Cauchy problem for the nonlocal NLS equation with symmetric nonzero boundary conditions

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We consider the Cauchy problem for the focusing nonlocal nonlinear Schrödinger (NNLS) equation (here and below $\bar{q}$ is the complex conjugate of q)

$$iq_t(x,t)+q_{xx}(x,t)+2q^2(x,t)\bar{q}(-x,t)=0, \qquad x\in\mathbb{R},\ t>0, \tag{1a}$$

$$q(x,0)=q_0(x), \qquad x\in\mathbb{R}, \tag{1b}$$

with symmetric nonzero boundary conditions $q(x,t)\to Ae^{2iA^2t}$, $x\to\pm\infty$, $t\geq 0$, with some $A>0$. The NNLS equation was introduced by Ablowitz and Musslimani in 2013 [1] as a new nonlocal reduction of the classical integrable Ablowitz-Kaup-Newell-Segur system.

The choice of the symmetric boundary conditions is inspired by considerable interest in describing the effect of modulation instability (MI) in recent years. For both focusing NNLS and classical NLS equations, the stationary wave Ae^{2iA^2t} is unstable under small perturbations. In [4] Santini considered the periodic Cauchy problem for (1a) and showed, using the perturbation approach, that Akhmediev-type rogue waves are relevant for describing the evolution of the solution in an intermediate region, i.e., for t such as $1\ll t\ll O(|\log\varepsilon|)$, where the perturbation of the stationary wave is $O(\varepsilon)$.

In our work we study the nonlinear stage of MI for the NNLS equation. We do so by analyzing the solution $q(x,t)$ as $t\to\infty$, which can't be done by linearizing equation (1a), cf. [4]. In the context of the NLS equation this problem was considered by Biondini and Mantzavinos [2]. They showed that in the solitonless case the large-time asymptotics of the modulus of the solution is formed solely by the boundary conditions of the problem, so the asymptotic stage of MI is universal. Using the inverse scattering transform method together with the Deift and Zhou method we showed, that the asymptotic stage of MI in the case of the NNLS equation is *not* universal and essentially depends on $q_0(x)$ [3]:

Theorem 1. *(Plane wave region)*
Let $q_0(x)$ is close, in a sense, to the constant A. Then the asymptotics of the solution $q(x,t)$ of problem (1) along the rays $\xi=\frac{x}{4t}=const$ with $|\xi|>\sqrt{2}A$ has the form:

$$q(x,t)=Ae^{-2\operatorname{Im}F_\infty(k_1)}e^{2i(A^2t+\operatorname{Re}F_\infty(k_1))}+o(1), \qquad t\to\infty,\quad \xi>\sqrt{2}A, \tag{2a}$$

$$q(-x,t)=Ae^{2\operatorname{Im}F_\infty(k_1)}e^{2i(A^2t+\operatorname{Re}F_\infty(k_1))}+o(1), \qquad t\to\infty,\quad -\xi<-\sqrt{2}A, \tag{2b}$$

where $k_1=\frac{1}{2}\left(-\xi-\sqrt{\xi^2-2A^2}\right)$ for $x>0$ and the complex constant $F_\infty(k_1)$ can be found in terms of $q_0(x)$.

Theorem 2. *(Modulated elliptic wave region)*
Under the same assumptions as in Theorem 1, the asymptotics of the solution $q(x,t)$ along the rays $\xi=\frac{x}{4t}=const$ with $0<|\xi|<\sqrt{2}A$ has the form:

$$\begin{aligned}q(x,t)=(A+\operatorname{Im}\alpha)e^{-2\operatorname{Im}G_\infty(k_0,\alpha)}&\frac{\Theta(\frac{\Omega t}{2\pi}+\frac{\omega}{2\pi}-\frac{1}{4}-v_\infty+c)\Theta(v_\infty+c)}{\Theta(\frac{\Omega t}{2\pi}+\frac{\omega}{2\pi}-\frac{1}{4}+v_\infty+c)\Theta(-v_\infty+c)}\\ &\times e^{2i(tH_\infty+\operatorname{Re}G_\infty(k_0,\alpha))}+o(1),\quad 0<\xi<\sqrt{2}A,\end{aligned} \tag{3a}$$

$$\begin{aligned}q(-x,t)=(A+\operatorname{Im}\alpha)e^{2\operatorname{Im}G_\infty(k_0,\alpha)}&\frac{\Theta(\frac{\Omega t}{2\pi}+\frac{\overline{\omega}}{2\pi}-\frac{1}{4}+\overline{v_\infty}-\overline{c})\Theta(-\overline{v_\infty}-\overline{c})}{\Theta(\frac{\Omega t}{2\pi}+\frac{\overline{\omega}}{2\pi}-\frac{1}{4}-\overline{v_\infty}-\overline{c})\Theta(\overline{v_\infty}-\overline{c})}\\ &\times e^{2i(tH_\infty+\operatorname{Re}G_\infty(k_0,\alpha))}+o(1),\quad 0>-\xi>-\sqrt{2}A.\end{aligned} \tag{3b}$$

Here Θ is the genus-1 theta function; constants α, Ω, v_∞ and c, do not depend on the initial data $q_0(x)$, while the complex constants ω, H_∞ and $G_\infty(k_0,\alpha)$ can be found in terms of $q_0(x)$.

References

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