Closed geodesics without self-intersections on regular tetrahedra in spaces of constant curvature

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Since regular triangles form a regular tiling of Euclidean plane it easily follows the full classification of closed geodesics on a regular tetrahedron in Euclidean space.

We described all simple (without self-intersections) closed geodesics on regular tetrahedra in three-dimensional hyperbolic and spherical spaces. In these spaces the tetrahedron's curvature is concentrated not only into its vertices but also into its faces. The intrinsic geometry of such tetrahedra depends on the value of faces angles.

The faces angle α of tetrahedron in hyperbolic space satisfies $0 < \alpha < \pi/3$, and in spherical space the faces angle α satisfies $\pi/3 < \alpha \leq 2\pi/3$.

A simple closed geodesic on a tetrahedron has the type (p, q) if it has p points on each of two opposite edges of the tetrahedron, q points on each of another two opposite edges, and $(p + q)$ points on each edges of the third pair of opposite one.

We prove that on a regular tetrahedron in hyperbolic space for any coprime integers (p, q), $0 \leq p < q$, there exists unique, up to the rigid motion of the tetrahedron, simple closed geodesic of type (p, q). These geodesics exhaust all simple closed geodesics on a regular tetrahedron in hyperbolic space. The number of simple closed geodesics of length bounded by L is asymptotic to constant (depending on α) times L^2, when L tends to infinity [1].

On a regular tetrahedron in spherical space with faces angle α, such that $\pi/3 < \alpha < 2\pi/3$, there exists the finite number of simple closed geodesic. The length of all these geodesics is less than 2π. For any coprime integers (p, q) we presented the numbers α_1 and α_2, depending on p, q and satisfying the inequalities $\pi/3 < \alpha_1 < \alpha_2 < 2\pi/3$, such that
1) if $\pi/3 < \alpha < \alpha_1$, then on a regular tetrahedron in spherical space with the faces angle α there exists unique simple closed geodesic of type (p, q), up to the rigid motion of this tetrahedron;
2) if $\alpha_2 < \alpha < 2\pi/3$, then on a regular tetrahedron with the faces angle α there is not simple closed geodesic of type (p, q) [2].

If $\alpha = 2\pi/3$, then a tetrahedron coincides with the unit two-dimensional sphere. Hence there are infinitely many simple closed geodesics on it and they are great circles of the sphere.

References

[1] A. A. Borisenko, D. D. Sukhorebska, "Simple closed geodesics on regular tetrahedra in hyperbolic space", Mat. Sb., 2020, 211(5), p.3-30. (in Russian). *English translation*: SB MATH, 2020, 211(5), DOI:10.1070/SM9212

[2] A. A. Borisenko, D. D. Sukhorebska, "Simple closed geodesics on regular tetrahedra in spherical space", Mat. Sb. (accepted).
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