a series of rectangular pulses with: $T=\tau=0.5$ (a), $T=0.5$, $\tau=5$ (b), $T=0.5$, $\tau=20$ (c).

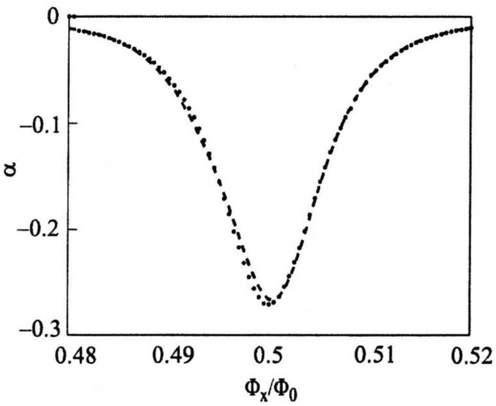

FIG. 4. Comparison of the theoretical (dashed) and numerical (marked by points) curves for excitation by a series of pulses with the parameters: $T=0.5$, $\tau=100$.

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