The modified Camassa-Holm equation on a nonzero background: large-time asymptotics for the Cauchy problem

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We consider the initial value (Cauchy) problem for the modified Camassa–Holm (mCH) equation:

$$m_t + \left((u^2 - u_x^2)m\right)_x = 0, \quad m := u - u_{xx}, \quad t > 0, \ -\infty < x < +\infty, \tag{0.1a}$$

$$u(x,0) = u_0(x), \qquad -\infty < x < +\infty, \tag{0.1b}$$

assuming that $u_0(x) \to 1$ as $x \to \pm\infty$ and that the time evolution preserves this behavior: $u(x,t) \to 1$ as $x \to \pm\infty$ for all $t > 0$.

The modified Camassa–Holm equation is an integrable modification, with cubic nonlinearity, of the Camassa–Holm (CH) equation $m_t + (um)_x + u_x m = 0, \quad m := u - u_{xx}$. It was introduced as a new integrable system by Fuchssteiner, Olver and Rosenau, and arises as a model equation in the theory of nonlinear water waves. The CH and mCH equations are both integrable in the sense that they have Lax pair representations, which allows developing the inverse scattering method, in one form or another, to study the properties of solutions of Cauchy problems for these equations.

In [1], we have developed the inverse scattering transform method in the form of Riemann–Hilbert (RH) problem for the Cauchy problem (0.1). In particular, we have obtain the parametric representation of the solution of this Cauchy problem in terms of the solution of an associated (singular) RH problem.

In [2], we have been aimed to obtain the leading asymptotic terms for the solution of the Cauchy problem (0.1), taking the formalism developed in [1] as the starting point. We have focused on the study of the solitonless case assuming that there are no residue conditions (inclusion of the discrete spectrum can then be made by, e.g., using the Blaschke–Potapov factors; by this approach, the original RH problem is reduced to a RH problem without residue conditions). For this purpose, we have reduced the original (singular) RH problem representation for the solution to the solution of a regular RH problem (i.e., to a RH problem with the jump and normalization conditions only). Then we have analyzed asymptotically, as $t \to +\infty$, the solution of this regular RH problem using the nonlinear steepest descent method as a tool. This method consists in successive transformations of the original RH problem, in order to reduce it to an explicitly solvable problem. The different steps include appropriate triangular factorizations of the jump matrix; "absorption" of the triangular factors with good large-time behavior; reduction, after rescaling, to a RH problem which is solvable in terms of certain special functions.

Finally, we have obtained the leading asymptotic terms for the solution $u(x,t)$ of the Cauchy problem (0.1), in the two sectors of the (x,t) half-plane, $1 < \zeta := \frac{x}{t} < 3$ and $\frac{3}{4} < \zeta < 1$, where the deviation from the background value is nontrivial (in the remaining sectors $\frac{x}{t} > 3$ and $\frac{x}{t} < \frac{3}{4}$, $u(x,t)$ decays rapidly to 1):

- $1 < \zeta < 3$: $u(x,t) = 1 + \frac{C_1}{\sqrt{t}} \cos\left\{C_2 t + C_3 \ln t + C_4\right\} + \mathrm{o}(t^{-1/2})$;
- $\frac{3}{4} < \zeta < 1$: $u(x,t) = 1 + \sum_{j=0,1} \frac{C_1^{(j)}}{\sqrt{t}} \cos\left\{C_2^{(j)} t + C_3^{(j)} \ln t + C_4^{(j)}\right\} + \mathrm{o}(t^{-1/2})$.

Here $C_i(\zeta)$, $C_i^{(j)}(\zeta)$ are functions of ζ specified in terms of the scattering data, which in turn are uniquely specified by the initial data.

References

[1] A. Boutet de Monvel, I. Karpenko, and D. Shepelsky, *A Riemann-Hilbert approach to the modified Camassa–Holm equation with nonzero boundary conditions*, J. Math. Phys. **61** (2020), no. 3, 031504, 24.

[2] ———, *The modified Camassa-Holm equation on a nonzero background: large-time asymptotics for the Cauchy problem*, preprint: arXiv:2011.13235.