On relation between statistical ideal and ideal generated by a modulus function

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The applicant's Ph.D. research deals with filters and ideals of sets and their applications in mathematical analysis and functional analysis. He authored and co-authored 4 articles [2, 3, 4, 6] related to this subject. The application is based on the last article [6], authored by the applicant solely.

Let Ω be a non-empty set. A non-empty family $\mathfrak{I} \subset 2^\Omega$ is called *an ideal* on Ω if $\mathfrak{I}$ satisfies: $\Omega \notin \mathfrak{I}$; if $A, B \in \mathfrak{I}$ then $A \cup B \in \mathfrak{I}$; if $A \in \mathfrak{I}$ and $D \subset A$ then $D \in \mathfrak{I}$. For a subset $A \subset \mathbb{N}$ denote $\alpha_A(n) := |A \cap [1, n]|$, where $|M|$ stands for a number of elements in the set $M \subset \mathbb{N}$. Let $A \subset \mathbb{N}$. *The natural density* of A is $d(A) := \lim\limits_{n\to\infty} \dfrac{\alpha_A(n)}{n}$. The ideal of sets $A \subset \mathbb{N}$ having $d(A) = 0$ is called *the statistical ideal*. We denote this ideal $\mathfrak{I}_s$. In [1] authors introduced the following generalization of the natural density. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an unbounded modulus function, that is $f(x) = 0$ if and only if $x = 0$; $f(x + y) \le f(x) + f(y)$ for all $x, y \in \mathbb{R}^+$; $f(x) \le f(y)$ if $x \le y$; f is continuous from the right at 0; $\lim\limits_{n\to\infty} f(n) = \infty$. Let f be a modulus function. The quantity $d_f(A) := \lim\limits_{n\to\infty} \dfrac{f(\alpha_A(n))}{f(n)}$ is called *the f-density* of $A \subset \mathbb{N}$. The ideal $\mathfrak{I}_f := \{A \subset \mathbb{N} : \ d_f(A) = 0\}$ is called *the f-ideal*. The ideal $\mathfrak{I}_f$ appears implicitly in [1] and explicitly in [4]. It is known that $\mathfrak{I}_f \subset \mathfrak{I}_s$. The aim of the paper is to present a complete description of those modulus functions f for which $\mathfrak{I}_f = \mathfrak{I}_s$. In [5] the question of description of those modulus functions f for which $\mathfrak{I}_f = \mathfrak{I}_s$ was approached (in a bit different terminology). The authors called a modulus function f *compatible* if for every $\varepsilon > 0$ there exists $\varepsilon' > 0$ and $n_0(\varepsilon) \in \mathbb{N}$ such that $\dfrac{f(\varepsilon' n)}{f(n)} < \varepsilon$ for all $n \ge n_0$. [5, Proposition 2.7] says that if f is a compatible function then $\mathfrak{I}_s \subset \mathfrak{I}_f$. Also in [5, Proposition 2.9] the inverse implication was claimed to be also true. Unfortunately, the proof of [5, Proposition 2.9] contains a logical error. In the paper [6], with the help of the quantitive parameter $h_f(t) := \limsup\limits_{n\to\infty} \dfrac{f(n)}{f(tn)}$, the applicant gives the promised complete description of those modulus functions f for which $\mathfrak{I}_f = \mathfrak{I}_s$, which, in particular, fixes the gap in the proof of the above mentioned result from [5, Proposition 2.9].

Theorem 1. *For an unbounded modulus function f the following statements are equivalent:* (1) $\mathfrak{I}_s = \mathfrak{I}_f$, (2) $\lim\limits_{t\to\infty} h_f(t) = 0$, *and* (3) f *is compatible.*

References

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